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# AN INTERARRIVAL HYPEREXPONENTIAL MACHINE INTERFERENCE WITH BALKING, RENEGING, STATE-DEPENDENT, SPARES, AND AN ADDITIONAL SERVER FOR LONGER QUEUES

**RAGAB OMARAH AL-SEEDY and FAWZIAH M. AL-IBRAHEEM**

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Abstract. The aim of this paper is to treat the analytical solution of the truncated interarrival hyperexponential machine interference queue: $H_2/M/1/m+Y/m+Y$ in case of two branches with the following concepts: balking, reneging, state-dependent, spares, and an additional server for longer queues. Our research treats the general case for the values of $m$ and $Y$ considering the discipline FIFO. And some special cases have been verified.

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**1. Introduction.** The truncated interarrival hyperexponential queue $H_2/M/1/m+y/m+y$ (in case of two branches) treated numerically by Gupta [6]. Al-Seedy [2] treated analytically in some cases of $Y=0$ and $m=1, 2, 3$. Also Al-Seedy [3] studied the general case of $m$ but with $Y=0$ with the two concepts of balking and reneging. Abou-El-Ata [1] treated the analytical solution of the truncated interarrival hyperexponential machine interference queue $H_2/M/C/m/m$ with both balking and reneging concepts only.

In this paper, we treat the analytical solution of the truncated interarrival hyperexponential machine interference queue $H_2/M/1/m+Y/m+Y$ with the concepts of balking, reneging, state-dependent, spares, and an additional server for longer queues. The discipline considered is FIFO.

**2. Analyzing the problem.** Consider the two-channels truncated interarrival hyperexponential queue having two branches with rates $\alpha\lambda_1$ and $(1-\alpha)\lambda_2$, and the service time is an exponential with service rate $\mu$.

Assume the balking concept with probability

$$\beta = \text{Prob}\{\text{a unit joins the queue}\}, \tag{2.1}$$

where

$$0 \le \beta < 1 \quad \text{if } n = 1(1)\overline{m+Y}, \qquad \beta = 1 \quad \text{if } n = 0. \tag{2.2}$$

Also, the reneging concept in case of $n$ units with probability is

$$r(n) = \text{Prob}\{\text{a unit leaves the queue}\} = (n-1)\delta, \tag{2.3}$$

where

$$r(n) = 0 \quad \text{for } n = 0,1, \qquad 0 < r(n) \le 1 \quad \text{for } n = 2(1)\overline{m+Y}, \tag{2.4}$$

and $\delta$ is the rate of time $t$, having the probability density function $f(t) = \delta e^{-\delta t}$.

The interarrival rates $\lambda_n$ in case of machine with spares are

$$\lambda_n = \begin{cases} m\lambda, & n = 0, \\ m\beta\lambda, & 1 \le n \le Y, \\ (m-n+y)\beta\lambda, & Y < n < Y+m, \\ 0, & n \ge Y+m. \end{cases} \tag{2.5}$$

But the service time rates in case of the concepts of state-dependent and an additional server for longer queues are

$$\mu_n = \begin{cases} \mu_1, & 0 \le n \le k_1, \\ \mu_2, & k_1 \le n \le k_2, \\ \mu_2 + \mu_3 = \mu, & k_2 \le n \le m+Y. \end{cases} \tag{2.6}$$

Define the probabilities:

$P_{n,j}$ = Prob$\{n$ units in the system and $j$ the arrival branch occupied by the next arrival unit$\}$ where $n \ge 0$ and $j = 1,2$.

The steady-state difference equations of the queue $H_2/M/1/m+Y/m+Y$ considering all the concepts of balking, reneging, state-dependent, spares, and an additional server for longer queues, are

- for $n = 0$,

$$\begin{aligned} m\lambda_1 P_{0,1} &= \mu_1 P_{1,1}, \\ m\lambda_2 P_{0,2} &= \mu_1 P_{1,2}, \end{aligned} \tag{2.7}$$

- for $n = 1$,

$$\begin{aligned} [m\beta\lambda_1 + \mu_1]P_{1,1} &= \alpha m\lambda_1 P_{0,1} + \alpha m\lambda_2 P_{0,2} + (\mu_1 + \delta)P_{2,1}, \\ [m\beta\lambda_2 + \mu_1]P_{1,2} &= (1-\alpha)m\lambda_1 P_{0,1} + (1-\alpha)m\lambda_2 P_{0,2} + (\mu_1 + \delta)P_{2,2}, \end{aligned} \tag{2.8}$$

