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SPATIAL DECAY ESTIMATES FOR A CLASS OF NONLINEAR DAMPED HYPERBOLIC EQUATIONS

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ABSTRACT. This paper is concerned with investigating the spatial decay estimates for a class of nonlinear damped hyperbolic equations. In addition, we compare the solutions of two-dimensional wave equations with different damped coefficients and establish an explicit inequality which displays continuous dependence on this coefficient.

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1. Introduction. Spatial decay estimates for several types of partial differential equations and systems have been the subject of extensive investigations in the literature for close to a century and a half. These studies were motivated by a desire to formulate Saint-Venant and Phragmén-Lindelöf type principles in elasticity and heat conduction. Roughly speaking, these estimates assert that the solution of the problem decays exponentially with distance from the boundary on which a mechanical or thermal “load” has been applied. In the case of elliptic problems, this work has been directed toward establishing a rational form of Saint-Venant’s principle and has included studies in linear elasticity (see Toupin [18] and Knowles [9]), in nonlinear plane elasticity (see Roseman [16]) and in linear viscoelasticity (see Edelstein [4]). In a recent paper, Tahamtani [17] derived an explicit Saint-Venant type decay estimate for solutions of the Dirichlet problem for nonlinear biharmonic equations defined in a semi-infinite cylinder in $\mathbb{R}^n$ with homogeneous Dirichlet data on the lateral surface of the cylinder.

A spatial decay estimate for transient heat conduction was first given by Edelstein [3]. The result has been consistently improved by the studies completed by Knowles [10], Horgan et al. [7], and Chiriţă [2].

Very little attention has been devoted to the study of hyperbolic differential equations. Horgan and Knowles [6] and Horgan [5] pointed out the paucity of Saint-Venant type results for hyperbolic system of the kind describing elastic wave propagation. The only previous work known to us on questions like this for the hyperbolic differential equations is that of Quintanilla [15]. He considered the transient solutions of the damped wave equation and established a spatial decay estimate of the kind described by Knowles [10] for the heat conduction equation. The results we present here generalize the work in [15] to nonlinear damped hyperbolic equations and obtain stronger results involving an exponential decay of energy functional.

Alternatively, the results may be viewed as theorems of Phragmén-Lindelöf type [1, 8, 14] for nonlinear damped hyperbolic equations.

In this paper, we show that if the solution is bounded in an energy norm, then it must decay exponentially in energy norm as the distance from the near end tends to infinity. Finally, we compare the solutions of two damped wave equations with different damped coefficients and establish an explicit inequality which displays continuous dependence on this coefficient.

2. Preliminaries. In this paper, we derive a spatial decay estimate for a functional defined on the solutions of the equation

$$\alpha u_{tt} + \nu f(u_t) = \Delta u, \tag{2.1}$$

where α and ν are two positive numbers, Δ is the Laplace operator, and f is a nonlinear function satisfying the inequalities

$$f(v)v \geq c_1|v|^p, \qquad |f(v)| \leq c_2|v|^{p-1}, \tag{2.2}$$

for $p \geq 2$, $c_1 > 0$, $c_2 > 0$.

Our attention is focused on the initial-boundary problem for (2.1) in the space-time region $\Omega \times (0,\infty)$, where

$$\Omega = \{(x_1,x') \in \mathbb{R}^n : x_1 \in \mathbb{R}^+,\ x' = (x_2,x_3,\dots,x_n) \in \sigma_{x_1} \subset \mathbb{R}^{n-1}\} \tag{2.3}$$

is the semi-infinite prismatic cylinder and σ_{x_1} denotes the open, bounded, and simply connected cross section of Ω. In addition, $u(x_1,x',t)$ is required to satisfy the initial and boundary conditions

$$u(x_1,x',0) = 0, \quad u_t(x_1,x',0) = 0, \quad (x_1,x') \in \Omega, \tag{2.4}$$

$$u(x_1,x',t) = 0, \quad x' \in \partial\sigma_{x_1},\ x_1 \geq 0,\ t \geq 0, \tag{2.5}$$

$$u(0,x',t) = g(x',t), \quad x' \in \sigma_{x_1},\ x_1 = 0,\ t \geq 0, \tag{2.6}$$

where the function $g(x',t)$ is a prescribed function and vanishes on the boundary $\partial\sigma_{x_1}$. For convenience, we introduce the notation

$$\Omega_\tau = \{(x_1,x') : 0 < \tau < x_1\}, \qquad \sigma_\tau = \{(x_1,x') : x_1 = \tau\}. \tag{2.7}$$

