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A REGULARIZATION OF FREDHOLM TYPE SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. We present a method to regularize first and second kind integral equations of Fredholm type with singular kernel. By appropriate application of the Poincaré-Bertrand formula we change such integral equations into a second kind Fredholm's integral equation with at most weakly singular kernel.

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1. Introduction. As many mathematical models in applied problems in physics and engineering lead to a first or second kind Fredholm's integral equation with singular kernel [1], considering this problem for the following investigation is justified. According to Fubini's theorem [5, 6] over a bounded region in $\mathbb{R}^2$, to calculate a repeated integral we can integrate in either order. This result clearly holds for any continuous function $f(x,y)$. Even more important is the fact that the Fubini's theorem holds for discontinuous $f(x,y)$, for example, if integrals in either order are weakly singular or only one of them is singular. If both integrals appearing in the repeated integral are singular then the Fubini's theorem no longer holds. So, by the Poincaré-Bertrand formula [2, 4] we have

$$\int_S \frac{dt}{t-t_0}\int_S \frac{\phi(t,t_1)}{t_1-t}dt_1 = -\pi^2\phi(t_0,t_0)+\int_S dt_1\int_S \frac{\phi(t,t_1)}{(t-t_0)(t_1-t)}dt, \tag{1.1}$$

where S, the boundary of a bounded region D in $\mathbb{R}^2$, is a closed curve. $C^{(k,h)}(\Omega)$ is the class of all functions defined over a domain Ω that along with its partial derivatives up to k are continuous of Hölder exponent $0<h<1$. A region $\Omega\subset\mathbb{R}^2$ belongs to class $A^{(k,h)}$ if it satisfies the following four conditions:

(1) $\partial\Omega$, the closed boundary of Ω, can be represented as a finite sum of pieces, where each piece can be represented as a parametric function $x_l = x_l(\mu)$, $l=1,2$, on a bounded interval I in $\mathbb{R}$.

(2) The functions x_l, $l=1,2$ define a one-to-one correspondence between $\bar{I}$ and the corresponding piece of $\partial\Omega$ and also $x_l\in C^{(k,h)}(\bar{I})$, where $\bar{I}$ is the closure of I and $k\geq 1$.

(3) $J=[(dx_2/d\mu)^2+(dx_1/d\mu)^2]^{1/2}>0$, for $\mu\in\bar{I}$.

(4) According to [4], the fourth condition in our case reduces to the following:

$$\cos\widehat{\nu x_1}=\frac{(dx_2/d\mu)}{J},\qquad \cos\widehat{\nu x_2}=\frac{(dx_1/d\mu)}{J}, \tag{1.2}$$

where ν is the outer unit normal to the $\partial\Omega$.

When $\partial\Omega$, the boundary of the Ω, also belongs to $A^{(1,h)}$ it is called a Lyapunov curve.

To use the Poincaré-Bertrand formula (1.1) we assume that S belongs to $A^{(1,h)}$ and also $\phi(t,t_1) \in C^{(0,h)}(S)$, $t_0 \in S$.

Now as it is clear, the integral term on the right-hand side of (1.1) is at most weakly singular. Using this regularized formula we are going to solve some important first and second kind Fredholm's integral equations in which the kernels are singular. Before starting using (1.1), in the following we show its equivalent formulation on an interval (a,b).

We parameterize S, where the parameter is taken to be the arc length. So we can write $t = \psi(\tau)$, $t_1 = \psi(\tau_1)$, $t_0 = \psi(\tau_0)$, where $t, t_1, t_0 \in S$, $0 \le \tau \le l$, l is the total length of S, ψ is the parameterization function.

On substituting $t = \psi(\tau)$, $t_1 = \psi(\tau_1)$, and $t_0 = \psi(\tau_0)$ in (1.1), we obtain

$$\begin{aligned}
&\int_0^l \frac{\psi'(\tau)d\tau}{\psi(\tau)-\psi(\tau_0)} \int_0^l \frac{\phi(\psi(\tau),\psi(\tau_1))}{\psi(\tau_1)-\psi(\tau)} \psi'(\tau_1)d\tau_1 \\
&\quad = -\pi^2\phi(\psi(\tau_0),\psi(\tau_0)) + \int_0^l \psi'(\tau_1)d\tau_1 \int_0^l \frac{\phi(\psi(\tau),\psi(\tau_1))}{(\psi(\tau)-\psi(\tau_0))(\psi(\tau_1)-\psi(\tau))}\psi'(\tau)\,d\tau.
\end{aligned} \tag{1.3}$$