- for $2 \le n \le Y$,

$$\begin{aligned} &[m\beta\lambda_1 + \mu_1 + (n-1)\delta]P_{n,1} \\ &\qquad = \alpha m\beta\lambda_1 P_{n-1,1} + \alpha m\beta\lambda_2 P_{n-1,2} + (\mu_1 + n\delta)P_{n+1,1}, \\ &[m\beta\lambda_2 + \mu_1 + (n-1)\delta]P_{n,2} \\ &\qquad = (1-\alpha)m\beta\lambda_1 P_{n-1,1} + (1+\alpha)m\beta\lambda_2 P_{n-1,2} + (\mu_1 + n\delta)P_{n+1,2}, \end{aligned} \tag{2.9}$$

- for $Y < n < k_1 - 1$,

$$\begin{aligned}
&[(m-n+Y)\beta\lambda_1+\mu_1+(n-1)\delta]P_{n,1}\\
&\qquad=\alpha(m-n+Y+1)\beta\lambda_1P_{n-1,1}+\alpha(m-n+Y+1)\beta\lambda_2P_{n-1,2}\\
&\qquad\quad+(\mu_1+n\delta)P_{n+1,1},\\
&[(m-n+Y)\beta\lambda_2+\mu_1+(n-1)\delta]P_{n,2}\\
&\qquad=(1-\alpha)(m-n+Y+1)\beta\lambda_1P_{n-1,1}+(1-\alpha)(m-n+Y+1)\beta\lambda_2P_{n-1,2}\\
&\qquad\quad+(\mu_1+n\delta)P_{n+1,2},
\end{aligned}\tag{2.10}$$

- for $n = k_1 - 1$,

$$\begin{aligned}
&[(m-k_1+Y+1)\beta\lambda_1+\mu_1+(k_1-2)\delta]P_{K_1-1,1}\\
&\qquad=\alpha(m-k_1+Y+2)\beta\lambda_1P_{k_1-2,1}+\alpha(m-k_1+Y+2)\beta\lambda_2P_{k_1-2,2}\\
&\qquad\quad+[\mu_2+(k_1-1)\delta]P_{k_1,1},\\
&[(m-k_1+Y+1)\beta\lambda_2+\mu_1+(k_1-2)\delta]P_{k_1-1,2}\\
&\qquad=(1-\alpha)(m-k_1+Y+2)\beta\lambda_1P_{k_1-2,1}+(1-\alpha)(m-k_1+Y+2)\beta\lambda_2P_{k_1-2,2}\\
&\qquad\quad+[\mu_2+(k_1-1)\delta]P_{k_1,2},
\end{aligned}\tag{2.11}$$

- for $k_1 \le n < k_2 - 1$,

$$\begin{aligned}
&[(m-n+Y)\beta\lambda_1+\mu_2+(n-1)\delta]P_{n,1}\\
&\qquad=\alpha(m-n+Y+1)\beta\lambda_1P_{n-1,1}+\alpha(m-n+Y+1)\beta\lambda_2P_{n-1,2}\\
&\qquad\quad+(\mu_2+n\delta)P_{n+1,1},\\
&[(m-n+Y)\beta\lambda_2+\mu_2+(n-1)\delta]P_{n,2}\\
&\qquad=(1-\alpha)(m-n+Y+1)\beta\lambda_1P_{n-1,1}\\
&\qquad\quad+(1-\alpha)(m-n+Y+1)\beta\lambda_2P_{n-1,2}+(\mu_2+n\delta)P_{n+1,2},
\end{aligned}\tag{2.12}$$

- for $n = k_2 - 1$,

$$\begin{aligned}
&[(m-k_2+Y+1)\beta\lambda_1+\mu_2+(k_2-2)\delta]P_{K_2-1,1}\\
&\qquad=\alpha(m-k_2+Y+2)\beta\lambda_1P_{k_2-2,1}+\alpha(m-k_2+Y+2)\beta\lambda_2P_{k_2-2,2}\\
&\qquad\quad+(\mu+(k_2-1)\delta)P_{k_2,1},\\
&[(m-k_2+Y+1)\beta\lambda_2+\mu_2+(k_2-2)\delta]P_{k_2-1,2}\\
&\qquad=(1-\alpha)(m-k_2+Y+2)\beta\lambda_1P_{k_2-2,1}+(1-\alpha)(m-k_2+Y+2)\beta\lambda_2P_{k_2-2,2}\\
&\qquad\quad+(\mu+(k_2-1)\delta)P_{k_2,2},
\end{aligned}\tag{2.13}$$

