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SUBGROUPS OF FINITE INDEX IN AN ADDITIVE GROUP OF A RING

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Abstract. If K is an infinite field and $G \subseteq K$ is a subgroup of finite index in an additive group, then $K^* = G^*G^{*-1}$ where G^* denotes the set of all invertible elements in G and G^{*-1} denotes all inverses of elements of G^*. Similar results hold for various fields, division rings and rings.

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1. Introduction. Let R be a ring (not necessarily commutative) with a unit element 1 and R^* denotes the multiplicative group of invertible elements of R. In [9] Leep and Shapiro proved that if G is a subgroup of index 3 in the multiplicative group F^*, then $G+G=F$. In [2] Berrizbetia proved that if F is a field and $G \subseteq F^*$ is a subgroup of finite index n, then there is a positive integer m, that depends on n, so that if $\operatorname{char} F = 0$ or $\operatorname{char} F \geq m$, then $G-G=F$. In [1] Bergelson and Shapiro proved that, for various ring R, if G is a subgroup of finite index of R^*, then $G-G=R$. In [14] Turnwald proved that if G is a subgroup of finite index n in the multiplicative group of a division ring F then $G-G=F$ or $|F| < (n+1)^4+4n$, and if $|F| > (n-1)^2$ and -1 is a sum of elements of G then every element of F has this property; the bound $(n-1)^2$ is optimal for infinitely many n. The theories which have important role in studying of the above were Ramsey theory, measure theory and number theory, (cf. [4, 7, 15]). Furthermore in [1] the roles of multiplication and addition were switched, and it was shown that

Proposition 1.1 (see [1, Proposition 2.14]). *Let K be an infinite field and $G \subseteq K$ a subgroup of finite index in additive group. Then $G^*G^{*-1} = K^*$ where $G^* = G\setminus\{0\}$; that is, for every $c \in K^*$ there exist $g_1, g_2 \in G$ such that $c = g_1/g_2$.*

Corollary 1.2. *If D is an infinite division ring then the above result is satisfied.*

In this paper, the roles of multiplication and addition are switched and it is shown that Proposition 1.1 and Corollary 1.2 hold for various fields, division rings and rings.

Now let $G \subseteq R$ be a subgroup of finite index in an additive group, G^* be the set of all invertible elements in G, $G^{*-1} = \{g^{-1} : g \in G^*\}$ and $G^*G^{*-1} = \{g_1 g_2{}^{-1} : g_1, g_2 \in G^*\}$.

2. G^*G^{*-1}-ring. Let K be a ring or field and $G \subseteq K$ be a subgroup of finite index in an additive group, then it is not necessary that $G^*G^{*-1} = R^*$ or even G^*, G^{*-1}, and G^*G^{*-1} have group structure. Note the following statements.

(i) Let $F = F_{p^2}$, and $G = F_p$, then $G^* = F_p^*$ and $G^*G^{*-1} = F_p^* \neq F^*$.

(ii) Let α be a root of the polynomial x^3+x+1 over the splitting field $Z_2(\alpha) = F_8$, where F_8 has 8 elements. Put $G = \{0, \alpha, \alpha^2, \alpha+\alpha^2\}$ so $G^* = \{\alpha, \alpha^2, \alpha+\alpha^2\}$. It is clear that G^* is not a group, because G^* does not contain the unit element 1. But G^*G^{*-1} is a subgroup of the multiplicative group F_8^*. Furthermore, $G^*G^{*-1} = F_8^*$.

(iii) Let β be a root of the polynomial $x^4 - x + 1$ over the splitting field $Z_2(\beta) = F_{16}$. Put $G = \{0, 1, \beta, 1+\beta\}$, so $G^* = \{1, \beta, 1+\beta\}$, $G^{*-1} = \{1, \beta^3+1, \beta^3+\beta^2+\beta\}$ and therefore $G^*G^{*-1} = \{1, \beta^3+1, \beta^3+\beta^2+\beta, \beta, 1+\beta+\beta^2+\beta^3, 1+\beta, \beta^3\}$. It is clear that G^* is not a group but G^*G^{*-1} is a proper subgroup of F_{16}^*.

(iv) Let $R = \mathbb{Z}/n\mathbb{Z}$ where n is not a prime number. If $G \subset R$ is a proper subgroup in an additive group then it is clear that $G^* = \varnothing$ and $G^*G^{*-1} = \varnothing \neq R^*$.

