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ON THE CORRECT FORMULATION OF A NONLINEAR DIFFERENTIAL EQUATION IN BANACH SPACES

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ABSTRACT. We study the existence and uniqueness of the initial value problems in a Banach space E for the abstract nonlinear differential equation $(d^{n-1}/dt^{n-1})(du/dt + Au) = B(t)u + f(t, W(t))$, and consider the correct solution of this problem. We also give an application of the theory of partial differential equations.

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1. Introduction. Let E be a Banach space. Suppose that $\{(B_i(t),\ i = 1, 2, \dots, \nu),\ B(t),\ t \in I = [0, T_0]\}$ are families of closed linear operators defined on dense sets $S_1, S_2, \dots, S_\nu$, F in E, independent of t. Let $-A$ be a closed linear operator defined on a dense set S in E such that $S \subset F$, $S \subset S_i$, $(i = 1, 2, \dots, \nu)$.

Suppose that the range of these operators are in E, therefore consider the abstract nonlinear differential equation

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{du}{dt} + Au\right) = B(t)u + f(t, W), \tag{1.1}$$

$$u|_{t=0} = g_0,\ \frac{du}{dt}\bigg|_{t=0} = g_1,\ \frac{d^2u}{dt^2}\bigg|_{t=0} = g_2,\ \dots,\ \frac{d^{n-1}u}{dt^{n-1}}\bigg|_{t=0} = g_{n-1}, \tag{1.2}$$

where all the elements $g_0, g_1, g_2, \dots, g_{n-1} \in S$, $W = (B_1(t)u, B_2(t)u, \dots, B_\nu(t)u)$ and f is a given abstract nonlinear function defined on $I \times E^\nu$ with values in E. Without loss of generality, we assume that

$$u|_{t=0} = \frac{du}{dt}\bigg|_{t=0} = \frac{d^2}{dt^2}\bigg|_{t=0} = \cdots = \frac{d^{n-1}}{dt^{n-1}}\bigg|_{t=0} = \theta, \tag{1.3}$$

where θ is the zero element of the Banach space E. Let f be uniformly Hölder continuous for all $t \in I$, that means

$$\|f(t, W) - f(t^*, W)\| \le K|t - t^*|^\beta, \tag{1.4}$$

for all t and t^* in I and all W in E^ν, the constant K and β are positive and $\beta < 1$, where $\|\cdot\|$ is the norm in E. For all $W, W^* \in E^\nu$, $W = (w_1, w_2, \dots, w_\nu)$, $W^* = (w_1^*, w_2^*, \dots, w_\nu^*)$, and $t \in I$, the function f satisfies the Lipschitz condition

$$\|f(t, W) - f(t, W^*)\| \le K_1 \sum_{i=1}^{\nu} \left\|w_i - w_i^*\right\|. \tag{1.5}$$

For every $z \in S_i \cap F$ the functions $B_i(t)z$ and $B(t)z$ are uniformly Hölder continuous for $t \in I$ and $i = 1,2,3,\dots,\nu$ with exponents β' and β'', respectively—without loss of generality, we can suppose that $\beta = \beta' = \beta''$. The space of continuous functions $u(t)$ with $t \in I$ and $u(t) \in E$ is denoted by $C^E(I)$. The norm in this space is defined by

$$\|u\|_{C^E(I)} = \max_{t\in I} \|u(t)\|. \tag{1.6}$$

Suppose that $-A$ generates a semigroup $\{T(t), t \in I\}$ strongly continuous for all $t \geq 0$, this class of semigroup is called C_0 (see [11, 12]). Furthermore, suppose that $T(t)v \in S$ for all $v \in E$, $t > 0$ (see [6]).

Assume that if there exist $0 < \delta < 1$ and a positive constant M, then

$$||B(t_2)T(t_1)v|| \leq \frac{M}{t_1^\delta}\|v\|, \tag{1.7}$$

$$||B_i(t_2)T(t_1)v|| \leq \frac{M}{t_1^\delta}\|v\|, \tag{1.8}$$

where M is a positive constant and $0 < \delta < 1$ for all $v \in E$, $t_2 \in I$, $t_1 \in (0,T_0]$ with $i = 1,2,\dots,\nu$. In this paper, we prove the existence and the uniqueness of the solution of the Cauchy problem (1.1) and (1.2). The correct formulation of the considered problem is also proved, finally we give an application of the theory of partial differential equations.