We describe the quantity

$$\lambda^p(D) = \inf_{v \in C_0^1(D)} \left(\int_D |\nabla v|^p\,dx\right)\left(\int_D |v|^p\,dx\right)^{-1}, \tag{2.8}$$

where $C_0^1(D)$ is the set of functions that are continuously differentiable with compact support in D. In [13] examples are given, where for an analogous λ^p, a lower estimate can be found by means of the first eigenvalues of some elliptic boundary-value problem on D. We note that for $p = 2$, (2.8) is the Poincaré-Friedrich's inequality

$$\int_D v^2\,dx \leq \lambda^{-2}(D) \int_D |\nabla v|^2\,dx, \tag{2.9}$$

see [11]. Young's inequality is used often in this article. It states that

$$x^{1/p}y^{1/q} \le \frac{1}{p\varepsilon}x + \frac{1}{q}\varepsilon^p y, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.10}$$

for x, $y > 0$ and arbitrary $\varepsilon > 0$.

3. A decay theorem. In this section, we state a spatial decay estimate for the solution of the problem defined by (2.1), (2.4), (2.5), and (2.6). We recall that the following equalities:

$$\begin{aligned} \nabla\cdot(u\nabla u) - \nabla u\nabla u &= \nu u f(u_t) + \frac{d}{dt}(\alpha u u_t) - \alpha u_t^2, \\ \nabla\cdot(u_t\nabla u) - \nabla u\nabla u_t &= \nu u_t f(u_t) + \frac{d}{dt}\left(\frac{\alpha}{2}u_t^2\right) \end{aligned} \tag{3.1}$$

are satisfied for all solutions of the nonlinear equation (2.1). Let $\delta = \nu/(1+\alpha)$; we may consider

$$F(\tau, t_1) := -\int_0^{t_1} \nabla\cdot[(u_t + \delta u)\nabla u]dt, \quad x \in \Omega,\ 0 \le t_1 < t. \tag{3.2}$$

To obtain our estimates, it is suitable to recall that (see [15, Lemma 2.1, page 80])

$$\int_{\Omega_\tau} F(\tau, t_1) \ge J_1 + J_2, \tag{3.3}$$

where

$$J_1 := \int_0^{t_1}\int_{\Omega_\tau} (\delta|\nabla u|^2 - c_2\delta\nu|u||u_t|^{p-1} + c_1\nu|u_t|^p - \delta\alpha u_t^2)\,dx\,dt, \tag{3.4}$$

$$J_2 := \int_{\Omega_\tau}\left(\frac{\alpha}{2(1+\alpha)}u_t^2 + \frac{1}{2}|\nabla u|^2 - \frac{\delta\nu}{2}u^2\right)dx. \tag{3.5}$$

Applying Hölder's and Young's inequalities we can estimate the second and fourth terms of (3.4) as follows:

$$\begin{aligned} I_1 &:= c_2\delta\nu\int_0^{t_1}\int_{\Omega_\tau}|u||u_t|^{p-1}\,dx\,dt \\ &\le c_2\frac{1}{p\varepsilon}\delta\nu\int_0^{t_1}\int_{\Omega_\tau}|u|^p\,dx\,dt + c_2\frac{p-1}{p}\varepsilon^p\delta\nu\int_0^{t_1}\int_{\Omega_\tau}|u_t|^p\,dx\,dt, \end{aligned} \tag{3.6}$$

$$\begin{aligned} I_2 &:= \delta\alpha\int_0^{t_1}\int_{\Omega_\tau}u_t^2\,dx\,dt \\ &\le \frac{p-2}{p}\varepsilon^p\delta\alpha\int_0^{t_1}\int_{\Omega_\tau}1\,dx\,dt + \frac{2}{p\varepsilon}\delta\alpha\int_0^{t_1}\int_{\Omega_\tau}|u_t|^p\,dx\,dt. \end{aligned} \tag{3.7}$$

