

ON THE CLOSURE OF THE SUM OF CLOSED SUBSPACES

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ABSTRACT. We give necessary and sufficient conditions for the sum of closed subspaces of a Hilbert space to be closed. Specifically, we show that the sum will be closed if and only if the angle between the subspaces is not zero, or if and only if the projection of either space into the orthogonal complement of the other is closed. We also give sufficient conditions for the sum to be closed in terms of the relevant orthogonal projections. As a consequence, we obtain sufficient conditions for the existence of an optimal solution to an abstract quadratic programming problem in terms of the kernels of the cost and constraint operators.

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1. Introduction. In this research/expository article, we suppose H is an arbitrary Hilbert space (real or complex) with K and N closed subspaces of H . We consider the question of when the (ordinary) sum $K + N$ is closed in H . It is obviously true if either $K \subseteq N$ or $N \subseteq K$. It is also true if either K or N is finite-dimensional. If K or N is of codimension 1 (i.e., one of the subspaces is a hyperplane through the origin), then $K + N$ is clearly closed. However, it is not closed in general (see [Example 2.2](#)).

In what follows, we will give necessary and sufficient conditions for $K + N$ to be closed. Although we have found no such conditions in the published literature, conditions were evidently provided without proof in unpublished course notes of Carl Pearcy [3]. In this paper, we state and prove a collection of similar necessary and sufficient conditions in [Theorem 2.1](#). Our first equivalent condition involves orthogonal complements and projections. Let $L = K^\perp$ denote the orthogonal complement of K in H , and let $M = N^\perp$ denote the orthogonal complement of N in H . Also, let $E_L : H \rightarrow L$ be the corresponding orthogonal projection onto L and $E_M : H \rightarrow M$ be the corresponding orthogonal projection onto M . In [Section 2](#), we see that $E_L(N)$ is closed if and only if $E_M(K)$ is closed, and more important for our purposes, that $K + N$ is closed if and only if each of these subspaces is closed (see [Theorem 2.1\(ii\)](#)). Thus, the closure of $N + K$ is equivalent to the closure of the orthogonal projection of N into L (resp., K into M). Our next equivalent condition involves the (cosine of the) angle $\theta(K, N)$ between K and N . Its definition is given in [Section 2](#), where we also see that $K + N$ is closed if and only if $\theta(K, N) < 1$, that is, the angle between K and N is not equal to 0 (see [Theorem 2.1\(iii\)](#)). Finally, in [Section 2](#) we give some sufficient conditions for [Theorem 2.1](#) to hold in terms of the orthogonal projections $E_K : H \rightarrow K$ and $E_N : H \rightarrow N$ (see [Theorem 2.3](#)).

In [Section 3](#), we give an application of our main results. This involves an abstract positive semidefinite quadratic programming problem given by minimizing the

quadratic objective function $\langle x, Qx \rangle$ subject to the linear equality constraint $Ax = b$, for $x \in H$, where Q and A are bounded linear operators, and Q is also selfadjoint and positive semidefinite. For such a problem, we let K denote the kernel of Q , N the kernel of A and F the feasible region, where $F = N + x$, for any $x \in F$. Then it is known that this problem admits an optimal solution if the (positive-definite) restriction of Q to $L = K^\perp$ is strictly positive definite (i.e., coercive) and the projection $E_L(F)$ of F into L is closed [5]. The restriction $Q|_L$ is well known to be strictly positive definite if and only if its (positive) spectrum is bounded away from zero. That leaves the question of when the projection $E_L(F)$ is closed. Our main results give equivalent conditions for this to happen. In particular, it happens precisely when $K + N$ is closed, or if $\theta(K, N) < 1$.