- for $k_2 \le n < m+Y$,

$$\begin{aligned}
&[(m-n+Y)\beta\lambda_1+\mu+(n-1)\delta]P_{n1} \\
&\quad=\alpha(m-n+Y+1)\beta\lambda_1P_{n-1,1}+\alpha(m-n+Y+1)\beta\lambda_2P_{n-1,2} \\
&\qquad+(\mu+n\delta)P_{n+1,1}, \\
&[(m-n+Y)\beta\lambda_2+\mu+(n-1)\delta]P_{n2} \\
&\quad=(1-\alpha)(m-n+Y+1)\beta\lambda_1P_{n-1,1} \\
&\qquad+(1-\alpha)(m-n+Y+1)\beta\lambda_2P_{n-1,2}+(\mu+n\delta)P_{n+1,2},
\end{aligned} \tag{2.14}$$

- for $n=m+Y$,

$$\begin{aligned}
&[\mu+(m+Y-1)\delta]P_{m+Y,1}=\alpha\beta\lambda_1P_{m+Y-1,1}+\alpha\beta\lambda_2P_{m+Y-1,2}, \\
&[\mu+(m+Y-1)\delta]P_{m+Y,2}=(1-\alpha)\beta\lambda_1P_{m+Y-1,1}+(1-\alpha)\beta\lambda_2P_{m+Y-1,2}.
\end{aligned} \tag{2.15}$$

Write

$$\rho_i=\frac{\lambda_1}{\mu_1},\quad \rho_i^*=\frac{\lambda_1}{\mu_2},\quad \bar{\rho}_i=\frac{\lambda_1}{\mu},\quad \gamma=\frac{\delta}{\mu_1},\quad \gamma^*=\frac{\delta}{\mu_2},\quad \bar{\gamma}=\frac{\delta}{\mu},\quad i=1,2. \tag{2.16}$$

And add every two equations in each step to simplify the required solution. We find

$$\begin{aligned}
&m\rho_1P_{0,1}=P_{1,1},\quad n=0, \\
&\quad(mB\rho_1+1)P_{1,1}+(mB\rho_2+1)P_{1,2} \\
&\qquad=(1+\gamma)P_{2,2}+m\rho_1P_{0,1}+m\rho_2P_{0,2}+(\gamma+1)P_{2,1},\quad n=1, \\
&\quad[(mB\rho_1+1+(n-1)\gamma)]P_{n,1}+[(mB\rho_2+1+(n-1)\gamma)]P_{n,2} \\
&\qquad=mB\rho_1P_{n-1,1}+(1+n\gamma)P_{n+1,1}+mB\rho_2P_{n-1,2+(1+n\gamma)P_{n+1,2}},\quad 2\le n\le Y, \\
&\quad[(m-n+Y)B\rho_1+1+(n-1)\gamma]P_{n,1}+[(m-n+Y)B\rho_2+1+(n-1)\gamma]P_{n,2} \\
&\qquad=(m-n+Y+1)B\rho_1P_{n-1,1}+(m-n+Y+1)B\rho_2P_{n-1,2} \\
&\qquad\quad+(1+n\gamma)P_{n+1,1}+(1+n\gamma)P_{n+1,2},\quad Y<n<k_1-1, \\
&\quad\Big[(m-k_1+1+Y)B\rho_1^*+\frac{\mu_1}{\mu_2}+(k_1-2)\gamma^*\Big]P_{k_1-1,1} \\
&\qquad\quad+\Big[(m-k_1+1+Y)B\rho_2^*+\frac{\mu_1}{\mu_2}+(k_1-2)\gamma^*\Big]P_{k_1-1,2} \\
&\qquad=(m-k_1+2+Y)B\rho_2^*P_{k_1-2,2}+(1+(k_1-1)\gamma^*)P_{k_1,1} \\
&\qquad\quad+(1+(k_1-1)\gamma^*)P_{k_1,2}+(m-k_1+2+Y)B\rho_1^*P_{k_1-2,1},\quad n=k_1-1,
\end{aligned}$$

