IJMMS 27:2 (2001) 77–82
PII. S0161171201005907
http://ijmms.hindawi.com

ON IDEALS OF IMPLICATIVE SEMIGROUPS

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(Received 2 October 2000)

ABSTRACT. We introduce the notion of ideals in implicative semigroups, and then state the characterizations of the ideals.

2000 Mathematics Subject Classification. 20M12, 06F05, 06A06, 06A12.

1. Introduction. The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [3]. The first is a generalization of implicative semilattice (see Nemitz [6] and Blyth [2]) and has a close relation with implication in mathematical logic and set theoretic difference (see Birkhoff [1] and Curry [4]). For the general development of implicative semilattice theory the ordered filters play an important role which is shown by Nemitz [6]. Motivated by this, Chan and Shum [3] established some elementary properties, and constructed quotient structure of implicative semigroups via ordered filters. Jun et al. [5] discussed ordered filters of implicative semigroups. In this paper, we introduce the notion of ideals in implicative semigroups. By introducing special subsets of an implicative semigroups, we provide a condition for the special subset to be an ideal. We establish two characterizations of ideals.

2. Preliminaries. We recall some definitions and results. By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*) we mean a set S with a partial ordering $\le$ and a binary operation $\cdot$ such that for all $x,y,z \in S$, we have

(1) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$,

(2) $x \le y$ implies $x\cdot z \le y\cdot z$ and $z\cdot x \le z\cdot y$,

(3) $x\cdot y \le x$ and $x\cdot y \le y$.

An n.p.o. semigroup $(S;\le,\cdot)$ is said to be *implicative* if there is an additional binary operation $*: S\times S \to S$ such that for any elements x,y,z of S,

(4) $z \le x*y$ if and only if $z\cdot x \le y$.

The operation $*$ is called *implication*. From now on, an implicative n.p.o. semigroup is simply called an *implicative semigroup*.

An implicative semigroup $(S;\le,\cdot,*)$ is said to be *commutative* if it satisfies

(5) $x\cdot y = y\cdot x$ for all $x,y\in S$, that is, $(S,\cdot)$ is a commutative semigroup.

In any implicative semigroup $(S;\le,\cdot,*)$, $x*x = y*y$ for every $x,y\in S$ and this element is the greatest element, written 1, of $(S,\le)$.

PROPOSITION 2.1 (see [3, Theorem 1.4]). *Let S be an implicative semigroup. Then for every $x,y,z\in S$, the following hold:*

(6) $x\le 1$, $x*x = 1$, $x = 1*x$,

(7) $x \le y*(x\cdot y)$,

(8) $x \le x * x^2$,
(9) $x \le y * x$,
(10) *if* $x \le y$ *then* $x * z \ge y * z$ *and* $z * x \le z * y$,
(11) $x \le y$ *if and only if* $x * y = 1$,
(12) $x * (y * z) = (x \cdot y) * z$,
(13) *if* S *is commutative then* $x * y \le (s \cdot x) * (s \cdot y)$ *for all* s *in* S.

Now we note important elementary properties of a commutative implicative semigroup, which follows from (5), (6), and (12).

Observation 2.2. If S is a commutative implicative semigroup, then for any $x, y, z \in S$,
(14) $x * (y * z) = y * (x * z)$,
(15) $y * z \le (x * y) * (x * z)$,
(16) $x \le (x * y) * y$.

3. Ideals of implicative semigroups. In what follows let S denote an implicative semigroup unless otherwise specified. We begin by defining the notion of ideals of S.

Definition 3.1. A subset I of S is called an *ideal* of S if
(I1) $x \in S$ and $a \in I$ imply $x * a \in I$,
(I2) $x \in S$ and $a, b \in I$ imply $(a * (b * x)) * x \in I$.