2. Main results. We begin this section by defining the quantity $\theta(K, N)$, which is essentially the angle between the subspaces K and N . Let

$$S(K, N) = \{(x, y) : x \in K \cap (K \cap N)^\perp, y \in N \cap (K \cap N)^\perp, \|x\| = \|y\| = 1\}. \tag{2.1}$$

Note that $S(K, N) = S(N, K)$ and $S(K, N) = \emptyset$ if and only if either $N \subseteq K$ or $K \subseteq N$. Let

$$\theta(K, N) = \begin{cases} \sup \{|\langle x, y \rangle| : (x, y) \in S(K, N)\}, & \text{for } S(K, N) \neq \emptyset, \\ -\infty, & \text{for } S(K, N) = \emptyset, \end{cases} \tag{2.2}$$

so that $-\infty \leq \theta(K, N) = \theta(N, K) \leq 1$. In particular, if $N \cap K = \{0\}$, then $(K \cap N)^\perp = H$, $N = N \cap (K \cap N)^\perp$, $K = K \cap (K \cap N)^\perp$, and

$$\theta(K, N) = \sup \{|\langle x, y \rangle| : x \in N, y \in K, \|x\| = \|y\| = 1\}. \tag{2.3}$$

If $L = K^\perp$ and $M = N^\perp$ are hyperplanes through the origin, then $\theta(K, N)$ is the (cosine of the) conventional angle between the one-dimensional subspaces K and N .

THEOREM 2.1. *The following statements are equivalent:*

- (i) $E_L(N)$ is closed in L .
- (ii) $K + N$ is closed in H .
- (iii) $\theta(K, N) < 1$.
- (iv) $E_M(K)$ is closed in M .

PROOF. Given the expository nature of this paper, we find it instructive to show all the implications involved.

(i) \Leftrightarrow (ii). Suppose $E_L(N)$ is closed. Let $x^k = v^k + \eta^k$, where $v^k \in N$ and $\eta^k \in K$, for all $k = 1, 2, \dots$, and $x^k \rightarrow x \in H$. To show that $x \in N + K$. Since $L = K^\perp$, we have that

$$E_L(x^k) \rightarrow E_L(x) \in L, \tag{2.4}$$

where

$$E_L(x^k) = E_L(v^k + \eta^k) = E_L(v^k) + E_L(\eta^k) = E_L(v^k) \in E_L(N), \quad \forall k = 1, 2, \dots \tag{2.5}$$

By hypothesis, $E_L(x) \in E_L(N)$. Thus, there exists $y \in N$ such that $E_L(x) = E_L(y)$, that is, $E_L(x - y) = 0$, so that $\eta = x - y \in K = L^\perp$. Therefore, $x = y + \eta$, that is, $x \in N + K$.

Now suppose $N + K$ is closed in H . Let $\{\xi^k\}$ be a sequence in $E_L(N)$ which converges to $\xi \in L$. To show that $\xi \in E_L(N)$, for each k , there exists $y^k \in N$ such that $E_L(y^k) = \xi^k$. Let $\eta^k = \xi^k - y^k$, for all k . Then

$$E_L(\eta^k) = E_L(\xi^k) - E_L(y^k) = E_L(E_L(y^k)) - E_L(y^k) = E_L(y^k) - E_L(y^k) = 0, \tag{2.6}$$

so that $\eta^k \in K$, for all $k = 1, 2, \dots$. Since $\xi^k = y^k + \eta^k \in N + K$, for all K , and $\xi^k \rightarrow \xi$, it follows, by hypothesis, that $\xi \in N + K$. Hence, there exist $v \in N$ and $\eta \in K$ for which $\xi = v + \eta$. Therefore,

$$\begin{aligned} E_L(E_L(y^k)) &= E_L(\xi^k) \rightarrow E_L(\xi) = E_L(v) + E_L(\eta) = E_L(v), \\ E_L(E_L(y^k)) &= E_L(y^k) = \xi^k \rightarrow \xi, \end{aligned} \tag{2.7}$$

so that $\xi = E_L(v)$, for $v \in N$, that is, $\xi \in E_L(N)$.

(ii) \Leftrightarrow (iii). First observe that if $N \subseteq K$ or $K \subseteq N$, then $N + K$ is closed in H and $\theta(N, K) = -\infty$. Thus, we may assume that $N \not\subseteq K$ and $K \not\subseteq N$.