$$
\begin{aligned}
&[(m-n+Y)B\rho_1^*+1+(n-1)\delta^*]P_{n,1}+[(m-n+Y)B\rho_2^*+1+(n-1)\gamma^*]P_{n,2}\\
&\quad=(m-n+Y+1)B\rho_1^*P_{n-1,1}+(1+n\gamma^*)P_{n+1,2}\\
&\qquad+(m-n+Y+1)B\rho_2^*P_{n-1,2}+(1+n\gamma^*)P_{n+1,1},\quad k_1\le n<k_2-1,\\
&\Big[(m-k_2+1+Y)B\bar\rho_1+\frac{\mu_2}{\mu}+(k_2-2)\bar\gamma\Big]P_{k_2-1,1}\\
&\qquad+\Big[(m-k_2+1+Y)B\bar\rho_2+\frac{\mu_2}{\mu}+(k_2-2)\bar\gamma\Big]P_{k_2-1,2}\\
&\quad=(m-k_2+2+Y)B\bar\rho_1P_{k_1-2,1}+(m-k_2+2+Y)B\bar\rho_2P_{k_2-2,2}\\
&\qquad+(1+(k_2-1)\bar\gamma)P_{k_2,1}+(1+(k_2-1)\bar\gamma)P_{k_2,2},\quad n=k_2-1,\\
&[(m-n+Y)B\bar\rho_1+1+(n-1)\bar\gamma]P_{n,1}+[(m-n+Y)B\bar\rho_2+1+(n-1)\bar\gamma]P_{n,2}\\
&\quad=(m-n+Y+1)B\bar\rho_1P_{n-1,1}+(m-n+Y+1)B\bar\rho_2P_{n-1,2}\\
&\qquad+(1+n\bar\gamma)P_{n+1,1}+(1+n\bar\gamma)P_{n+1,2},\quad k_2\le n<m+Y,\\
&[1+(m+y-1)\bar\gamma]P_{m+Y,1}+[1+(m+Y-1)\bar\gamma]P_{m+Y,2}\\
&\quad=B\bar\rho_1P_{m+Y-1,1}+B\bar\rho_2P_{m+Y-1,2},\quad n=Y+m.
\end{aligned}
\tag{2.17}
$$

These are $(m+Y+1)$ equations in the unknowns $P_{n,j}$. To solve them for $P_{n,j}$, we need the formula which gives the sum of the probabilities at every branch. We introduce the formula in the following lemma.

**Lemma 2.1.** *For the truncated hyperexponential machine interference queue $H_2/M/1/m+Y/m+Y$ with the concepts of balking, reneging, state-dependent, spares, and an additional server for longer queues, there is*

$$
\begin{aligned}
\sum_{n=0}^{m+Y}P_{n,1}=&\frac{1}{mB\{\alpha\lambda_2+(1-\alpha)\lambda_1\}}\\
&\times\Big[\alpha m\lambda_2\{\beta+(1-\beta)P_{0,2}\}+\alpha\beta\lambda_2Y\sum_{n=Y}^{m+Y}P_{n,2}\\
&\quad-m\lambda_1(1-\beta)(1-\alpha)P_{0,1}-\alpha\beta\lambda_2\sum_{n=Y}^{m+Y}nP_{n,2}\\
&\quad+\beta\lambda_1(1-\alpha)\sum_{n=Y}^{m+Y}nP_{n,1}-\beta\lambda_1Y(1-\alpha)\sum_{n=Y}^{m+Y}P_{n,1}\Big]=\eta,\\
&\sum_{n=0}^{m+Y}P_{n,2}=1-\eta.
\end{aligned}
\tag{2.18}
$$

**Proof.** Adding either the nine first relations or the nine second relations, we get

$$m\beta\lambda_1(1-\alpha)\sum_{n=0}^{m+Y}P_{n,1}-\alpha m\beta\lambda_2\sum_{n=0}^{m+Y}P_{n,2}$$
$$=\alpha m\lambda_2(1-\beta)P_{0,2}-m\lambda_1(1-\beta)(1-\alpha)P_{0,1}+\alpha\beta\lambda_2 Y\sum_{n=Y}^{m+Y}P_{n,2}-\alpha\beta\lambda_2\sum_{n=Y}^{m+Y}nP_{n,2}$$
$$+\beta\lambda_1(1-\alpha)\sum_{n=Y}^{m+Y}nP_{n,1}-\beta\lambda_1 Y(1-\alpha)\sum_{n=Y}^{m+Y}P_{n,1}. \tag{2.19}$$

But

$$\sum_{n=Y}^{m+Y}(P_{n,1}+P_{n,2})=1. \tag{2.20}$$

Multiply (2.20) by $\alpha m\lambda_2 B$ then add to (2.19), we obtain relation (2.18), that is, the concepts of reneging, state-dependent, and an additional server for longer queues are not affecting the results of the lemma. □

Now to solve the set of equations (2.17), first we need to solve $P_{n,1}$ (the first branch probabilities) in terms of $P_{n,2}$ (the second branch probabilities) by the following method:

We arrange the coefficients of equations (2.17) in a square matrix of order $(m+Y+1)$ after replacing (2.18) instead of the last equation (2.17). By doing elementary row-operations. We get the following solution's formula:

From the last row $i=1$, we get

$$P_{m+Y,1}=\frac{\alpha}{1-\alpha}P_{m+Y,2}=A,\quad n=m+Y, \tag{2.21}$$

where $i=m+Y+1-n$.