(v) Let $S = \mathbb{Z}/n\mathbb{Z}$ where n is a natural number, $R = S[x]$ and H is a proper subgroup of S. If $G = \{f(x) = a_0 + a_1x + \cdots + a_kx^k : a_i \in S,\ a_0 \in H\}$, so $G \subseteq R$ is a subgroup of finite index in an additive group and $G^* = \varnothing$. If the square of every prime number does not divide n and $a_0 \in S$ but $a_t \in H$, for finitely many $t > 0$, then $G \subseteq R$ is a subgroup of finite index in an additive group, $G^* = R^*$ and $G^*G^{*-1} = R^*$.

(vi) Let $R = \mathbb{Z}[x]$ and $G = \{f(x) = a_0 + a_1x + \cdots + a_nx^n : a_0 \in m\mathbb{Z}\ (m > 1),\ a_i \in \mathbb{Z}\ (i \geq 1)\}$. So $G \subseteq R$ is a subgroup of index m in an additive group. It is clear that $G^* = \varnothing = G^*G^{*-1} \neq R^* = \{1, -1\}$. If for finitely many nonzero indices i's, $a_i \in m_i\mathbb{Z}\ (m_i > 1)$, then $G^* = R^*$ and $G^*G^{*-1} = R^*$.

(vii) Let $\mathbb{Q}$ be the set of rational numbers and v_2 the 2-adic valuation on $\mathbb{Q}$. Then $R = \{x \in \mathbb{Q} : v_2(x) \geq 0\} = \{m/n \in \mathbb{Q} : (n, 2) = 1\}$ is a valuation ring (cf. [3, 10, 11, 12] or [13]). If $G = \{2m/n \in R : (n, 2) = 1\}$, then G is a subgroup of finite index 2 in an additive group where $0 + G$ and $1 + G$ are two distinct cosets G in R. It is easy to see that $G^* = \varnothing$, $R^* = \{m/n : m = 2k+1,\ n = 2l+1\}$, and $G^*G^{*-1} \neq R^*$.

The above statement can be shown for any p-adic valuation ring in $\mathbb{Q}$.

By Proposition 1.1, Corollary 1.2, and the previous statements, the following question may be raised.

Question 2.1. If F is a finite field or a ring and G is a subgroup of finite index in an additive group, must $G^*G^{*-1} = F^*$?

We will answer the question for all finite fields and some rings.

If F is a finite field and $|F|$ is sufficiently large to the index of G, in other words, G is sufficiently large, then $G^*G^{*-1} = F^*$, see the following result.

Theorem 2.2. (i) *Let D be a division ring with* $\operatorname{char} D = p$ *and $G \subseteq D$ be a subgroup of index p^k in an additive group. If $|D| \geq p^{2k+1}$, then $G^*G^{*-1} = D^*$.*

(ii) *If G is a subgroup of finite index $n \geq p^k$ in a division ring D and $|D| = p^{2k}$, then $G^*G^{*-1} \neq D^*$.*

Proof. (i) Fix $c \in D^*$. Let the g_i's be distinct elements in G^* ($|G| > p^{k+1}$). We form the cosets $(cg_1 + G), \ldots, (cg_{p^{k+1}} + G)$. By the pigeonhole principle at least two cosets are equal. So $cg_i + G = cg_j + G \Rightarrow 0 \neq c(g_i - g_j) \in G \Rightarrow cg' = g \Rightarrow c = g/g' \in G^*G^{*-1}$.

(ii) Since $|D| = p^{2k}$ hence $|D^*| = p^{2k} - 1$. By hypothesis $[D : G] = |D|/|G| \geq p^k$, so $|G| \leq p^k$ and therefore, $|G^{*-1}| = |G^*| \geq p^k - 1$, so we have, $|G^*G^{*-1}| \leq (p^k - 1)^2 = p^{2k} - 2p^k + 1 < p^{2k} - 1 = |D^*|$ so $G^*G^{*-1} \neq D^*$. □

Remark 2.3. Theorem 2.2(ii) gives a bound for $|D|$ in part (i) which is optimum.

We now give the result which generalizes Proposition 1.1, Corollary 1.2, and Theorem 2.2(i).