Example 3.2. Consider an implicative semigroup $S := \{1, a, b, c, d, 0\}$ with Cayley tables (Tables 3.1 and 3.2) and Hasse diagram (Figure 3.1) as follows:

Table 3.1

$\cdot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

Table 3.2

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

We know that $\{1, a, b\}$ is an ideal of S, but $\{1, a\}$ is not an ideal of S, since $(a * (a * b)) * b = b \notin \{1, a\}$.

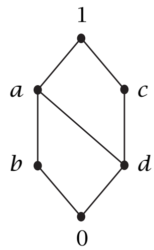

FIGURE 3.1

LEMMA 3.3. *Every ideal of S contains* 1.

PROOF. The proof follows from (6) and (I1). □

LEMMA 3.4. *If I is an ideal of S, then $(a*x)*x \in I$ for all $a \in I$ and $x \in S$.*

PROOF. The proof follows by taking $b = a$ and $a = 1$ in (I2). □

COROLLARY 3.5. *Let I be an ideal of S. If $a \in I$ and $a \le x$, then $x \in I$.*

PROOF. Let $a \in I$ and $x \in S$ be such that $a \le x$. Using (6) and Lemma 3.4, we have $x = 1*x = (a*x)*x \in I$. This completes the proof. □

LEMMA 3.6. *Let I be a subset of S such that*
(I3) $1 \in I$,
(I4) $x*(y*z) \in I$ *and* $y \in I$ *imply* $x*z \in I$ *for all* $x,y,z \in S$. *If* $a \in I$ *and* $a \le x$, *then* $x \in I$.

PROOF. Let $a \in I$ and $x \in S$ be such that $a \le x$. Then $x*(a*1) = x*1 = 1 \in I$ by (6) and (I3), and so $x = x*1 \in I$ by (I4). This completes the proof. □

The following is a characterization of ideals.

THEOREM 3.7. *Let S be a commutative implicative semigroup. A subset I of S is an ideal of S if and only if it satisfies conditions (I3) and (I4).*

PROOF. Let I be an ideal of S. Then $1 \in I$ by Lemma 3.3. Let $x,y,z \in S$ be such that $x*(y*z) \in I$ and $y \in I$. Using Lemma 3.4, we get $(y*z)*z \in I$. It follows from (6), (15), and (I2) that

$$x*z = 1*(x*z) = \big(((y*z)*z)*((x*(y*z))*(x*z))\big)*(x*z) \in I. \tag{3.1}$$

Conversely, assume that I satisfies conditions (I3) and (I4). Let $x \in S$ and $a \in I$. Since $x*(a*a) = x*1 = 1 \in I$ by (I3), it follows from (I4) that $x*a \in I$, that is, (I1) holds. Since $(a*x)*(a*x) = 1 \in I$, we have $(a*x)*x \in I$ by (I4). Note from (15) that

$$((a*x)*x)*((b*(a*x))*(b*x)) = 1, \tag{3.2}$$

that is,

$$(a*x)*x \le (b*(a*x))*(b*x) \tag{3.3}$$

for all $b \in I$. Thus, by Lemma 3.6, we have $(b*(a*x))*(b*x) \in I$. Using (I4), we conclude that $(b*(a*x))*x \in I$ which proves (I2). Hence I is an ideal of S. □

For any $u, v \in S$, consider a set

$$S(u,v) = \{z \in S \mid u * (v * z) = 1\}. \tag{3.4}$$

In Example 3.2, the set $S(1,a) = \{1,a\}$ is not an ideal of S. Hence we know that $S(u,v)$ may not be an ideal of S in general.