At $i=2$,

$$B\bar{\rho}_1P_{m+Y-1,1}-[1+(m+Y-1)\bar{y}]P_{m+Y,1}=[1+(m+Y-1)\bar{y}]P_{m+Y,2}-B\bar{\rho}_1P_{m+Y-1,2},$$
$$B\bar{\rho}_1P_{m+Y-1,1}-[1+(m+Y-1)\bar{y}]A+[1+(m+Y-1)\bar{y}]P_{m+Y,2}-B\bar{\rho}_1P_{m+Y-1,2}=B, \tag{2.22}$$

$$P_{m+Y-1,1}=\frac{B}{\beta\bar{\rho}_1},\quad n=m+Y-1. \tag{2.23}$$

At $i=3$,

$$P_{m+Y-2,1}=\frac{1}{2\beta\rho_1}\Bigg[\{1+(m+Y-2)\bar{y}\}P_{m+Y-1,2}$$
$$-2\beta\bar{\rho}_2P_{m+Y-2,2}+\frac{1+(m+Y-2)\bar{y}}{\beta\bar{\rho}_1}B\Bigg],\quad n=m+Y-2. \tag{2.24}$$

And so on.

Then we can find the following formula for the solutions to the rows $4 \le i \le m + Y - (k_2 - 2)$, that is, $k_2 - 1 \le n \le m + Y - 3$:

$$\begin{aligned} p_{n,1} = {} & \frac{1}{(m+Y-n)B\bar{\rho}_1} \\ & \times \Bigg[ B \frac{(1+n\bar{\gamma})_{(m+Y-n-1)\bar{\gamma}}}{(m+Y-n-1)!(B\bar{\rho}_1)^{(m+Y-n-1)}} \\ & \quad - B\bar{\rho}_2 \sum_{r=1}^{m+Y-n-1} \frac{(r+1)(1+n\gamma)_{(m+Y-n-r-1)\bar{\gamma}}}{[m+Y-n-1]_{(m+Y-n-r-1)}(B\bar{\rho}_1)^{(m+Y-n-r-1)}} P_{(m+Y-r-1,2)} \\ & \quad + \sum_{r=1}^{m+Y-n-1} \frac{(1+n\gamma)_{(m+Y-n-r-1)\bar{\gamma}}}{[m+Y-n-1]_{(m+Y-n-r-1)}(B\bar{\rho}_1)^{(m+Y-n-r-1)}} P_{(m+Y-r,2)} \Bigg], \\ & \qquad k_2 - 1 \le n \le m + Y - 3. \end{aligned} \tag{2.25}$$

In particular for $n = k_2 - 1$, we find the value of $P_{k_2-1,1}$

$$\begin{aligned} & (m-k_2+Y+1)B\bar{\rho}_1 P_{k_2-1,1} \\ & \quad = T_1 = B \frac{(1+(k_2-1)\bar{\gamma})_{(m+Y-k_2)\bar{\gamma}}}{(m+Y-k_2)!(B\bar{\rho}_1)^{(m+Y-k_2)}} \\ & \qquad - B\bar{\rho}_2 \sum_{r=1}^{m+Y-k_2} \frac{(r+1)(1+(k_2-1)\bar{\gamma})_{(m+Y-k_2-r)\bar{\gamma}}}{[m+Y-k_2]_{(m+Y-k_2-r)}(B\bar{\rho}_1)^{(m+Y-k_2-r)}} P_{(m+Y-r-1,2)} \\ & \qquad + \sum_{r=1}^{m+Y-k_2} \frac{(1+(k_2-1)\bar{\gamma})_{(m+Y-k_2-r+1)\bar{\gamma}}}{[m+Y-k_2]_{(m+Y-k_2-r)}(B\bar{\rho}_1)^{(m+Y-k_2-r)}} P_{(m+Y-r,2)}. \end{aligned} \tag{2.26}$$

Now from row $i = m + Y - (k_2 - 3)$, the value of $P_{k_2-2,1}$ is

$$\begin{aligned} p_{k_2-2,1}(m-k_1+Y+2)B\rho_1^* = {} & \{1+(k_2-2)\gamma^*\}P_{k_2-1,2} - (m-k_2+Y+2)B\rho_2^* P_{k_2-2,2} \\ & + T_1 \frac{1+(k_2-2)\gamma^*}{(m-k_2+Y+1)B\rho_1} = D. \end{aligned} \tag{2.27}$$