Next we show that, without loss of generality, we may also assume that $N \cap K = \{0\}$. If not, then we may write

$$N = N_1 \oplus (N \cap K), \quad K = K_1 \oplus (N \cap K), \tag{2.8}$$

where N_1 (resp., K_1) is the orthogonal complement of $N \cap K$ in N (resp., K). Then

$$\begin{aligned} N + K &= (N_1 \oplus (N \cap K)) + (K_1 \oplus (N \cap K)) \\ &= (N_1 + K_1) \oplus ((N \cap K) + (N \cap K)) \\ &= (N_1 + K_1) \oplus (N \cap K). \end{aligned} \tag{2.9}$$

Thus, $N + K$ is closed if and only if $N_1 + K_1$ is closed, where $N_1 \cap K_1 = \{0\}$.

Suppose $N + K$ is closed with $N \cap K = \{0\}$. Consider the canonical Hilbert space mapping

$$f : N \oplus K \rightarrow N + K, \tag{2.10}$$

given by

$$f(v, \eta) = v + \eta, \quad \forall v \in N, \forall \eta \in K. \tag{2.11}$$

This mapping is clearly linear and onto. It is also one-to-one since $N \cap K = \{0\}$. We next show that it is also bounded.

Let $v \in N$ and $\eta \in K$, then

$$\begin{aligned} \|f(v, \eta)\|^2 &= \|v + \eta\|^2 = \langle v + \eta, v + \eta \rangle = \langle v, v \rangle + \langle v, \eta \rangle + \langle \eta, v \rangle + \langle \eta, \eta \rangle \\ &= \langle v, v \rangle + 2\langle v, \eta \rangle + \langle \eta, \eta \rangle = \|v\|^2 + 2\langle v, \eta \rangle + \|\eta\|^2 \\ &\leq \|v\|^2 + 2|\langle v, \eta \rangle| + \|\eta\|^2 \leq \|v\|^2 + 2\|v\|\|\eta\| + \|\eta\|^2. \end{aligned} \tag{2.12}$$

Moreover,

$$2\|v\|\|\eta\| \leq \|v\|^2 + \|\eta\|^2. \tag{2.13}$$

Hence,

$$\|f(v, \eta)\|^2 \leq 2(\|v\|^2 + \|\eta\|^2) = 2\|(v, \eta)\|^2, \tag{2.14}$$

so that $\|f\| \leq \sqrt{2}$, that is, f is bounded.

By the open mapping theorem [2, page 57], f has a bounded linear inverse, that is, there exists $c > 0$ such that

$$c\|(\nu, \eta)\| \leq \|f(\nu, \eta)\| = \|\nu + \eta\| \leq \sqrt{2}\|(\nu, \eta)\|. \tag{2.15}$$

Consequently,

$$c^2(\|\nu\|^2 + \|\eta\|^2) = c^2\|(\nu, \eta)\|^2 \leq \|\nu + \eta\|^2 = \|\nu\|^2 + 2\langle \nu, \eta \rangle + \|\eta\|^2. \tag{2.16}$$

Hence,

$$c^2(\|\nu\|^2 + \|\eta\|^2) \leq \|\nu\|^2 + 2\langle \nu, \eta \rangle + \|\eta\|^2, \tag{2.17}$$

that is,

$$(c^2 - 1)(\|\nu\|^2 + \|\eta\|^2) \leq 2\langle \nu, \eta \rangle, \tag{2.18}$$

for all such ν, η .