2. The solution of the problem. In this section, we discuss the existence and uniqueness of the solution of the initial value problem (1.1) and (1.3). Define on $C^E(I)$, a distance function (metric) ρ by

$$\rho(u_1,u_2) = \max_{t\in I} e^{-\lambda t}\|u_1(t)-u_2(t)\|, \tag{2.1}$$

where $u_1,u_2 \in C^E(I)$, $\lambda > 1$ being a fixed number. It is clear that $(C^E(I),\rho)$ is a metric space, (see [2, 3]).

Theorem 2.1. *The abstract initial value problem (1.1) and (1.3) has a weak solution in the metric space $(C^E(I),\rho)$ for every $t \in I$.*

Proof. From equation (1.1), let

$$\frac{du}{dt} = -Au + v. \tag{2.2}$$

The desired solution u of the above equation can be written in the form (see [1, 11, 12])

$$u(t) = \int_0^t T(t-s)v(s)\,ds, \tag{2.3}$$

where v satisfies

$$\frac{d^{n-1}v}{dt^{n-1}} = B(t)\int_0^t T(t-s)v(s)\,ds + f(t,W), \tag{2.4}$$

$$W = (w_1,w_2,\dots,w_\nu), \qquad w_i(t) = B_i(t)\int_0^t T(t-s)v(s)\,ds. \tag{2.5}$$

Integrating (2.4) $(n-1)$ times, we get

$$\begin{aligned} v = &\int_0^t \int_s^t \int_s^{y_{n-1}} \cdots \int_s^{y_3} \int_s^{y_2} B(y_1) T(y_1 - s) v(s)\, dy_1\, dy_2 \cdots dy_{n-1}\, ds \\ &+ \int_0^t \int_0^{\xi_{n-1}} \cdots \int_0^{\xi_3} \int_0^{\xi_2} f(\xi_1, W(\xi_1))\, d\xi_1\, d\xi_2 \cdots d\xi_{n-1}. \end{aligned} \tag{2.6}$$

Let Q be an operator defined on $C^E(I)$ by

$$\begin{aligned} Qv = &\int_0^t \int_s^t \int_s^{y_{n-1}} \cdots \int_s^{y_3} \int_s^{y_2} B(y_1) T(y_1 - s) v(s)\, dy_1\, dy_2 \cdots dy_{n-1}\, ds \\ &+ \int_0^t \int_0^{\xi_{n-1}} \cdots \int_0^{\xi_3} \int_0^{\xi_2} f(\xi_1, W(\xi_1))\, d\xi_1\, d\xi_2 \cdots d\xi_{n-1}. \end{aligned} \tag{2.7}$$

We prove that Q is a contraction mapping. We notice that

$$\|Qv - Qv^*\| \le K_1 \int_0^t \int_s^t \int_s^{y_{n-1}} \cdots \int_s^{y_2} (y_1 - s)^{-\delta} \|v(s) - v^*(s)\|\, dy_1 \cdots dy_{n-1}\, ds \tag{2.8}$$

which gives

$$\rho(Qv, Qv^*) \le \frac{K_1 T_0^{n-\delta} \Gamma(1-\delta)}{\Gamma(n-\delta)\lambda^{n-2}} \rho(v, v^*). \tag{2.9}$$

For a sufficiently large λ we deduce that Q is a contraction operator therefore there exists a unique fixed point such that (see [2, 3, 1]) $Qv = v \in C^E(I)$, which proves the existence and uniqueness of a weak solution u in $C^E(I)$. □

We prove that $|f(t, W(t))|$ is bounded on the interval $[0, T_0]$.