Then

$$P_{k_2-2,1} = \frac{D}{(m-k_1+Y+2)B\rho_1^*}. \tag{2.28}$$

From the row $i = m + Y - (k_2 - 4)$, the value of $P_{k_2-3,1}$ is

$$\begin{aligned} P_{k_2-3,1} = \frac{1}{(m-k_2+Y+3)B\rho_1^*} \Bigg[ & \{1+(k_2-3)\gamma^*\}P_{k_2-2,2} - (m-k_2+Y+3)B\rho_2^* P_{k_2-3,2} \\ & + \frac{1+(k_2-3)\gamma^*}{(m-k_2+Y+2)B\rho_1^*} D \Bigg], \end{aligned} \tag{2.29}$$

then we can obtain the formula for the solutions to the rows $m+Y-(k_2-5) \le i \le m+Y-(k_1-2)$, that is, $k_1-1 \le n \le k_2-4$ as

$$\begin{aligned} P_{n,1} &= \frac{1}{(m+Y-n)B\rho_1^*} \\ &\quad \times \Bigg[ D\frac{(1+n\gamma^*)_{(k_1-n-2)\gamma^*}}{[m+Y-n-1]_{(k_2-n-2)}(B\rho_1^*)^{(k_2-n-2)}} \\ &\quad - B\rho_2^* \sum_{r-1}^{k_2-n-2} \frac{(m-k_2+Y+r+2)(1+n\gamma^*)_{(k_2-n-r-2)\gamma^*}}{[m+Y-n-1]_{(k_2-n-r-2)}(B\rho_1)^{(k_2-n-r-2)}} P_{k_2-r-2,2} \\ &\quad + \sum_{r-1}^{k_2-n-2} \frac{(1+n\gamma^*)_{(k_1-n-r-1)\gamma^*}}{[m+Y-n-1]_{(k_2-n-r-2)}(B\rho_1^*)^{(k_2-n-r-2)}} P_{k_2-r-1,2} \Bigg]. \end{aligned} \tag{2.30}$$

In particular, for $n = k_1-1$, we find the value of $P_{k_2-1,1}$

$$\begin{aligned} &(m-k_1+Y+1)B\rho_1^* P_{k_1-1,1} \\ &\quad = D\frac{(1+(k_1-1)\gamma^*)(k_2-k_1-1)\gamma^*}{[m+Y-k_1]_{(k_2-k_1-1)}(B\rho_1^*)^{(k_2-k_1-1)}} \\ &\quad - B\rho_2^* \sum_{r=1}^{k_2-k_1-1} \frac{(m-k_2+Y+r+2)(1+(k_1-1)\gamma^*)_{(k_2-k_1-r-1)\gamma^*}}{[m+Y-k_1]_{(k_2-k_1-r-1)}(B\rho_1^*)^{(k_2-k_1-r-1)}} P_{k_2-r-2,2} \\ &\quad + \sum_{r=1}^{k_2-k_1-1} \frac{(1+(k_1-1)\gamma^*)_{(k_2-k_1-r)\gamma^*}}{[m+Y-k_1]_{(k_2-k_1-r-1)}(B\rho_1^*)^{(k_2+k_1-r-1)}} P_{k_2-r-1,2} = T_2. \end{aligned} \tag{2.31}$$

Now from row $i = m+Y-(k_1-3)$, the value of $P_{k_1-2,1}$ is

$$P_{k_1-2,1} = \frac{D_1}{(m-k_1+Y+2)B\rho_1}, \tag{2.32}$$

where

$$D_1 = \{1+(k_1-2)\gamma\}P_{k_1-1,2} - (m-k_1+Y+2)B\rho_2 P_{k_1-2,2} + T_2 \frac{1+(k_1-2)\gamma}{(m-k_1+Y+1)B\rho_1^*}. \tag{2.33}$$

From row $i = m+Y-(k_1-4)$, the value of $P_{k_1-3,1}$ is

$$\begin{aligned} P_{k_1-3,1} = \frac{1}{(m-k_1+Y+3)B\rho_1} \Bigg[ &\{1+(k_1-3)\gamma\}P_{k_1-2,2} - (m-k_1+Y+3)B\rho_2 P_{k_1-3,2} \\ &+ \frac{1+(k_1-3)\gamma}{(m-k_1+Y+2)B\rho_1} D_1 \Bigg]. \end{aligned} \tag{2.34}$$