Now assume in addition that $\|\nu\| = \|\eta\| = 1$. Then

$$2(c^2 - 1) = (c^2 - 1)(\|\nu\|^2 + \|\eta\|^2) \leq 2\langle \nu, \eta \rangle, \tag{2.19}$$

that is,

$$1 - c^2 \geq \langle -\nu, \eta \rangle, \tag{2.20}$$

where $-\nu$ is an arbitrary element of N of norm 1, since ν is such. If $\langle \nu, \eta \rangle \leq 0$, then $\langle -\nu, \eta \rangle = |\langle \nu, \eta \rangle|$, so that $1 - c^2 \geq \langle \nu, \eta \rangle$, and $0 < c \leq 1$. If $\langle \nu, \eta \rangle \geq 0$, then $\langle \nu, \eta \rangle = |\langle \nu, \eta \rangle|$, so that $1 - c^2 \geq -|\langle \nu, \eta \rangle|$. Letting $\nu' = -\nu$, we get

$$1 - c^2 \geq -\langle \nu', \eta \rangle = \langle \nu, \eta \rangle = |\langle \nu, \eta \rangle|. \tag{2.21}$$

Hence, in either case,

$$|\langle \nu, \eta \rangle| \leq 1 - c^2 < 1, \tag{2.22}$$

for all $\nu \in N, \eta \in K$ with $\|\nu\| = \|\eta\| = 1$. Thus, $\theta(N, K) < 1$.

Now suppose $-\infty \leq \theta = \theta(N, K) < 1$. Observe that N, K are weakly closed in H , as well as closed. Also, from the definition of θ we have that

$$|\langle \nu, \eta \rangle| \leq \theta\|\nu\|\|\eta\|, \tag{2.23}$$

so that

$$-2\theta\|\nu\|\|\eta\| \leq 2\langle \nu, \eta \rangle \leq 2\theta\|\nu\|\|\eta\|, \quad \forall \nu \in N, \forall \eta \in K. \tag{2.24}$$

Suppose $\{\nu^k\}$ is a sequence in N and $\{\eta^k\}$ is a sequence in K such that $x^k = \nu^k + \eta^k \rightarrow x$ in H . To show that $x \in N + K$. First, we show that

$$\sup_k \|\nu^k\| < \infty, \quad \sup_k \|\eta^k\| < \infty. \tag{2.25}$$

We have

$$\begin{aligned} \|x^k\|^2 &= \|\nu^k + \eta^k\|^2 = \|\nu^k\|^2 + \|\eta^k\|^2 + 2\langle \nu^k, \eta^k \rangle \\ &\geq \|\nu^k\|^2 + \|\eta^k\|^2 - 2\theta\|\nu^k\|\|\eta^k\| \\ &= (\theta\|\nu^k\| - \|\eta^k\|)^2 + (1 - \theta^2)\|\nu^k\|^2, \end{aligned} \tag{2.26}$$

that is,

$$\|v^k + \eta^k\|^2 \geq (\theta\|v^k\| - \|\eta^k\|)^2 + (1 - \theta^2)\|v^k\|^2. \tag{2.27}$$

Interchanging v^k and η^k , we also obtain

$$\|v^k + \eta^k\|^2 \geq (\theta\|\eta^k\| - \|v^k\|)^2 + (1 - \theta^2)\|\eta^k\|^2, \tag{2.28}$$

where $1 - \theta^2 > 0$ by hypothesis. We see from these inequalities that if either $\{v^k\}$ or $\{\eta^k\}$ is unbounded, we obtain a contradiction, since the convergent sequence $\{v^k + \eta^k\}$ must be bounded.

By the Banach-Alaoglu theorem [2], passing to subsequences if necessary, we may assume that there exists $v \in N$ and $\eta \in K$ such that $v^k \rightharpoonup v$ and $\eta^k \rightharpoonup \eta$ (weak convergence), as $k \rightarrow \infty$. Thus, $v^k + \eta^k \rightharpoonup v + \eta$ and $v^k + \eta^k \rightarrow x$, as $k \rightarrow \infty$. Necessarily, $x = v + \eta \in N + K$, since the weak topology on H separates points.