Theorem 2.2. *If the function $f(t, W(t))$ satisfies the conditions (1.4) and (1.8), then $|f(t, W(t))|$ is bounded for all $t \in I$.*

Proof. From condition (1.4), it is clear that

$$\begin{aligned} \|f(t, W(t)) - f(t, \theta, \ldots, \theta)\| = \|f(t, W(t)) - f(t, W(0))\| &\le K \sum_{i=1}^{\nu} \|w_i(t)\| \\ &= K \sum_{i=1}^{\nu} \left\| \int_0^t B_i(t) T(t-s) v(s)\, ds \right\|. \end{aligned} \tag{2.10}$$

From (1.8), we get the required result. □

Theorem 2.3 (see [1, 4, 11]). *The function $u(t)$ is an element of S for every $t \in I$ and so $u \in C^S[0, T_0]$.*

Proof. To prove this theorem, it is enough to show that $v(t)$ satisfies the Lipschitz condition in $t \in I$

$$\begin{aligned} v(t_2)-v(t_1) &= \int_0^{t_2}\int_{t_1}^{t_2}\int_s^{y_{n-1}}\cdots\int_s^{y_3}\int_s^{y_2} B(y_1)T(y_1-s)v(s)\,dy_1\,dy_2\cdots dy_{n-1}\,ds \\ &\quad+\int_{t_1}^{t_2}\int_s^{t_2}\int_s^{y_{n-1}}\cdots\int_s^{y_3}\int_s^{y_2} B(y_1)T(y_1-s)v(s)\,dy_1\,dy_2\cdots dy_{n-1}\,ds \\ &\quad+\int_{t_1}^{t_2}\int_0^{\xi_{n-1}}\cdots\int_0^{\xi_3}\int_0^{\xi_2} f(\xi_1,W(\xi_1))\,d\xi_1\,d\xi_2\cdots d\xi_{n-1}\,ds. \end{aligned} \tag{2.11}$$

The theorem is proved by using the above equation and (1.6).

To complete the proof of the existence and uniqueness of the solution (strongly) we prove that each of the following derivative

$$\frac{du}{dt},\ \frac{d^2u}{dt^2},\ \ldots,\ \frac{d^{n-1}u}{dt^{n-1}} \tag{2.12}$$

belong to $C^G(I)$, let $\Psi_1(t)=B(t)u(t)$ and $\Psi_2(t)=f(t,W(t))$. From (1.1), we can write formally

$$\begin{aligned} \frac{d^r u(t)}{dt^r} &= \int_0^{t}\int_s^{y_{n-r-1}}\cdots\int_s^{y_3}\int_s^{y_2} T(y_1-s)\Psi_1(s)\,dy_1\,dy_2\cdots dy_{n-r-1}\,ds \\ &\quad+\int_0^{t}\int_s^{\xi_{n-r-1}}\cdots\int_s^{\xi_3}\int_s^{\xi_2} T(\xi_1-s)\Psi_2(s)\,d\xi_1\,d\xi_2\cdots d\xi_{n-r-1}\,ds. \end{aligned} \tag{2.13}$$

To get the required result, we must prove that Ψ_1 and Ψ_2 satisfies a uniform Hölder condition for $t \in I$. Suppose that $t_2 > t_1$, therefore it is easy to show that

$$\begin{aligned} \Psi_1(t_2)-\Psi_1(t_1) &= \int_0^{t_1} B(t_2)T(t_1-s)[T(t-t)-J]v(s)\,ds \\ &\quad+\int_0^{t_1}[B(t_2)-B(t_1)]T(t_1-s)v(s)\,ds \\ &\quad+\int_{t_1}^{t_2} B(t_2)T(t-s)v(s)\,ds, \end{aligned} \tag{2.14}$$

where J is the identity operator on E,

$$\begin{aligned} \|\Psi_2(t_2)-\Psi_2(t_1)\| &\le \|f(t_2,W(t_2))-f(t_1,W(t_2))\|+\|f(t_1,W(t_2))-f(t_1,W(t_1))\| \\ &\le K_1(t_2-t_1)^{\beta}+K_2\sum_{i=1}^{\nu}\|A_i(t_2)u(t_2)-A_i(t_1)u(t_1)\|, \end{aligned} \tag{2.15}$$

where K_1 and K_2 are positive constants. Similarly, as in [9], we can prove that Ψ_1 and Ψ_2 satisfies Hölder condition in $t \in I$, therefore $(du/dt) \in C^S(I)$ and (dv/dt) is continuous for all $t \in I$.