Using the quantity in (2.8), we find from (3.6)

$$I_1 \le c_2\frac{1}{p\varepsilon}\delta\nu\lambda^{-p}(\Omega_\tau)\int_0^{t_1}\int_{\Omega_\tau}|\nabla u|^p\,dx\,dt + c_2\frac{p-1}{p}\varepsilon^p\delta\nu\int_0^{t_1}\int_{\Omega_\tau}|u_t|^p\,dx\,dt. \tag{3.8}$$

Dropping the first term on the right-hand side of (3.4), then from (3.4), (3.7), and (3.8) we obtain

$$\begin{aligned} J_1 \geq{}& \nu\left(1-c_2\frac{1}{p\varepsilon}\delta\lambda^{-p}(\Omega_\tau)\right)\int_0^{t_1}\int_{\Omega_\tau}|\nabla u|^p\,dx\,dt \\ &+\nu\left(c_1-c_2\frac{p-1}{p}\varepsilon^p\delta\right)\int_0^{t_1}\int_{\Omega_\tau}|u_t|^p\,dx\,dt-\nu\int_0^{t_1}\int_{\Omega_\tau}|\nabla u|^p\,dx\,dt \\ &-\frac{2}{p\varepsilon}\delta\alpha\int_0^{t_1}\int_{\Omega_\tau}|u_t|^p\,dx\,dt-C_{\Omega_\tau}\frac{p-2}{p}\varepsilon^p\delta\alpha t_1, \end{aligned} \tag{3.9}$$

where C_{Ω_τ} is a positive constant depending only on Ω_τ.

Next, utilizing the Poincaré-Friedrich's inequality (2.9) we get from (3.5)

$$J_2 \geq \int_{\Omega_\tau}\left[\frac{1}{2}(1-\delta\nu\lambda^{-2}(\Omega_\tau))|\nabla u|^2+\frac{\alpha}{2(1+\alpha)}u_t^2\right]dx. \tag{3.10}$$

From (3.9), (3.10), and (3.3) it follows that

$$\begin{aligned} \int_{\Omega_\tau}F(\tau,t_1) \geq{}& N_1(\tau)\int_{\Omega_\tau}H^2(u_t,\nabla u)dx+N_2(\tau)\int_0^{t_1}\int_{\Omega_\tau}H^p(u_t,\nabla u)dx\,dt \\ &-\beta\left\{\int_{\Omega_\tau}H^2(u_t,\nabla u)dx+\int_0^{t_1}\int_{\Omega_\tau}H^p(u_t,\nabla u)dx\,dt\right\}-C_{\Omega_\tau}\frac{p-2}{p}\varepsilon^p\delta\alpha t_1, \end{aligned} \tag{3.11}$$

where

$$\begin{gathered} H^i(u_t,\nabla u):=\frac{1}{2}|\nabla u|^i+\frac{\alpha}{2(1+\alpha)}|u_t|^i,\quad \text{for } i=2,p, \\ N_1(\tau)=\min\{1,(1-\delta\nu\lambda^{-2}(\Omega_\tau))\}, \\ N_2(\tau)=\min\left\{2\nu\left(1-c_2\frac{1}{p\varepsilon}\delta\lambda^{-p}(\Omega_\tau)\right),\frac{1+\alpha}{\alpha}2\nu\left(c_1-c_2\frac{p-1}{p}\varepsilon^p\delta\right)\right\}, \\ \beta=\max\left\{1,2\nu,\frac{4(1+\alpha)}{p\varepsilon}\delta\right\}. \end{gathered} \tag{3.12}$$

We may always take $\varepsilon>0$ so large and $\delta>0$ so small that

$$1-\delta\nu\lambda^{-2}(\Omega_\tau)>0,\qquad 1-c_2\frac{1}{p\varepsilon}\delta\lambda^{-p}(\Omega_\tau)>0,\qquad c_1-c_2\frac{p-1}{p}\varepsilon^p\delta>0. \tag{3.13}$$

We may define

$$E_u(\tau,t)=\int_{\Omega_\tau}\left\{\int_0^{t_1}H^p(u_t,\nabla u)dt+H^2(u_t,\nabla u)\right\}dx \tag{3.14}$$

as the strain energy contained in Ω_τ. Inserting (3.14) in (3.11) we get

$$\int_{\Omega_\tau}F(\tau,t_1)\geq(\gamma(\tau)-\beta)E_u(\tau,t)-C_{\Omega_\tau}\frac{p-2}{p}\varepsilon^p\delta\alpha t_1, \tag{3.15}$$

where $\beta<\gamma(\tau)=\min\{N_1(\tau),N_2(\tau)\}$. Our next objective is to estimate the left-hand side of (3.3). Due to the boundary conditions (2.5) and (2.6) and the divergence theorem, we obtain

$$\int_{\Omega_\tau} F(\tau,t_1) = -\int_0^{t_1}\int_{\sigma_\tau}(u_t+\delta u)u_{x_1}\,dx'\,dt. \tag{3.16}$$