(iii) \Leftrightarrow (iv). Interchange K and N in (i), (ii), and (iii). □

EXAMPLE 2.2. For each $j = 1, 2, \dots$, let $\psi_j = \arcsin(1/j)$ so that $\cos \psi_j = \sqrt{j^2 - 1}/j$. Define

$$H = \left\{ [y_i \ u_i]_{i=1}^\infty : y_i, u_i \in \mathbb{R}, \sum_{i=1}^\infty (y_i^2 + u_i^2) < \infty \right\}, \tag{2.29}$$

$$K = \left\{ [y_i \ 0]_{i=1}^\infty : y_i \in \mathbb{R}, \sum_{i=1}^\infty y_i^2 < \infty \right\}, \quad N = \{ [y_i \ u_i]_{i=1}^\infty \in H : y_i = u_i \cot \psi_i, \forall i \}.$$

Clearly, K and N are closed subspaces of the real Hilbert space H with $K \cap N = \{0\}$ and

$$L = K^\perp = \left\{ [0 \ u_i]_{i=1}^\infty : u_i \in \mathbb{R}, \sum_{i=1}^\infty u_i^2 < \infty \right\}. \tag{2.30}$$

Now, for each $j = 1, 2, \dots$, define $x^j = (x_i^j)_{i=1}^\infty$ by

$$x_i^j = \begin{cases} [0 \ 0], & \text{for } i \neq j, \\ [1 \ 0], & \text{for } i = j, \end{cases} \tag{2.31}$$

so that $x^j \in K$ and $\|x^j\| = 1$. Similarly, define $v^j = (v_i^j)_{i=1}^\infty$ by

$$v_i^j = \begin{cases} [0 \ 0], & \text{for } i \neq j, \\ [\cos \psi_j \ \sin \psi_j], & \text{for } i = j, \end{cases} \tag{2.32}$$

so that $v^j \in N$ and $\|v^j\| = 1$. Note that

$$v_j^j = [\cos \psi_j \ \sin \psi_j] = \sin \psi_j [\cot \psi_j \ 1], \quad \forall j. \tag{2.33}$$

We then have

$$|\langle x^j, v^j \rangle| = \left| \sum_{i=1}^\infty \langle x_i^j, v_i^j \rangle \right| = |\langle [1 \ 0], [\cos \psi_j \ \sin \psi_j] \rangle| = \cos \psi_j, \quad \forall j = 1, 2, \dots \tag{2.34}$$

Consequently, we see that

$$\theta(N, K) \geq \sup_j \cos \psi_j = 1, \tag{2.35}$$

that is, $\theta(N, K) = 1$, so that $E_L(N)$ is not closed by [Theorem 2.1](#).

We next show that $E_L(N)$ is not closed. Suppose it is. Observe that $E_L(N)$ is the set of $([0 \ u_j])_{j=1}^\infty \in L$ for which there exists $\gamma_j \in \mathbb{R}$ such that $\gamma_j = u_j \cot \psi_j, \forall j$, and (γ_j) is square summable. For each $j = 1, 2, \dots$, let $u_j = 1/j$, so that $u = ([0 \ u_j]) \in L$. Set $\gamma_j = u_j \cot \psi_j$, and define

$$x_i^j = \begin{cases} [\gamma_i \ u_i], & \text{for } i \leq j, \\ [0 \ 0], & \text{for } i > j, \end{cases} \tag{2.36}$$

so that $x^j \in N$, and $u^j = E_L(x^j) \in E_L(N)$, where

$$u_i^j = \begin{cases} [0 \ u_i], & \text{for } i \leq j, \\ [0 \ 0], & \text{for } i > j, \end{cases} \tag{2.37}$$

for all $j = 1, 2, \dots$. Clearly, $\{u^j\}$ is a Cauchy sequence in $E_L(N)$. Since $E_L(N)$ is closed, then there exists $u \in E_L(N)$ such that $u^j \rightarrow u$, as $j \rightarrow \infty$. Therefore, there exists $\gamma = (\gamma_j)$ such that $\gamma_j = u_j \cot \psi_j$, for all j , and $\sum \gamma_j^2 < \infty$, that is, $([\gamma_j \ u_j]) \in H$. But

$$\|([\gamma_j \ u_j])\|^2 = \sum_{j=1}^\infty (\gamma_j^2 + u_j^2) = \sum_{j=1}^\infty (\cot^2 \psi_j + 1)u_j^2 = \sum_{j=1}^\infty \frac{u_j^2}{\sin^2 \psi_j} = \infty. \tag{2.38}$$

This is a contradiction. Hence, $E_L(N)$ is not closed in this example.