Now, $Au(t)$ can be written in the form

$$Au(t) = \int_0^{t}\int_s^{t}\int_s^{y_{n-1}}\cdots\int_s^{y_3}\int_s^{y_2} AT(y_1-s)[\Psi_1(s)+\Psi_2(s)]\,dy_1\,dy_2\cdots dy_{n-1}\,ds. \tag{2.16}$$

Thus differentiate $(n-1)$ times we get

$$\frac{d^{n-1}}{dt^{n-1}}[Au] = \int_0^t AT(y_1-s)[\Psi_1(s)+\Psi_2(s)]\,ds = A\frac{d^{n-1}u}{dt^{n-1}}. \tag{2.17}$$

Therefore,

$$\frac{d^n u}{dt^n} = \frac{d^{n-1}v}{dt^{n-1}} - A\frac{d^{n-1}u}{dt^{n-1}} \tag{2.18}$$

is continuous on I. Consequently,

$$u(t) = \int_0^t T(t-s)v(s)\,ds \tag{2.19}$$

represent the unique solution of the considered Cauchy problem (compare with [8, 10, 11]). □

3. Correct solution. In this section, we prove the correct formulation of the considered initial value problem (1.1) and (1.2). In other words, we prove the continuous dependent of the solution of the problem on the initial conditions. Let $\{u^m\}$ be a sequence of solutions of the initial value problem

$$\frac{d^n u^m}{dt^n} + A\frac{d^{n-1}u^m}{dt^{n-1}} = B(t)u^m + f(t,W^m), \tag{3.1}$$

$$u^m|_{t=0} = g_0^m \in S, \quad \left.\frac{du^m}{dt}\right|_{t=0} = g_1^m, \ \ldots, \ \left.\frac{d^{n-1}u^m}{dt^{n-1}}\right|_{t=0} = g_{n-1}^m, \tag{3.2}$$

where W^m is the sequence $W^m = (B_1(t)u^m, B_2(t)u^m, \ldots, B_\nu(t)u^m)$.

Theorem 3.1 (see [7, 9]). *Let the sequences $\{g_0^m\},\{g_1^m\},\{g_2^m\},\ldots,\{g_{n-1}^m\},\{Ag_{n-1}^m\}$, be convergent in E to $g_0,g_1,g_2,\ldots,g_{n-1},Ag_{n-1}$, respectively. If the sequences $\{B(t)g_0^m\}$, $\{B(t)g_1^m\}$, $\{B(t)g_2^m\},\ldots,\{B(t)g_{n-1}^m\}$, $\{B_i(t)g_0^m\}$, $\{B_i(t)g_1^m\}$, $\{B_i(t)g_2^m\},\ldots,\{B_i(t)g_{n-1}^m\}$ are uniformly convergent with respect to $t\in I$ in E to $B(t)g_0,B(t)g_1,B(t)g_2,\ldots,$ $B(t)g_{n-1},B_i(t)g_0,B_i(t)g_1,B_i(t)g_2,\ldots,B_i(t)g_{n-1}$, $i=1,2,\ldots,\nu$, respectively, where $g_0,g_1,g_2,\ldots,g_{n-1}$ are elements in G, then the sequence of solutions $\{u^m(t)\}$ of the problem (3.1) and (3.2) converges in the metric space $C^E(I)$ to the solution $u(t)$ of the considered problem with the initial conditions,*

$$u|_{t=0} = g_0, \quad \left.\frac{du}{dt}\right|_{t=0} = g_1, \ \ldots, \ \frac{d^{n-1}u}{dt^{n-1}} = g_{n-1}, \tag{3.3}$$

Proof. Let

$$\zeta^m(t) = u^m(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}g_k^m \tag{3.4}$$

substitute in equation (1.8), we get

$$\frac{d^n\zeta^m}{dt^n} + A\frac{d^{n-1}\zeta^m}{dt^{n-1}} = B(t)\zeta^m + f^*(t,W^m) \tag{3.5}$$

with the initial conditions

$$\zeta^m|_{t=0} = 0, \frac{d\zeta^m}{dt}\bigg|_{t=0} = 0, \ \dots, \ \frac{d^{n-1}\zeta^m}{dt^{n-1}}\bigg|_{t=0} = 0, \tag{3.6}$$