Using the Schwarz inequality we find

$$\left|\int_{\Omega_\tau} F(\tau,t_1)\right| \le \int_0^{t_1}\left[\int_{\sigma_\tau}(u_{x_1}^2\,dx')^{1/2}\left\{\int_{\sigma_\tau}(u_t^2\,dx')^{1/2}+\delta\int_{\sigma_\tau}(u^2\,dx')^{1/2}\right\}\right]dt. \tag{3.17}$$

From Poincaré-Friedrich's and the arithmetic-geometric mean inequalities we deduce

$$\left|\int_{\Omega_\tau} F(\tau,t_1)\right| \le \frac{1}{2\varepsilon}(1+\delta^2\lambda^{-2}(\sigma_\tau))\int_0^{t_1}\int_{\sigma_\tau}|\nabla u|^2\,dx'\,dt+\frac{\varepsilon}{2}\int_0^{t_1}\int_{\sigma_\tau}u_t^2\,dx'\,dt. \tag{3.18}$$

Using Hölder's and Young's inequalities

$$\begin{aligned}\left|\int_{\Omega_\tau} F(\tau,t_1)\right| \le{}& \frac{1}{p\varepsilon^2}(1+\delta^2\lambda^{-2}(\sigma_\tau))\int_0^{t_1}\int_{\sigma_\tau}|\nabla u|^p\,dx'\,dt\\ &+\frac{1}{p}\int_0^{t_1}\int_{\sigma_\tau}|u_t|^p\,dx'\,dt+\tilde{C}_{\sigma_\tau}\frac{p-2}{2p}\varepsilon^{p-1}(1+\varepsilon^2+\delta^2\lambda-2(\sigma_\tau))t_1,\end{aligned} \tag{3.19}$$

where $\tilde{C}_{\sigma_\tau}$ is a positive constant depending only on σ_τ.

We add appropriate terms into the right-hand side of (3.19) in order to put it into the energy term (3.14). Thus,

$$\left|\int_{\Omega_\tau} F(\tau,t_1)\right| \le -N_0(\tau)\frac{\partial}{\partial\tau}E_u(\tau,t)+\tilde{C}_{\sigma_\tau}\frac{p-2}{2p}\varepsilon^{p-1}(1+\varepsilon^2+\delta^2\lambda^{-2}(\sigma_\tau))t_1, \tag{3.20}$$

where $N_0(\tau)=\max\{1,(2(1+\alpha)/\alpha p),(2/p\varepsilon^2)(1+\delta^2\lambda^{-2}(\sigma_\tau))\}$ and

$$\frac{\partial}{\partial\tau}E_u(\tau,t) = -\int_{\sigma_\tau}\left\{\int_0^{t_1}H^p(u_t,\nabla u)dt+H^2(u_t,\nabla u)\right\}dx'. \tag{3.21}$$

Combining the estimates (3.15) and (3.20), we find

$$E_u(\tau,t)+\omega(\tau)\frac{\partial}{\partial\tau}E_u(\tau,t)\le M_{t_1}(\tau), \tag{3.22}$$

where $\omega(\tau):=N_0(\tau)(\gamma(\tau)-\beta)^{-1}$, and

$$M_{t_1}(\tau):=\frac{p-2}{2p}\varepsilon^{p-1}\left[(1+\varepsilon^2+\delta^2\lambda^{-2}(\sigma_\tau))\tilde{C}_{\sigma_\tau}+2\varepsilon\delta\alpha C_{\Omega_\tau}\right](\gamma(\tau)-\beta)^{-1}t_1. \tag{3.23}$$

Inequality (3.22) immediately implies that

$$\begin{aligned}E_u(\tau,t)\le{}& E_u(0,t)\exp\left[-\int_0^\tau\omega^{-1}(s)ds\right]+\exp\left[-\int_0^\tau\omega^{-1}(s)ds\right]\\ &\times\int_0^\tau\exp\left[\int_0^s\omega^{-1}(r)dr\right]\omega^{-1}(s)M_{t_1}(s)ds.\end{aligned} \tag{3.24}$$