We next show that $N + K$ is not closed. Let $[\gamma_i, u_i]$ be an arbitrary element of N , so that $\sum_{i=1}^\infty u_i^2 \leq \infty$, in particular. Once again let

$$x_i^j = \begin{cases} [\gamma_i \ u_i], & \text{for } i \leq j, \\ [0 \ 0], & \text{for } i > j, \end{cases} \tag{2.39}$$

so that $x^j \in N$, and

$$z^j = \begin{cases} [\gamma_i \ 0], & \text{for } i \leq j, \\ [0 \ 0], & \text{for } i > j, \end{cases} \tag{2.40}$$

so that $-z^j \in K$, for all j . Consequently, $u^j \in N + K$, for all j , where

$$u^j = \begin{cases} [0 \ u_i], & \text{for } i \leq j, \\ [0 \ 0], & \text{for } i > j. \end{cases} \tag{2.41}$$

It follows that the sequence $\{u^j\}$ is Cauchy since $\sum_{j=1}^\infty u_i^2 < \infty$. By hypothesis, there exists $u \in H$ such that $u^j \rightarrow u$ as $j \rightarrow \infty$. Necessarily, $u_i^j \rightarrow u_i$, for all i . Hence,

$$u = ([0, u_1], [0, u_2], \dots). \tag{2.42}$$

If $u \in K + N$, then there exists $z = ([z_1, 0], [z_2, 0], \dots) \in K$ and $x \in N$ such that $z + x = u$. Thus,

$$x = u - z = -([z_1, u_1], [z_2, u_2], \dots). \tag{2.43}$$

Since this belongs to N , we must have

$$z_i = u_i \cot \psi_i, \quad \forall i. \tag{2.44}$$

Consequently,

$$\|x\|^2 = \sum_{j=1}^{\infty} (\cot \psi_j^2 + 1) u_j = \infty, \tag{2.45}$$

which is a contradiction. Therefore, $K + N$ is not closed.

THEOREM 2.3. *The following are sufficient for $K + N$ to be closed:*

- (i) $E_N E_K = 0$ or $E_K E_N = 0$.
- (ii) $E_N E_K - E_K E_N = 0$.
- (iii) $\sup\{\|E_N \eta\| : \eta \in K \cap (N \cap K)^\perp, \|\eta\| = 1\} < 1$.
- (iiib) $\sup\{\|E_K \nu\| : \nu \in N \cap (N \cap K)^\perp, \|\nu\| = 1\} < 1$.

PROOF. (i) If $E_N E_K = 0$, then

$$E_K E_N = (E_N E_K)^* = E_K^* E_N^* = 0, \tag{2.46}$$

so that $L + M = L \oplus M$, and is therefore closed in H . Similarly for $E_K E_N = 0$.

(ii) We have

$$E_N E_K = E_K E_N = E_{N \cap K}, \tag{2.47}$$

(where, in general, E_X denotes the orthogonal projection of H onto the closed subspace X) and

$$N + K = (N \cap K) \oplus (N \cap (N \cap K)^\perp + K \cap (N \cap K)^\perp). \tag{2.48}$$

We also have

$$E_N (I - E_N E_K) = E_{N \cap (N \cap K)^\perp}, \tag{2.49}$$

where I is the identity operator on H and

$$I - E_N E_K = E_{(N \cap K)^\perp}. \tag{2.50}$$

Thus, by hypothesis,

$$E_N (I - E_N E_K) = E_N - E_N^2 E_K = E_N - E_N E_K E_N = (I - E_N E_K) E_N. \tag{2.51}$$

Similarly,

$$E_K (I - E_N E_K) = E_{K \cap (N \cap K)^\perp}. \tag{2.52}$$

But

$$E_N (I - E_N E_K) (I - E_N E_K) E_K = 2(E_N E_K - E_K E_N) = 0, \tag{2.53}$$

by hypothesis. (Note that part (i) is valid for any closed subspace N and K and their corresponding projections E_N and E_K .) Hence, by (i) applied to E_N and $E_{(N \cap K)^\perp}$, we obtain that

$$(N \cap (N \cap K)^\perp) + (K \cap (N \cap K)^\perp) \tag{2.54}$$

is closed in H . Consequently, $N + K$ is closed in H .