Lemma 2.4. *Let R be a ring and let S be a subset of R with invertible differences, that is, $a-b \in R^*$ for any distinct elements $a,b \in S$.*

(i) *Suppose $G \subseteq R$ is a subgroup of index n in an additive group. If $|S| > n^2$ then $G^*G^{*-1} = R^*$.*

(ii) *If $|S| = \infty$ then $G^*G^{*-1} = R^*$.*

Proof. (i) Let $r \in R^*$ be any element. By the pigeonhole principle there exist $s,t \in S$ such that $s-t = a$ and $rs-rt$ both lie in G^*. So $r = ba^{-1} \in G^*G^{*-1}$, as claimed.

(ii) This part is an immediate consequence of part (i). □

Apply the lemma with $S = K$ for the proof of Proposition 1.1, with $S = D$ for the proof of Proposition 1.1 and Theorem 2.2(i).

We now state the following definition which is a key concept in the paper. This is the analog of [1, Definition 0.1].

Definition 2.5. A ring R is a G^*G^{*-1}-ring, if $G^*G^{*-1} = R^*$ for every subgroup $G \subseteq R$ of finite index in an additive group.

If R is a ring which is a divisible group, then R has no additive subgroups of finite index (cf. [6]). Combining this statement, Lemma 2.4, and Definition 2.5 we obtain the following result.

Proposition 2.6. *If D is an infinite division ring, then every ring R which contains a copy of D is a G^*G^{*-1}-ring. In particular $D[x]$, $D[[x]]$, $M_n(D)$, $M_n(D[x])$, and $M_n(D[[x]])$ are G^*G^{*-1}-rings.*

Proof. If $\operatorname{char}(D)$ is zero then every ring that contains a copy of D is a divisible group and hence $R^* = G^*G^{*-1}$. If the $\operatorname{char}(D) \neq 0$, Lemma 2.4 implies that $R^* = G^*G^{*-1}$. □

Remark 2.7. The converse of Definition 2.5 does not necessarily hold. Let $R = Q[x]$, $G = Q$, then $G^*G^{*-1} = R^*$. But G is not of finite index in an additive group.

3. Properties of G^*G^{*-1}-ring.

In this section, some properties of the G^*G^{*-1}-ring is verified.

Proposition 3.1. *Let R be a commutative ring and I an ideal of R such that R/I is a G^*G^{*-1}-ring. If I does not contain any additive subgroup of finite index and every element of $1+I$ is invertible, then R is a G^*G^{*-1}-ring.*

Proof. Let $G \subseteq R$ be a subgroup of finite index in an additive group. Since $(G+I)/G \cong I/(G\cap I)$ so $|I/(G\cap I)| < \infty$ and hence $I \cap G = I$. Choose $x \in R^*$, then $x+I = (g_1+I)(g_2+I)^{-1}$. It is easily seen that $g_1, g_2 \in G^*$. So $x = g_1g_2^{-1}+a$ for some $a \in I$. But $x = (g_1+ag_2)g_2^{-1}$ where $g_1+ag_2 \in G^*$, that is, $R^* = G^*G^{*-1}$. □

Let R be a commutative ring and $R[x]$ the polynomial ring over R. The element $f(x) = a_0+a_1x+\cdots+a_nx^n \in R[x]$ is invertible if and only if $a_0 \in R^*$ and each a_i $(i>0)$ is nilpotent. So by Proposition 3.1 we have the following result.

Theorem 3.2. *Let R be a commutative ring, $R[x]$ the polynomial ring and $I = \{g(x) = a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \ (i \geq 1)$ is nilpotent$\}$. If I does not contain any additive subgroup of finite index and R is a G^*G^{*-1}-ring, then $R[x]$ also has that property.*

Proof. We have $(R[x]/I)^* = ((R+I)/I)^* = \{r+I \mid r \in R^*\}$. Suppose $G/I \subseteq R[x]/I$ is a subgroup of finite index in an additive group, then G is an additive subgroup of finite index in $R[x]$ and $|R/(G\cap R)| = |(R+G)/G| < \infty$. By hypothesis for $r \in R^*$ there exist $g_1, g_2 \in G^* \cap R^*$ such that $r = g_1g_2^{-1}$. So $r+I = g_1g_2^{-1}+I = (g_1+I)(g_2^{-1}+I) \in (G/I)^*(G/I)^{*-1}$, that is, $(R[x]/I)^* = (G/I)^*(G/I)^{*-1}$. Now Proposition 3.1 completes the proof. □

As an immediate consequence we obtain the following result.