Now if we suppose that $\int_0^\infty\omega^{-1}(\tau)d\tau=\infty$ and for fixed t_1, $\lim_{\tau\to\infty}M_{t_1}(\tau)=0$, then by l'Hôpital's rule we have

$$\lim_{\tau\to\infty} \exp\left[-\int_0^\tau \omega^{-1}(s)ds\right]\int_0^\tau \exp\left[\int_0^s \omega^{-1}(r)dr\right]\omega^{-1}(s)M_{t_1}(s)ds = 0. \tag{3.25}$$

Inequality (3.24) implies that $\lim_{\tau\to\infty}\sup E_u(\tau,t) \le 0$. Thus we may state the following result.

Theorem 3.1. *Let u be a solution of the initial-boundary value problem (2.1), (2.2), (2.4), (2.5), and (2.6). If the cylinder Ω satisfies $\int_0^\infty \omega^{-1}(\tau)d\tau = \infty$ and $\lim_{\tau\to\infty} M_{t_1}(\tau) = 0$, then the following estimate holds for all $\tau > 0$,*

$$\begin{aligned}
&\int_{\Omega_\tau}\left\{\int_0^{t_1} H^p(u_t,\nabla u)dt + H^2(u_t,\nabla u)\right\}dx \\
&\quad\le \exp\left[-\int_0^\tau \omega^{-1}(s)ds\right]\int_{\Omega_0}\left\{\int_0^{t_1} H^p(u_t,\nabla u)dt + H^2(u_t,\nabla u)\right\}dx \\
&\qquad + \exp\left[-\int_0^\tau \omega^{-1}(s)ds\right]\int_0^\tau \exp\left[\int_0^s \omega^{-1}(r)dr\right]\omega^{-1}(s)M_{t_1}(s)ds.
\end{aligned} \tag{3.26}$$

4. Continuous dependence on the damped coefficient. If $f(u_t) = u_t$ in (2.1), we denote by $v(x_1,x',t)$ the solution of the linear equation

$$\alpha u_{tt} + \nu u_t = \Delta u \tag{4.1}$$

that satisfies the initial and boundary conditions in (2.4), (2.5), and (2.6) with ν replaced by the constant $\tilde{\nu}$. For u to be the solution of (4.1), (2.4), (2.5), and (2.6) and v to be the solution of (4.1), (2.4), (2.5), and (2.6) with damping coefficient $\tilde{\nu}$ in (4.1), we establish an explicit inequality which displays continuous dependence on the coefficient ν.

If we now set $w = u - v$, then w satisfies

$$\begin{gathered}
\alpha w_{tt} + \nu w_t + (\nu-\tilde{\nu})v_t = \Delta w, \qquad (x_1,x',t)\in\Omega\times(0,\infty), \\
w(x_1,x',0) = 0, \quad w_t(x_1,x',0) = 0, \quad (x_1,x')\in\Omega, \\
w(x_1,x',t) = 0, \quad x'\in\partial\sigma_{x_1},\ x_1\ge 0,\ t\ge 0, \\
w(0,x',t) = 0, \quad x'\in\sigma_{x_1},\ x_1 = 0,\ t\ge 0.
\end{gathered} \tag{4.2}$$

Using the methods of [12], we can treat the case in which $v \ne u$ on the end $x_1 = 0$. Calculations similar to those used in Section 3 lead to the equalities

$$\begin{aligned}
\int_0^{t_1} \nabla\cdot(w\nabla w)dt &= \alpha w w_t + \frac{\nu}{2}w^2 + \int_0^{t_1}\left(|\nabla w|^2 - \alpha w_t^2 + (\nu-\tilde{\nu})w v_t\right)dt, \\
\int_0^{t_1} \nabla\cdot(w_t\nabla w)dt &= \frac{\alpha}{2}w_t^2 + \frac{1}{2}|\nabla w|^2 + \int_0^{t_1}\left(\nu w_t^2 + (\nu-\tilde{\nu})w_t v_t\right)dt.
\end{aligned} \tag{4.3}$$