Then we can obtain the formula for the solutions for the rows $m+Y-(k_1-5) \le i \le m$, that is, $Y+1 \le n \le k_1-4$ as

$$\begin{aligned} P_{n,1} = \frac{1}{B\rho_1}\Bigg[ & D_1 \frac{(1+n\gamma)_{(k_1-n-2)\gamma}}{[m+Y-n-1]_{(k_1-n-2)}(\beta\rho_1)^{(k_1-n-2)}} \\ & - B\rho_2 \sum_{r-1}^{k_1-n-2} \frac{(m-k_1+Y+r+2)(1+n\gamma)_{(k_1-n-r-2)\gamma}}{[m+Y-n-1]_{(k_1-n-r-2)}(B\rho_1)^{(k_1-n-r-2)}} P_{k_1-r-2,2} \\ & + \sum_{r-1}^{k_1-n-2} \frac{(1+n\gamma)_{(k_1-n-r-1)\gamma}}{[m+Y-n-1]_{(k_1-n-r-2)}(B\rho_1)^{(k_1-n-r-2)}} P_{k_1-r-1,2} \Bigg]. \end{aligned} \tag{2.35}$$

In particular for $n = Y+1$, we find the value of $P_{Y+1,1}$

$$\begin{aligned} (m-1)B\rho_1 P_{Y+1,1} = {} & D_1 \frac{(1+(Y+1)\gamma)_{(k_1-Y-3)\gamma}}{[m-2]_{(k_1-Y-3)}(B\rho_1)^{(k_1-Y-3)}} \\ & - B\rho_2 \sum_{r=1}^{k_1-Y-3} \frac{(m-k_1+Y+r+2)(1+(Y+1)\gamma)_{(k_1-Y_1-r-3)\gamma}}{[m-2]_{(k_1-Y_1-r-3)}(B\rho_1)^{(k_1-Y-r-3)}} P_{k_1-r-2,2} \\ & + \sum_{r=1}^{k_1-Y-3} \frac{(1+(Y+1)\gamma)_{(k_1-Y-r-2)\gamma}}{[m-2]_{(k_1-Y-r-3)}(B\rho_1)^{(k_1-Y-r-3)}} P_{k_1-r-1,2} = T_3. \end{aligned} \tag{2.36}$$

Now from row $i = m+1$, the value of $P_{Y,1}$ is

$$P_{Y,1} = \frac{D_2}{mB\rho_1}, \tag{2.37}$$

where

$$D_2 = P_{Y,1} m B\rho_1 = (1+Y\gamma)P_{Y+1,2} - mB\rho_2 P_{Y,2} + \frac{(1+Y\gamma)}{(m-1)B\rho_1} T_3. \tag{2.38}$$

From row $i = m+2$, the value of $P_{Y-1,1}$ is

$$P_{Y-1,1} = \frac{1}{mB\rho_1}\left[\{1+(Y-1)\gamma\}P_{Y,2} - mB\rho_2 P_{Y-1,2} + \frac{\{1+(Y-1)\gamma\}}{mB\rho_1} D_2\right]. \tag{2.39}$$

Then we can obtain the formula for the solutions at the rows $m+3 \le i \le m+Y$, that is, $1 \le n \le Y-2$ as

$$\begin{aligned} P_{n,1} = \frac{1}{mB\rho_1}\Bigg[ & D_2 \frac{(1+n\gamma)(Y-n)\gamma}{(m\beta\rho_1)^{(Y-n)}} - mBP_2 \sum_{r-1}^{Y-n} \frac{(1+n\gamma)(Y-n-r)\gamma}{(mB\rho_1)^{(Y-n-r)}} PY-r,2 \\ & + \sum_{r-1}^{Y-n} \frac{(1+n\gamma)(Y-n-r)\gamma}{(mB\rho_1)^{(Y-n-r)}} PY-r+1,2 \Bigg]. \end{aligned} \tag{2.40}$$

In particular for $n = 1$, we find the value of $P_{1,1}$.

At the first row $i = m+Y+1$ we find the value of $P_{0,1}$ as

$$P_{0,1} = \frac{1}{m\rho_1} P_{1,1}, \tag{2.41}$$

where

$$\begin{aligned} [L]_\alpha &= L\cdot(L-1)\cdot(L-2)\cdots(L-\alpha-1), \\ (L)_\alpha &= L\cdot(L+1)\cdot(L+2)\cdots(L+\alpha+1). \end{aligned} \tag{2.42}$$

Thus we succeed to deduce the first branch probabilities $P_{n,1}$ in terms of the second branch probabilities $P_{n,2}$.

Substituting relations from (2.21), (2.23), (2.24), (2.25), (2.28), (2.29), (2.30), (2.32), (2.34), (2.35), (2.37), (2.39), (2.40), and (2.41) in the second relations of (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) and by using the relation of (2.18), the second branch probabilities $P_{n,2}$ can be obtained in an explicit form. And so the first branch probabilities $P_{n,1}$ can be obtained in an explicit form too.