Substituting $\psi(\tau)-\psi(\tau_0) = \psi'(\theta_0)(\tau-\tau_0)$ and $\psi(\tau_1)-\psi(\tau) = \psi'(\theta_1)(\tau_1-\tau)$ in this result, where θ_0 is between τ and τ_0 and θ_1 is between τ_1 and τ, yields

$$\begin{aligned}
&\int_0^l \frac{\psi'(\tau)d\tau}{\psi'(\theta_0)(\tau-\tau_0)} \int_0^l \frac{\psi'(\tau_1)}{\psi'(\theta_1)(\tau_1-\tau)} \phi(\psi(\tau),\psi(\tau_1))d\tau_1 \\
&= -\pi^2\phi(\psi(\tau_0),\psi(\tau_0)) + \int_0^l \psi'(\tau_1)\,d\tau_1 \int_0^l \frac{\psi'(\tau)}{\psi'(\theta_0)(\tau-\tau_0)\psi'(\theta_1)(\tau_1-\tau)}\phi(\psi(\tau),\psi(\tau_1))d\tau.
\end{aligned} \tag{1.4}$$

This is, clearly, equivalent to the following result:

$$\int_0^l \frac{d\tau}{\tau-\tau_0} \int_0^l \frac{K(\tau,\tau_1)}{\tau_1-\tau} d\tau_1 = -\pi^2 K(\tau_0,\tau_0) + \int_0^l d\tau_1 \int_0^l \frac{K(\tau,\tau_1)}{(\tau-\tau_0)(\tau_1-\tau)} d\tau, \tag{1.5}$$

where

$$K(\tau,\tau_1) = \frac{\psi'(\tau)\psi'(\tau_1)}{\psi'(\theta_0)\psi'(\theta_1)}\phi(\psi(\tau),\psi(\tau_1)). \tag{1.6}$$

This is simply transformed to an interval (a,b), which is an equivalent formulation of (1.1).

Problem 1 (singular Fredholm's integral equation of the first kind)**.** We consider

$$\int_a^b \frac{K(x,\xi)}{x-\xi} y(\xi)\,d\xi = f(x), \quad x \in (a,b), \tag{1.7}$$

where $f(x)$ is continuous in $[a,b] \subset \mathbb{R}$, a, b finite, $K(x,\xi)$ is at least Hölder continuous in $D \subset \mathbb{R}^2$. To see an example of (1.7) we recall that whenever we obtain the solution of Dirichlet problem as potential of simple layer, we have actually obtained a Fredholm's integral equation of the first kind whose kernel is logarithmic and hence by differentiating this equation we get to a similar equation as (1.7). For example, consider

$$\Delta u(x) = 0, \quad x \in D, \tag{1.8}$$

$$u(x) = \phi(x), \quad x \in S. \tag{1.9}$$

Thus, its solution as potential of simple layer is as follows:

$$u(x) = \int_S \sigma(\xi) \frac{1}{2\pi} L_n |x - \xi| d\xi, \quad x \in D, \tag{1.10}$$

where the density $\sigma(\xi)$ is unknown function. Applying the boundary condition (1.9) on (1.10), we get

$$\int_S \sigma(\xi) \frac{1}{2\pi} L_n |\eta - \xi| d\xi = \phi(\eta), \quad \eta \in S. \tag{1.11}$$

Clearly, equation (1.11) is a Fredholm's integral equation of the first kind for σ and its kernel has a weak singularity. Differentiating (1.11) gives

$$\int_S \sigma(\xi) \frac{1}{2\pi} \frac{K(\eta,\xi)}{|\eta - \xi|} d\xi = \phi'(\eta), \tag{1.12}$$

where $K(\eta,\xi)$ is a continuous and bounded function in the domain. Obviously, equation (1.12) is similar to (1.7).