Similar to the definition of F in (3.2), we may define

$$\begin{aligned}
\Phi(\tau,t_1) = &\int_0^{t_1}\int_{\Omega_\tau}\left(\tilde{\delta}|\nabla w|^2 + (\nu-\tilde{\nu})(w_t+\tilde{\delta}w)v_t + (\nu-\alpha\tilde{\delta})w_t^2\right)dx\,dt \\
&+ \int_{\Omega_\tau}\left(\frac{1}{2}|\nabla w|^2 + \frac{\alpha}{2}w_t^2 + \alpha\tilde{\delta}w w_t + \frac{\nu}{2}w^2\right)dx,
\end{aligned} \tag{4.4}$$

where $\tilde{\delta}$ is a positive constant to be specified later. By similar calculation techniques of the previous section, from (4.4) we deduce

$$\begin{aligned} M_0(\tau)\int_{\sigma_\tau}&\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx' \\ &\geq \int_{\Omega_\tau}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dx+\int_0^{t_1}\int_{\Omega_\tau}[\tilde{\delta}w_t^2+(\nu-\tilde{\nu})[w_t+\tilde{\delta}w]v_t]\,dx\,dt, \end{aligned} \tag{4.5}$$

where $M_0(\tau)=\max\{1,\varepsilon(1+\alpha)/\alpha,\varepsilon^{-1}(1+\tilde{\delta}^2\lambda^{-2}(\sigma_\tau))\}$. Making use of Schwarz inequality together with Poincaré-Friedrich's and arithmetic-geometric mean inequalities we have

$$\begin{aligned} (\nu-\tilde{\nu})&\int_0^{t_1}\int_{\Omega_\tau}[w_t+\tilde{\delta}w]v_t\,dx\,dt \\ &\leq \frac{\varepsilon}{2}\tilde{\delta}^2\lambda^{-2}(\Omega_\tau)\int_0^{t_1}\int_{\Omega_\tau}|\nabla w|^2\,dx\,dt \\ &\quad+\frac{\varepsilon}{2}\int_0^{t_1}\int_{\Omega_\tau}w_t^2\,dx\,dt+\frac{1}{2\varepsilon}(\nu-\tilde{\nu})^2\int_0^{t_1}\int_{\Omega_\tau}v_t^2\,dx\,dt. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6)

$$\begin{aligned} M_0(\tau)\int_{\sigma_\tau}&\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx' \\ &\geq \left(\nu-\frac{\varepsilon}{2}\tilde{\delta}\lambda^{-2}(\Omega_\tau)\right)\int_0^{t_1}\int_{\Omega_\tau}|\nabla w|^2\,dx\,dt+\tilde{\delta}\int_0^{t_1}\int_{\Omega_\tau}w_t^2\,dx\,dt \\ &\quad+\int_{\Omega_\tau}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dx-\nu\int_0^{t_1}\int_{\Omega_\tau}|\nabla w|^2\,dx\,dt-\frac{\varepsilon}{2}\int_0^{t_1}\int_{\Omega_\tau}w_t^2\,dx\,dt \\ &\quad-\int_{\Omega_\tau}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dx-\frac{1}{2\varepsilon}(\nu-\tilde{\nu})^2\int_0^{t_1}\int_{\Omega_\tau}v_t^2\,dx\,dt. \end{aligned} \tag{4.7}$$

Taking $\tilde{\delta}>0$ so small that $\nu>(\varepsilon/2)\tilde{\delta}\lambda^{-2}(\Omega_\tau)$, we obtain

$$\begin{aligned} M_0(\tau)\int_{\sigma_\tau}&\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx' \\ &\geq M_1(\tau)\int_{\Omega_\tau}\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx \\ &\quad-\tilde{\beta}\int_{\Omega_\tau}\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx \\ &\quad-\frac{1}{2\varepsilon}(\nu-\tilde{\nu})\int_0^{t_1}\int_{\Omega_\tau}v_t^2\,dx\,dt, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} M_1(\tau)&=\min\left\{1,\frac{[2(1+\alpha)\tilde{\delta}]}{\alpha},(2\nu-\varepsilon\tilde{\delta}\lambda^{-2}(\Omega_\tau))\right\}, \\ \tilde{\beta}&=\max\left\{1,2\nu,\frac{[\varepsilon(1+\alpha)]}{\alpha}\right\}. \end{aligned} \tag{4.9}$$