## 3. Measures of effectiveness.

The expected number of units in the system and in the queue are

$$L = \sum_{n=0}^{m+Y} n\left(P_{n,1}+P_{n,2}\right), \qquad L_q = \sum_{n=1}^{m+Y} (n-1)\left(P_{n,1}+P_{n,2}\right). \tag{3.1}$$

Also, the probability that there are no units in the system is

$$P_0 = P_{0,1} + P_{0,2}. \tag{3.2}$$

## 4. Particular cases

**Case 4.1** (the queue $H2/M/1/2/2(\beta,\delta)$). Let $Y = 0$ and $k_1 = k_2 = m$, (i.e., $\mu_1 = \mu_2 = \mu$ and $\mu_3 = 0$) in the above relations, we get

$$\begin{aligned} P_{0,1} &= \frac{\rho_2\{1+\gamma-a\beta\rho_2(1-\alpha)\}}{\rho_1(ab+d)}, \qquad P_{0,2} = \frac{a\beta\rho_2(1-\alpha)}{ab+d}, \\ P_{1,1} &= \frac{2\rho_2\{1+\gamma-a\beta\rho_2(1-\alpha)\}}{ab+d}, \qquad P_{1,2} = \frac{2a\beta\rho_1\rho_2(1-\alpha)}{ab+d}, \\ P_{2,1} &= \frac{2\alpha\beta\rho_1\rho_2}{ab+d}, \qquad P_{2,2} = \frac{2\beta\rho_1\rho_2(1-\alpha)}{ab+d}, \\ P_0 &= \frac{1}{\rho_1(ab+d)}\{\rho_2(1+\gamma)+a\beta(1-\alpha)(\rho_1^2-\rho_2^2)\}, \\ L &= \frac{2\rho_2}{ab+d}\{1+\gamma+a\beta(1-\alpha)(\rho_1-\rho_2)+2\beta\rho_1\}, \qquad L_q = \frac{2\beta\rho_1\rho_2}{ab+d}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} a &= \frac{(1+\gamma)(1+\beta\rho_1)}{\beta\{\beta\rho_1\rho_2+\alpha\rho_1+(1-\alpha)\rho_2\}}, \\ b &= \beta(2\rho_2+1)\{\alpha\rho_2+(1-\alpha)\rho_1\} - \beta\rho_2^2\{4\alpha(1-\beta)+3\beta-2\}, \\ d &= 2\beta\rho_1\rho_2\{1-2\alpha(1-\beta)\} - \rho_2(1+\gamma)(2-3\beta). \end{aligned} \tag{4.2}$$

Also, the machine availability is

$$\text{M.A.} = \frac{m-L}{m} = 1 - \frac{\rho_2}{ab+d}\left[1+\gamma+a\beta(1-\alpha)(\rho_1-\rho_2)+2\beta\rho_1\right]. \tag{4.3}$$

And the operative efficiency is

$$\text{O.E.} = 1-\rho_0 = 1 - \frac{1}{\rho_1(ab+d)}\left\{\rho_2(1+\gamma)+a\beta(1-\alpha)(\rho_1^2-\rho_2^2)\right\}. \tag{4.4}$$

**Case 4.2** (the queue $M/M/1/m+y/m+y$)**.** Let

$$\begin{gathered}\beta = 1, \qquad \delta = 0, \quad (\text{i.e., } \gamma = 0), \qquad \alpha = \frac{1}{2}\left(\text{i.e., } \rho_1 = \rho_2 = \rho = \frac{\lambda}{\mu}\right), \\ k_1 = k_2 = m \quad (\text{i.e., } \mu_1 = \mu_2 = \mu,\ \mu_3 = 0).\end{gathered} \tag{4.5}$$

We obtain the results of Harris [5] when $c = 1$.

**Case 4.3** (the queue $M/M/1/2/2$)**.** Let

$$\begin{gathered}\beta = 1, \qquad \delta = 0 \quad (\text{i.e., } \gamma = 0), \qquad Y = 0, \\ \alpha = \frac{1}{2}\left(\text{i.e., } \rho_1 = \rho_2 = \rho = \frac{\lambda}{\mu}\right), \qquad k_1 = k_2 = m = 2.\end{gathered} \tag{4.6}$$

We get

$$\begin{gathered}P_0 = (2\rho^2+2\rho+1)^{-1}, \qquad L = 2\rho(2\rho+1)(2\rho^2+2\rho+1)^{-1}, \\ L_q = 2\rho^2(2\rho^2+2\rho+1)^{-1}.\end{gathered} \tag{4.7}$$

These results agree with that obtained by Harris [5] and Al-Seedy [4].

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Ragab Omarah Al-Seedy: Department of Mathematics, Teacher's College in Al-Ahsa, Saudi Arabia

Fawziah M. Al-Ibraheem: Department of Mathematics, Science College for Girls in Dammam, Saudi Arabia