where

$$\begin{aligned} f^*(t,W^m) &= \sum_{k=0}^{n-1}\left(B\frac{t^k}{k!}g_k^m\right) + f(t,W^m) - Ag_{n-1}^m, \\ W^m &= (B_1(t)u^m, B_2(t)u^m, \dots, B_\nu(t)u^m) \\ &= \left(B_1\zeta^m + \sum_{k=0}^{n-1} B_1(t)\frac{t^k}{k!}g_k^m, \dots, B_\nu(t)\zeta^m + \sum_{k=0}^{n-1} B_\nu(t)\frac{t^k}{k!}g_k^m\right). \end{aligned} \tag{3.7}$$

Set

$$\frac{d\zeta^m(t)}{dt} + A\zeta^m(t) = P^m(t), \tag{3.8}$$

therefore,

$$\frac{d^{n-1}}{dt^{n-1}}(P^m(t)) = B(t)\zeta^m(t) + f^*(t,W^m). \tag{3.9}$$

It is clear that

$$\begin{aligned} P^m(t) = &\int_0^t\int_s^t\int_s^{y_{n-1}}\cdots\int_s^{y_3}\int_s^{y_2} B(y_1)T(y_1-s)P^m(s)\,dy_1\,dy_2\cdots dy_{n-1}\,ds \\ &+ \int_0^t\int_0^{\xi_{n-1}}\cdots\int_0^{\xi_3}\int_0^{\xi_2} f^*(\xi_1,W^m(\xi_1))\,d\xi_1\,d\xi_2\cdots d\xi_{n-1}, \end{aligned} \tag{3.10}$$

we can easily deduce that

$$\begin{aligned} &||P^m(t)-P^r(t)|| \\ &\quad\le \int_0^t\int_s^t\int_s^{y_{n-1}}\cdots\int_s^{y_3}\int_s^{y_2} ||B(y_1)T(y-s)||\,\|P^m(s)-P^r(s)\|\,dy_1\,dy_2\cdots dy_{n-1}\,ds \\ &\qquad+ \int_0^t\int_0^{\xi_{n-1}}\cdots\int_0^{\xi_3}\int_0^{\xi_2}\Bigg[\sum_{k=0}^{n-1} B\frac{t^k}{k!}\left|\left|g_k^m-g_k^r\right|\right| \\ &\qquad\qquad + ||f(t,W^m)-f(t,W^r)|| + A\left|\left|\left(g_{n-1}^m-g_{n-1}\right)\right|\right|\Bigg] \\ &\qquad\qquad \times d\xi_1\,d\xi_2\cdots d\xi_{n-1}. \end{aligned} \tag{3.11}$$

Multiply by $e^{-\lambda t}$ and using the metric defined by equation (2.1), we get

$$\begin{aligned} \rho(P^m,P^r) \le K\rho\left(Ag_{n-1}^m, Ag_{n-1}^r\right) + K\left[\sum_{k=0}^{n-1}\rho\left(Bg_k^m, Bg_k^r\right)\right] \\ + K\sum_{k=1}^{\nu}\sum_{j=0}^{n-1}\rho\left(A_kg_j^m, A_kg_j^r\right). \end{aligned} \tag{3.12}$$

According to all the conditions before, the sequence $\{P^m\}$ is fundamental and hence converges to P in $C^E(I)$. But

$$\zeta^m(t) = \int_0^t T(t-s)P^m(s)\,ds. \tag{3.13}$$

Therefore, the sequence $\{u^m(t)\}$ uniformly converges with respect to $t \in I$ in E to the required solution (compare with [6, 5, 4, 9]). □