Let

$$E_w(\tau,t) := \int_{\Omega_\tau}\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx,$$

$$\frac{\partial}{\partial\tau}E_w(\tau,t) := -\int_{\sigma_\tau}\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\right\}dx'. \tag{4.10}$$

Upon inserting (4.10) in (4.8), we obtain the differential inequality (provided that $M_1(\tau)\ge\tilde{\beta}$)

$$\frac{\partial}{\partial\tau}E_w(\tau,t)+(M_1(\tau)-\tilde{\beta})M_0^{-1}(\tau)E_w(\tau,t)\le\frac{1}{2\varepsilon}(\nu-\tilde{\nu})^2\int_0^{t_1}\int_{\Omega_\tau}v_t^2\,dx\,dt. \tag{4.11}$$

It is well known that

$$\begin{aligned}\frac{\alpha}{2(1+\alpha)}&\int_0^{t_1}\int_{\Omega_\tau}v_t^2\,dx\,dt\\ &\le\int_{\Omega_\tau}\left\{\int_0^{t_1}\left(\frac{1}{2}|\nabla v|^2+\frac{\alpha}{2(1+\alpha)}v_t^2\right)dt+\frac{1}{2}|\nabla v|^2+\frac{\alpha}{2(1+\alpha)}v_t^2\right\}dx\\ &\le E_v(0,t)\exp\left[-\int_0^\tau\omega^{-1/2}(s)ds\right],\end{aligned} \tag{4.12}$$

where

$$E_v(0,t)=\int_{\Omega_0}\left\{\int_0^{t_0}\left(\frac{1}{2}|\nabla v|^2+\frac{\alpha}{2(1+\alpha)}v_t^2\right)dt+\frac{1}{2}|\nabla v|^2+\frac{\alpha}{2(1+\alpha)}v_t^2\right\}dx \tag{4.13}$$

is bounded (cf. [15, Theorem 3.1]), and $\omega(\tau)$ is some positive function. Thus inserting (4.12) into (4.11) leads to

$$\begin{aligned}\frac{\partial}{\partial\tau}&E_w(\tau,t)+(M_1(\tau)-\tilde{\beta})M_0^{-1}(\tau)E_w(\tau,t)\\ &\le\frac{1+\alpha}{\alpha\varepsilon}(\nu-\tilde{\nu})^2E_v(0,t)M_0^{-1}(\tau)\exp\left[-\int_0^\tau\omega^{-1/2}(s)ds\right].\end{aligned} \tag{4.14}$$

We now choose $(M_1(\tau)-\tilde{\beta})M_0^{-1}(\tau)=\omega^{-1/2}(\tau)$. But (4.14) may then be rewritten as

$$\frac{\partial}{\partial\tau}\left(\exp\left[\int_0^\tau\omega^{1/2}(s)ds\right]E_w(\tau,t)\right)\le\frac{1+\alpha}{\alpha\varepsilon}(\nu-\tilde{\nu})^2E_v(0,t)M_0^{-1}(\tau). \tag{4.15}$$

An integration leads to

$$E_w(\tau,t)\le\frac{1+\alpha}{\alpha\varepsilon}(\nu-\tilde{\nu})^2E_v(0,t)\left(\int_0^\tau M_0^{-1}(s)ds\right)\exp\left[-\int_0^\tau\omega^{1/2}(s)ds\right]. \tag{4.16}$$

We have thus established the following theorem.

Theorem 4.1. *Let u be the solution of the problem (4.1), (2.4), (2.5), and (2.6) and v the solution of the same problem with ν replaced by $\tilde{\nu}$. Then for arbitrary $\tau\ge 0$, $t\ge 0$*

the closeness of u and v in energy measure satisfies the following inequality:

$$\begin{aligned}\int_{\Omega_\tau}\Big\{\int_0^{t_1}\Big(\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\Big)dt+\frac{1}{2}|\nabla w|^2+\frac{\alpha}{2(1+\alpha)}w_t^2\Big\}dx \\ \le \frac{1+\alpha}{\alpha\varepsilon}(\nu-\tilde{\nu})^2E_v(0,t)\Big(\int_0^\tau M_0^{-1}(s)ds\Big)\exp\Big[-\int_0^\tau \omega^{1/2}(s)ds\Big].\end{aligned} \tag{4.17}$$

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