Corollary 3.3. *Let R be a commutative ring without any nonzero nilpotent elements. If R is a G^*G^{*-1}-ring then so is $R[x]$.*

The converse of Theorem 3.2 holds in general, see the following result.

Theorem 3.4. *Let R be a commutative ring. If $R[x]$ is a G^*G^{*-1}-ring then R also has that property.*

Proof. Let $G \subseteq R$ be a subgroup of finite index in an additive group. Put $H = \{a_0 + r_1x^1 + \cdots + r_kx^k : k$ is a nonnegative integer, $a_0 \in G \ r_i \in R \ i > 0\}$. It is easily seen that, $H \subseteq R[x]$ is a subgroup of finite index in an additive group and $H^* \cap R^* = G^*$. Since $(R[x])^* = H^*H^{*-1}$ therefore $R^* = (R[x])^* \cap R^* = (H^*H^{*-1}) \cap R^* = G^*G^{*-1}$. Thus the proof is complete. □

Here, we give a necessary condition for infinite R; $G^*G^{*-1} = R^*$, this condition is not sufficient. We also verify the behavior of G^*G^{*-1}-ring under homomorphisms.

Theorem 3.5. *If R is a G^*G^{*-1}-ring. Assume that S is a nontrivial homomorphic image of R with homomorphism $\varphi : R \to S$ then*

(i) *S is infinite.*

(ii) *Assume $\varphi^{-1}\{1_S\} = \{1_R\}$. If R^* is a G^*G^{*-1}-ring , then S^* also a G^*G^{*-1}-ring.*

Proof. (i) Suppose S is a finite ring. Let $G = \ker\varphi \cdot G \subseteq R$ is a subgroup of finite index in an additive group, because $R/G \cong S$. Then $G^*G^{*-1} = R^*$ therefore $1_R \in G^*G^{*-1}$ and $1_S = \varphi(1_R) \in \varphi(G^*G^{*-1}) = \varphi(G^*)\varphi(G^{*-1}) \subseteq \varphi(G)\varphi(G^{*-1}) = 0$ therefore $S = 0$ which is a contradiction, and the proof is complete.

(ii) Let $G \subseteq S$ be a subgroup of finite index in an additive group. Put $H = \varphi^{-1}(G)$, then H is a subgroup of R. Now define the following homomorphism

$$\alpha : R \longrightarrow \frac{S}{G}, \qquad \alpha(x) = \varphi(x) + G, \tag{3.1}$$

so α is also surjective and by the first isomorphism theorem $R/H \cong S/G$. Since S/G is finite then so is R/H and thus H is of finite index. Then by hypothesis $H^*H^{*-1} = R^*$ now we have $\varphi(H^*)\varphi(H^{*-1}) = \varphi(H^*H^{*-1}) = \varphi(R^*) = S^*$ so $G^*G^{*-1} = S^*$, and this implies that S^* is a G^*G^{*-1}-ring. □

Theorem 3.5 implies that if R is a finite ring, then R^* is not a G^*G^{*-1}-ring.

We now verify the behavior of G^*G^{*-1}-rings under products.

Theorem 3.6. *Suppose $R = R_1 \times R_2$, if R_1^* and R_2^* is G^*G^{*-1}-ring then so is R^*.*

Proof. Suppose $G \subseteq R = R_1 \times R_2$ is a subgroup of finite index of R. Put $A_1 = \{a \in R_1 : (a,0) \in G\}$. Now define

$$\alpha : R \longrightarrow \frac{R_1 \times R_2}{G}, \qquad \alpha(a) = (a,0) + G, \tag{3.2}$$

so $A_1 = \ker \alpha$. It implies that $A_1 \subseteq R_1$ is a subgroup of finite index in an additive group. Therefore $A_1^* A_1^{*-1} = R_1^*$. Similarly, we define A_2 in R_2, so $A_2^* A_2^{*-1} = R_2^*$. Now we have $A_1 \times A_2 = \{(a,b) \mid (a,0),(0,b) \in G\} = \{(a,0)+(0,b) \mid (a,0),(0,b) \in G\} \subseteq G+G \subseteq G$ and also $(A_1 \times A_2^*)(A_1 \times A_2^*)^{-1} = (A_1^* \times A_2^*)(A_1^{*-1} \times A_2^{*-1}) = A_1^* A_1^{*-1} \times A_2^* A_2^{*-1} = R_1^* \times R_2^* = R^*$. Since $A_1 \times A_2 \subseteq G$ then $G^*G^{*-1} = R^*$ and thus R^* is a G^*G^{*-1}-group. □