4. Application. Consider the Cauchy problem

$$\frac{\partial^{n-1}}{\partial t^{n-1}}(Lu) = \sum_{|\alpha| \le 2m-1} a_\alpha(x,t) D^\alpha u + f(x,t,W), \tag{4.1}$$

with the initial conditions

$$u(x,t)|_{t=0} = g_0(x), \ \frac{\partial u(x,t)}{\partial t}\bigg|_{t=0} = g_1(x), \ \ldots, \ \frac{\partial^{n-1} u(x,t)}{\partial t^{n-1}}\bigg|_{t=0} = g_{n-1}(x), \tag{4.2}$$

where

$$\begin{gathered} Lu = \frac{\partial u}{\partial t} + A(x,D)u, \qquad A(x,D) = \sum_{|\alpha| \le 2m} b_\alpha(x) D^\alpha, \\ W = \big(B_1(t)u, B_2(t)u, \ldots, B_\nu(t)u\big), \qquad B_i(t) = \sum_{|\alpha| \le 2m-1} C_{\alpha,i}(x,t) D^\alpha, \end{gathered} \tag{4.3}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $D_i = \partial/\partial x_i$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-dimensional multi-index, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let M be an open set in the n-dimensional Euclidean space $\mathbb{R}^n$, and let $L_2(M)$ be the space of all square integrable functions on M. We denote by $C^m(M)$ the set of all continuous real-valued functions in M together with all their m-partial derivatives, and we denote by $C_0^m(M)$ the subset of $C^m(M)$ consisting of all functions having a compact support. Let $H^m(M)$ be the complete space of $C^m(M)$ with respect to the norm (see [2, 3])

$$\|f\|_m = \left[\sum_{|\alpha| \le m} \int_M |D^\alpha f(x)|^2 dx \right]^{1/2}. \tag{4.4}$$

For any $0 < b < \infty$ denote by Ω_b the cylinder $\{(x,t) : x \in M, \ 0 < t < b\}$, and by Γ_b the boundary $\{(x,t) : x \in \partial M, \ 0 < t < b\}$, where ∂M is the boundary of M. We say that L is uniformly parabolic in $\bar{M}$, the closure of M if the coefficients b_α are continuous on $\bar{M}$ and if

$$(-1)^m \sum_{|\alpha| = 2m} b_\alpha(x) \eta^\alpha \ge C|\eta|^{2m}, \quad c > 0, \tag{4.5}$$

for all $x \in M$ and for all $\eta \in R^n$, where $\eta^\alpha = \eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_n^{\alpha_n}$, and $|\eta|^2 = \eta_1^2 + \eta_2^2 + \cdots + \eta_n^2$. Suppose also that the coefficients a_α, $C_{\alpha,i}$ are continuous on Ω_b and satisfies a uniform Hölder condition in $t \in [0,b]$. The Cauchy problem (4.1) and (4.2) can be written in the abstract form (1.1) and (1.2), where A is the operator with domain $S = H^{2m}(M) \cap H_0^m(M)$ given by

$$Au = A(x,D)u = \sum_{|\alpha| \le 2m} b_\alpha(x) D^\alpha u. \tag{4.6}$$

Let $E = L_2(M)$. Then the domain S is dense in $L_2(M)$. The operators $B(t), B_1(t), B_2(t), \ldots, B_\nu(t)$ are given by

$$B(t) = \sum_{|\alpha| \le 2m-1} a_\alpha(x,t) D^\alpha, \qquad B_i(t) = \sum_{|\alpha| \le 2m-1} C_{\alpha,i}(x,t) D^\alpha, \tag{4.7}$$

where $i = 1,2,\dots,\nu$. The domain of these operators can be taken $H^{2m-1}(M) \cap H_0^m(M)$ which is dense in $L_2(M)$ (see [1, 2, 10]). Therefore, we can assume that

$$S_1 = S_2 = \cdots = S_\nu = F = H^{2m-1}(M) \cap H_0^m(M). \tag{4.8}$$

Suppose that $g_0(x), g_1(x), \dots, g_{n-1}(x)$ are given functions in S. Since Lu is uniformly parabolic, it follows that $-A = -A(x,D)$ generates a semi-group $\{T(t)\}$ of class C_0. It can be proved that $T(t)$ satisfies the condition (1.7) and (1.8). Consequently, [2, 6] can be applied to the Cauchy problem (4.1) and (4.2). This means that the considered problem can be solved in S without any restrictions on the characteristic forms of the operators

$$\sum_{|\alpha| \le 2m} a_\alpha(x,t) D^\alpha, \qquad \sum_{|\alpha| \le 2m} C_{\alpha,i}(x,t) D^\alpha, \tag{4.9}$$

which depends only on the continuity of the functions $g_0, g_1, \dots, g_{n-1}$.

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