(iiia) For convenience, let

$$\beta = \sup \{ \|E_N \eta\| : \eta \in K \cap (N \cap K)^\perp, \|\eta\| = 1 \}. \quad (2.55)$$

(We may exclude the case where the defining set is empty. This happens only if $K \subseteq N$, in which case $K + N$ is trivially closed.) Then $\beta < 1$ and $\beta^2 < 1$, that is, $\delta = 1 - \beta^2 > 0$. Fix

$$\eta \in K \cap (N \cap K)^\perp, \quad \nu \in N \cap (N \cap K)^\perp, \quad (2.56)$$

such that $\|\eta\| = \|\nu\| = 1$. Then

$$|\langle \eta, \nu \rangle| = \|\eta\| \|\nu\| \cos \psi, \quad (2.57)$$

where ψ is the angle between η and ν , if this angle is at most $\pi/2$, or the supplement of this angle, if it is greater than $\pi/2$. (Alternately, we can replace η by $-\eta$ where necessary.) Thus, $0 \leq \psi \leq \pi/2$. By the law of cosines, we have

$$\|\eta + \nu\|^2 = \|\eta\|^2 + \|\nu\|^2 - 2\|\eta\|\|\nu\|\cos \psi = 2 - 2\cos \psi. \quad (2.58)$$

On the other hand, since $\nu \in N$, and $E_N \eta$ is the best approximation in N to η , we have that

$$\|\eta - \nu\|^2 \geq \|\eta - E_N \eta\|^2 = \|\eta\|^2 - \|E_N \eta\|^2 = 1 - \|E_N \eta\|^2, \quad (2.59)$$

by the Pythagorean identity, where $\eta - E_N \eta$ is orthogonal to $E_N \eta$. Thus,

$$1 - \|E_N \eta\|^2 \leq 2 - 2\cos \psi. \quad (2.60)$$

However, by definition of β , we have

$$\|E_N \eta\|^2 \leq \beta^2, \quad (2.61)$$

so that

$$1 - \|E_N \eta\|^2 \geq 1 - \beta^2 = \delta > 0. \quad (2.62)$$

Consequently,

$$0 < \delta \leq 2 - 2\cos \psi, \quad (2.63)$$

that is,

$$0 \leq \cos \psi \leq 1 - \frac{\delta}{2} < 1, \quad (2.64)$$

so that

$$|\langle \eta, \nu \rangle| \leq 1 - \frac{\delta}{2} < 1. \quad (2.65)$$

This completes the proof of (iiia), since η and ν are arbitrary.

(iiib) The proof in part (iiia) depends only on the fact that N and K are closed subspaces of H . Thus, simply interchange these spaces in the proof. \square

3. An application to quadratic programming. We consider the general infinite quadratic programming problem given by

$$\min \langle x, Qx \rangle \tag{3.1}$$

subject to

$$Ax = b, \quad x \in H, \tag{3.2}$$

where H and M are real Hilbert spaces, $b \in M$, the constraint operator $A : H \rightarrow M$ is a bounded linear operator and the cost operator $Q : H \rightarrow H$ is a nonzero, (selfadjoint) positive semidefinite, bounded linear operator. The feasible region

$$F = \{x \in H : Ax = b\} \tag{3.3}$$

is a closed, affine subset of H (which we assume to be nonempty), and the kernel

$$N = \{x \in H : Ax = 0\} \tag{3.4}$$

of A in H is a closed subspace of H . Of course, $F = N + x$, for any $x \in F$.