Theorem 3.7. *Let R be a ring, I its ideal and every element of $1+I$ is invertible. If R is G^*G^{*-1}-ring then R/I is also G^*G^{*-1}-ring.*

Proof. Let $G/I \subseteq R/I$ be a subgroup of finite index in an additive group, then $G \subseteq R$ is a subgroup of finite index in an additive group. Choose $r+I \in (R/I)^*$ where $r \in R^*$ and $r = g_1 g_2^{-1}$ where $g_i \in G^*$, $i = 1,2$. Therefore, $r+I = (g_1+I)(g_2+I)^{-1} \in (G/I)^*(G/I)^{*-1}$. □

Theorems 3.5, 3.6, the properties of isomorphism, Proposition 2.6, Artin-Wederburn theorem (cf. [8]), and Theorem 3.7 imply the following result.

Corollary 3.8. (i) *If $R \cong R_1 \times R_2$ then R_1^* and R_2^* are G^*G^{*-1}-rings if and only if R^* is a G^*G^{*-1}-ring.*

(ii) *Every semisimple ring which has no finite homomorphic image is a G^*G^{*-1}-ring.*

(iii) *Let R be a G^*G^{*-1}-ring and J the Jacobson radical of R. Then S is a G^*G^{*-1}-ring.*

Remark 3.9. If S is a G^*G^{*-1}-ring and R is a subring of S then R is not necessarily a G^*G^{*-1}-ring. So If $\varphi : R \to S$ is a monomorphism and S is a G^*G^{*-1}-ring then R is not necessarily a G^*G^{*-1}-ring.

We end this section by verifying whether $D^* = G^*G^{*-1}$ D is an infinite division ring and $G = F + [D,D]$ where F denotes the center of D and $[D,D]$ denotes the additive commutator subgroup of D, (cf. [5]). As an example see the following example.

Example 3.10. Suppose that $D = Q(i,j,k)$ is the rational quaternion, by a simple investigation one can see that $[D,D] = ai + bj + ck$ for $a,b,c \in Q$, therefore $G = F + [D,D] = D$ and so $G^*G^{*-1} = D^*$.

This also holds for real quaternions. But in general we have the following result.

This question is answered for a finite-dimensional division algebra (or more generally central algebra).

Lemma 3.11. *Let D be a finite-dimensional division (or, more generally, central simple) algebra with center F. Then $[D,D]$ coincides with the set of elements of D of trace 0.*

Proof. Let $d_1, d_2, \dots, d_{n^2}$ be a basis of D of F vector space; here $n = \deg(D)$. Let T_0 be the n^2-1-dimensional of F-subspace of D consisting of trace-zero elements. Clearly $[D,D] \subseteq T_0$. Thus it is enough to show that $\dim_F[D,D] \geq n^2-1$. Let K be a splitting field of D. Then $D \otimes_F K = M_n(K)$. It is easy to see that $[M_n(K), M_n(K)]$ is precisely the set of $n\times n$-matrices of trace zero. On the other hand, this set is spanned by elements $[d_i, d_j]$, as $i,j = 1,2,\dots,n^2$. Thus n^2-1 of these elements are linearly independent over K and, hence, over F. This proves the inequality $\dim_F[D,D] \geq n^2-1$, as desired. □

We therefore conclude that $\dim_F[D,D] = n^2-1$, while $\dim_F(D) = n^2$. Thus

$$F + [D,D] = \begin{cases} D, & \text{if } \operatorname{char}(F) \text{ does not divide } n, \\ [D,D], & \text{if } \operatorname{char}(F) \text{ divides } n. \end{cases} \tag{3.3}$$

If D is noncommutative (i.e., of degree $n \geq 2$) then the following lemma shows that $D^* = [D,D]^*([D,D])^{*-1}$ in any characteristic.