2. Solution for Problem 1. Multiplying both sides of (1.7) by $1/(t-x)$, integrating over $[a,b]$ with respect to x, we get

$$\int_a^b \frac{dx}{t-x} \int_a^b \frac{K(x,\xi)}{x-\xi} y(\xi)\, d\xi = \int_a^b \frac{f(x)}{t-x} dx. \tag{2.1}$$

Application of the Poincaré-Bertrand formula (1.1) to the left-hand side of (2.1) yields the following:

$$-\pi^2 K(t,t) y(t) + \int_a^b y(\xi)\, d\xi \int_a^b \frac{K(x,\xi)}{(t-x)(x-\xi)} dx = \int_a^b \frac{f(x)}{t-x} dx. \tag{2.2}$$

Assuming $K(t,t) \neq 0$, dividing the above equation by $-\pi^2 K(t,t)$ gives

$$y(t) = \int_a^b y(\xi)\, d\xi \frac{1}{\pi^2 K(t,t)} \int_a^b \frac{K(x,\xi)}{(t-x)(x-\xi)} dx - \frac{1}{\pi^2 K(t,t)} \int_a^b \frac{f(x)}{t-x} dx. \tag{2.3}$$

This is just a second kind Fredholm's integral equation. Now, as $K(x,\xi)$ is Hölder continuous, substituting

$$\frac{1}{(t-x)(x-\xi)} = \left(\frac{1}{t-x} + \frac{1}{x-\xi} \right) \frac{1}{t-\xi} \tag{2.4}$$

in (2.3) we obtain its kernel with a weak singularity and on the other hand, the integral term $\int_a^b (f(x)/(t-x))dx$ exists as it is a Cauchy type integral [3]. So, using this technique we have been able to change a first kind Fredholm's integral equation with a singular kernel into a second kind Fredholm's integral equation with a weak singular kernel. Thus, the *Fredholm's alternative* remains valid [4].

Problem 2 (singular Fredholm's integral equation of the second kind)**.** We consider

$$y(x)=\int_a^b \frac{K(x,\xi)}{x-\xi}y(\xi)d\xi+f(x),\quad x\in(a,b). \tag{2.5}$$

To solve Problem 2, we multiply both sides of (2.5) by $K(t,x)/(t-x)$, integrating over $[a,b]$ with respect to x, we get

$$\int_a^b \frac{K(t,x)}{t-x}y(x)dx=\int_a^b \frac{K(t,x)}{t-x}dx\int_a^b \frac{K(x,\xi)}{x-\xi}y(\xi)d\xi+\int_a^b \frac{K(t,x)}{t-x}f(x)dx. \tag{2.6}$$

Using again the Poincaré-Bertrand formula for the first term on the right-hand side yields

$$\begin{aligned}\int_a^b \frac{K(t,x)}{t-x}y(x)dx=-\pi^2K^2(t,t)y(t)+\int_a^b y(\xi)d\xi\int_a^b \frac{K(t,x)}{t-x}\frac{K(x,\xi)}{x-\xi}dx\\ +\int_a^b \frac{K(t,x)}{t-x}f(x)dx.\end{aligned} \tag{2.7}$$

On the other hand, if in (2.5) we replace x by t and ξ by x, we then get

$$\int_a^b \frac{K(t,x)}{t-x}y(x)dx=y(t)-f(t). \tag{2.8}$$

Substituting this result in (2.7) we obtain

$$y(t)-f(t)=-\pi^2K^2(t,t)y(t)+\int_a^b y(\xi)d\xi\int_a^b \frac{K(t,x)K(x,\xi)}{(t-x)(x-\xi)}dx+\int_a^b \frac{K(t,x)}{t-x}f(x)dx. \tag{2.9}$$

Therefore, since $1+\pi^2K^2(t,t)\neq 0$, we get the following result:

$$\begin{aligned}y(t)=\int_a^b y(\xi)d\xi\frac{1}{1+\pi^2K^2(t,t)}\int_a^b \frac{K(t,x)K(x,\xi)}{(t-x)(x-\xi)}dx\\ +\frac{f(t)+\int_a^b (K(t,x)/(t-x))f(x)dx}{1+\pi^2K^2(t,t)}.\end{aligned} \tag{2.10}$$

Clearly, this result given in (2.10) is a regular second kind Fredholm's integral equation, as its kernel is just weakly singular.

Remark 2.1. If one likes, for some reason, to transform this problem to a problem in the form of Problem 1 and using the regularization discussed in Section 2, then one should do the following.