Theorem 3.8. *Let S satisfy the left self-distributive law under $*$, that is, $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in S$. For any $u, v \in S$, the set $S(u,v)$ is an ideal of S.*

Proof. Let $x \in S$ and $a, b \in S(u,v)$. Then

$$\begin{aligned} u * (v * (x * a)) &= (u * (v * x)) * (u * (v * a)) = (u * (v * x)) * 1 = 1, \\ u * (v * ((a * (b * x)) * x)) &= (u * (v * (a * (b * x)))) * (u * (v * x)) \\ &= ((u * (v * a)) * (u * (v * (b * x)))) * (u * (v * x)) \\ &= (1 * ((u * (v * b)) * (u * (v * x)))) * (u * (v * x)) \\ &= (u * (v * x)) * (u * (v * x)) = 1. \end{aligned} \tag{3.5}$$

Hence $x * a \in S(u,v)$ and $(a * (b * x)) * x \in S(u,v)$, which shows that $S(u,v)$ is an ideal of S. □

Lemma 3.9. *Let S be an implicative semigroup. If $y \in S$ satisfies $y * z = 1$ for all $z \in S$, then $S(x,y) = S = S(y,x)$ for all $x \in S$.*

Proof. The proof is straightforward. □

Example 3.10. Let $S := \{1, a, b, c, d\}$ be an implicative semigroup with Cayley tables (Tables 3.3 and 3.4) and Hasse diagram (Figure 3.2) as follows:

Table 3.3

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

Table 3.4

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

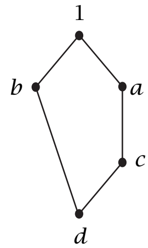

FIGURE 3.2

It is easy to check that S satisfies the left self-distributive law under $*$, that is, $x*(y*z) = (x*y)*(x*z)$ for all $x,y,z \in S$. By Lemma 3.9 we have $S(x,d) = S(d,x) = S$ for all $x \in S$. Furthermore we know that $S(1,1) = \{1\}$, $S(1,a) = S(a,1) = S(a,a) = S(a,b) = \{1,a\}$, $S(1,b) = S(b,1) = S(b,b) = \{1,b\}$, $S(1,c) = S(a,c) = S(c,1) = S(c,a) = S(c,c) = \{1,a,c\}$, $S(b,a) = \{1,a,b\}$, and $S(c,b) = S$ are ideals of S.

Using the set $S(u,v)$, we describe a characterization of ideals.

Theorem 3.11. *Let S be a commutative implicative semigroup and let I be a nonempty subset of S. Then I is an ideal of S if and only if $S(u,v) \subseteq I$ for all $u,v \in I$.*

Proof. Assume that I is an ideal of S and let $u,v \in I$. If $z \in S(u,v)$, then $u*(v*z) = 1 \in I$ and so $z = 1*z = (u*(v*z))*z \in I$ by (I2). Hence $S(u,v) \subseteq I$.

Conversely, suppose that $S(u,v) \subseteq I$ for all $u,v \in I$. Note that $1 \in S(u,v) \subseteq I$. Let $x,y,z \in S$ be such that $x*(y*z) \in I$ and $y \in I$. Since

$$(x*(y*z))*(y*(x*z)) = (y*(x*z))*(y*(x*z)) = 1, \tag{3.6}$$

we have $x*z \in S(x*(y*z),y) \subseteq I$. Applying Theorem 3.7, we conclude that I is an ideal of S. □

Theorem 3.12. *Let S be a commutative implicative semigroup. If I is an ideal of S, then*

$$I = \cup_{u,v\in I} S(u,v). \tag{3.7}$$

Proof. Let I be an ideal of S and let $x \in I$. Obviously, $x \in S(x,1)$ and so

$$I \subseteq \cup_{x\in I} S(x,1) \subseteq \cup_{u,v\in I} S(u,v). \tag{3.8}$$

Now let $y \in \cup_{u,v\in I} S(u,v)$. Then there exist $a,b \in I$ such that $y \in S(a,b)$. It follows from Theorem 3.11 that $y \in I$. Hence $\cup_{u,v\in I} S(u,v) \subseteq I$. This completes the proof. □

Corollary 3.13. *If I is an ideal of a commutative implicative semigroup S, then*

$$I = \cup_{w\in I} S(w,1). \tag{3.9}$$

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