Lemma 3.12. *Let D be a finite-dimensional division algebra of degree $n \geq 2$ with center F and let G be a d-dimensional F-vector subspace of D.*

(a) *Assume $2d > n^2$. Then $D^* = G^*G^{*-1}$.*

(b) *Assume $G = [D,D]$. Then $D^* = G^*G^{*-1}$.*

Proof. (a) Let $a \in D^*$. Since $2d > n^2$, the d-dimensional F-vector spaces G and aG have a nontrivial intersection in D, that is, $g_1 = ag_2$ for some $g_1, g_2 \in G^*$. Then $a = g_1 g_2{}^{-1}$, as desired.

(b) By Lemma 3.12, $d = \dim_F[D,D] = n^2-1$. Since D is noncommutative, $n \geq 2$ and thus $2d = 2n^2-2 > n^2$. Now apply part (a). □

Question 3.13. (1) If R is not a finite homomorphic image, must R^* be infinite? Must R contain an infinite subset with invertible differences?

(2) Is there a ring with no finite homomorphic image, but with some finite index subgroup G avoiding all units: $G^* = \varnothing$?

(3) If R is a G^*G^{*-1}-ring then must R^* be infinite? If R is a G^*G^{*-1}-ring must R contain an infinite sets with invertible differences?

(4) If R is a G^*G^{*-1}-ring, then must the matrix ring $M_n(R)$ also have that property? Conversely, if $M_n(R)$ is a G^*G^{*-1}-ring, then must R be a G^*G^{*-1}-ring?

(5) If R is a G^*G^{*-1}-ring and R is a subring of a ring S then must S also be a G^*G^{*-1}-ring?

(6) If R/I is a G^*G^{*-1}-ring and $1+I$ is invertible elements then must R also have that property?

(7) Let D be an infinite (algebraic) division algebra over its center F. If $G = F + [D,D]$. Is $D^* = G^*G^{*-1}$?

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References

[1] V. Bergelson and D. B. Shapiro, *Multiplicative subgroups of finite index in a ring*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 885–896. MR 93b:16001. Zbl 784.12002.

[2] P. Berrizbeitia, *Additive properties of multiplicative subgroups of finite index in fields*, Proc. Amer. Math. Soc. **112** (1991), no. 2, 365–369. MR 91i:12003. Zbl 726.12001.

[3] O. Endler, *Valuation Theory*, Universitext, Springer-Verlag, New York, 1972. MR 50#9847. Zbl 257.12111.

[4] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory*, Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication, John Wiley & Sons, New York, 1980. MR 82b:05001. Zbl 455.05002.

[5] R. Hazrat and M. Mahdavi-Hezavehi, *Some examples and counter example in division rings*, Tech. report, IPM-97-256.

[6] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York, 1980, reprint of the 1974 original. MR 82a:00006. Zbl 442.00002.

[7] K. F. Ireland and M. I. Rosen, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York, 1982, revised edition of Elements of number theory. MR 83g:12001. Zbl 482.10001.

[8] T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 1991. MR 92f:16001. Zbl 728.16001.

[9] D. B. Leep and D. B. Shapiro, *Multiplicative subgroups of index three in a field*, Proc. Amer. Math. Soc. **105** (1989), no. 4, 802–807. MR 89m:11127. Zbl 684.12015.

[10] D. Mojdeh, *Noninvariant matrix valuations and their associated valued division rings*, Indian J. Math. **38** (1996), no. 1, 57–66 (1997). MR 98m:16050. Zbl 902.16037.

[11] ______, *Initial ramification index of noninvariant valuations on finite-dimensional division algebras*, J. Sci. Islam. Repub. Iran **8** (1997), no. 3, 193–197. MR 98m:16019.

[12] ______, *Some examples in valuation theory*, J. Inst. Math. Comput. Sci. Math. Ser. **10** (1997), no. 1, 1–9. MR 98i:16018. Zbl 922.16010.

[13] O. F. G. Schilling, *The Theory of Valuations*, Mathematical Surveys, no. 4, American Mathematical Society, New York, 1950. MR 13,315b. Zbl 037.30702.

[14] G. Turnwald, *Multiplicative subgroups of finite index in a division ring*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 377–381. MR 94e:12002. Zbl 795.12004.

[15] S. Wagon, *The Banach-Tarski Paradox*, Encyclopedia of Mathematics and its Applications, vol. 24, Cambridge University Press, Cambridge, 1985. MR 87e:04007. Zbl 569.43001.

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