Multiplying both sides of (2.5) by $1/(t-x)$, integrating over $[a,b]$ with respect to x, we get

$$\int_a^b \frac{y(x)}{t-x}dx=\int_a^b \frac{dx}{t-x}\int_a^b \frac{K(x,\xi)}{x-\xi}y(\xi)d\xi+\int_a^b \frac{f(x)}{t-x}dx. \tag{2.11}$$

Using the Poincaré-Bertrand formula (1.1) for the first term on the right-hand side gives

$$\int_a^b \frac{y(x)}{t-x}dx=-\pi^2K(t,t)y(t)+\int_a^b y(\xi)d\xi\int_a^b \frac{K(x,\xi)}{(t-x)(x-\xi)}dx+\int_a^b \frac{f(x)}{t-x}dx, \tag{2.12}$$

where the term $\int_a^b (f(x)/(t-x))dx$ is a Cauchy type integral (i.e., its *Cauchy principal value (CPV)* exists). Assuming $K(t,t) \neq 0$, dividing both sides of (2.12) by $-\pi^2 K(t,t)$ yields

$$\begin{aligned} y(t) = &-\frac{1}{\pi^2}\int_a^b \frac{1}{K(t,t)}\frac{y(t)}{t-x}dx + \frac{1}{\pi^2 K(t,t)}\int_a^b y(\xi)d\xi \int_a^b \frac{K(x,\xi)}{(t-x)(x-\xi)}dx \\ &+ \frac{1}{\pi^2 K(t,t)}\int_a^b \frac{f(x)}{t-x}dx. \end{aligned} \tag{2.13}$$

Now, by comparing (2.5), (2.13) and equating their right-hand sides we obtain the following first kind Fredholm's integral equation in which the kernel is singular:

$$\begin{aligned} \int_a^b \frac{K(x,\xi)}{x-\xi}y(\xi)d\xi + f(x) = &-\frac{1}{\pi^2}\int_a^b \frac{1}{K(x,x)}\frac{y(\eta)}{x-\eta}d\eta + \frac{1}{\pi^2 K(x,x)}\int_a^b y(\xi)d\xi \\ &\times \int_a^b \frac{K(\eta,\xi)}{(x-\eta)(\eta-\xi)}d\eta \frac{1}{\pi^2 K(x,x)}\int_a^b \frac{f(\eta)}{x-\eta}d\eta. \end{aligned} \tag{2.14}$$

Therefore, we have

$$\begin{aligned} &\int_a^b \left[K(x,\xi) + \frac{1}{\pi^2 K(x,x)}\right]\frac{y(\xi)}{x-\xi}d\xi \\ &\quad = \frac{1}{\pi^2 K(x,x)}\int_a^b y(\eta)d\eta \int_a^b \frac{K(\xi,\eta)}{(x-\xi)(\xi-\eta)}d\xi + \frac{1}{\pi^2 K(x,x)}\int_a^b \frac{f(\eta)}{x-\eta}d\eta, \end{aligned} \tag{2.15}$$

$$\int_a^b \frac{\tilde{K}(x,\xi)}{x-\xi}y(\xi)d\xi = F(x), \quad x \in (a,b), \tag{2.16}$$

where $F(x) = (1/\pi^2 K(x,x))\int_a^b (f(\eta)/(x-\eta))d\eta$,

$$\tilde{K}(x,\xi) = K(x,\xi) + \frac{1}{\pi^2 K(x,x)} + \frac{x-\xi}{\pi^2 K(x,x)}\int_a^b \frac{K(\eta,\xi)}{(x-\eta)(\eta-\xi)}d\eta. \tag{2.17}$$

Hence, comparing (2.16) with (1.7) it is clear that *Problem 2* has changed to *Problem 1*, for which we have given regularization.

Remark 2.2. We believe that regularizing a singular integral equation can be possible whenever its operator is not unbounded for a constant kernel. In the following equation:

$$\int_0^x K(x,\xi)\frac{y(\xi)}{x-\xi}d\xi = f(x), \quad K(x,x) \neq 0, \tag{2.18}$$

even if K and y are constant, its operator is unbounded. As a particular case, we would like to know how to regularize it for $K \equiv 1$, $f(0) = 0$, or $f(0) \neq 0$.

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