Recall that Q is positive semidefinite if $\langle x, Qx \rangle \geq 0$, for all $x \in H$, and that Q is positive definite if $\langle x, Qx \rangle > 0$, for all $x \in H, x \neq 0$. We say that the operator Q is strictly positive definite if it is coercive, that is, if there exists $\sigma_Q > 0$ satisfying

$$\sigma_Q \|x\|^2 \leq \langle x, Qx \rangle, \quad \forall x \in H. \tag{3.5}$$

This condition is known (see [1, page 73]) to be necessary and sufficient for (3.1) to admit a (unique) optimal solution for any (nonempty) closed, convex subset F of H . Thus, even if F is a closed, convex set in H , and Q is only positive definite, problem (3.1) may not admit an optimal solution. See [4] for an example.

In this section, we establish sufficient conditions for (3.1) to admit an optimal solution.

Since Q is positive semidefinite, its kernel K is given by

$$K = \{x \in H : \langle x, Qx \rangle = 0\}. \tag{3.6}$$

Moreover, since Q is selfadjoint, it follows that K and $L = K^\perp$ are invariant under Q . Hence, Q also decomposes into $0 \oplus P$, where 0 is the zero operator on K and $P : L \rightarrow L$ is the restriction operator $Q|_L$. Note that P is a positive-definite, bounded linear operator on L . It need not be strictly positive definite.

Also, since $F \subseteq H$, we have that the image of F under E_K is

$$E_K(F) = \{\eta \in K : \eta + \xi \in F, \text{ for some } \xi \in L\}. \tag{3.7}$$

It is nonempty and convex in K , since this is the case for F in H . It is also true that F is closed in H ; however, $E_K(F)$ need not be closed in K .

Analogously, the image of F under E_L is

$$E_L(F) = \{\xi \in L : \eta + \xi \in F, \text{ for some } \eta \in K\}. \tag{3.8}$$

As with $E_K(F)$, the set $E_L(F)$ is nonempty and convex, but not necessarily closed in L . Moreover, $F \subseteq E_K(F) \oplus E_L(F)$. The same is true of $N, E_K(N)$, and $E_L(N)$.

We may now consider the following related problem:

$$\min_{\xi \in E_L(F)} \langle \xi, P\xi \rangle, \quad (3.9)$$

where, as we have seen, P is positive definite on L and $E_L(F)$ is a nonempty, convex subset of L , which may not be closed. Moreover,

$$\langle \xi, P\xi \rangle = \langle x, Qx \rangle, \quad (3.10)$$

for all $\xi \in L$, $\eta \in K$, and $x = \eta + \xi$.

Note that solving (3.9) is equivalent to solving (3.1) in the following sense. If $\xi^* \in E_L(F)$ is an optimal solution to (3.9), then there exists $\eta^* \in E_K(F)$ such that $x^* = \eta^* + \xi^* \in F$ and x^* is optimal for (3.1). Conversely, if $x^* \in F$ is optimal for (3.1), then $x^* = \eta^* + \xi^*$, for $\eta^* \in E_K(F)$ and $\xi^* \in E_L(F)$, where ξ^* is optimal for (3.9).

We are interested in when the feasible region $E_L(F)$ for (3.9) is closed.

LEMMA 3.1. *The following statements are equivalent for F, K , and N :*

- (i) $E_L(F)$ or $E_L(N)$ is closed in L .
- (ii) $E_M(K)$ is closed in M .
- (iii) $N + K$ is closed in H .
- (iv) $F + K$ is closed in H .
- (v) $\theta(N, K) < 1$.

PROOF. Apply Theorem 2.1 together with the fact that $F = N + x$, for any fixed $x \in F$. □

THEOREM 3.2. *If $Q|L$ is strictly positive definite and any one of the conditions in Lemma 3.1 holds, then (3.1) admits an optimal solution.*

PROOF. Observe that (3.9) admits an optimal solution if $P = Q|L$ is strictly positive definite and $E_L(F)$ is closed in L [1, page 73]. □

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