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# A NOTE ON MUES' CONJECTURE

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Abstract. We prove that Mues' conjecture holds for the second- and higher-order derivatives of a square and higher power of any transcendental meromorphic function.

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**1. Introduction, definitions, and results.** Let $f$ be a transcendental meromorphic function defined in the open complex plane $\mathscr{C}$. For a positive integer $l$ we denote by $N(r,\infty;f\mid\geq l)$ the counting function of the poles of $f$ with multiplicities not less than $l$, where a pole is counted according to its multiplicity. Also for $\alpha\in\mathscr{C}$, we denote by $N(r,\alpha;f\mid=1)$ the counting function of simple zeros of $f-\alpha$. We do not explain the standard definitions and notations of the value distribution theory as they are available in [1, 6].

In 1971, Mues [4] conjectured that for a positive integer $k$ the following relation might be true:

$$\sum_{a\neq\infty}\delta\left(a;f^{(k)}\right)\leq 1. \tag{1.1}$$

Mues [4] himself proved the following theorem.

**Theorem 1.1.** *If $N(r,f)-\bar{N}(r,f)=o\{N(r,f)\}$, then for $k\geq 2$*

$$\sum_{a\neq\infty}\delta\left(a;f^{(k)}\right)\leq 1. \tag{1.2}$$

In this direction Ishizaki [3] proved the following result.

**Theorem 1.2.** *If for some $l(\geq 2)$ $N(r,\infty;f\mid\geq l)=o\{N(r,f)\}$, then for all $k\geq l$*

$$\sum_{a\neq\infty}\delta\left(a;f^{(k)}\right)\leq 1. \tag{1.3}$$

Yang and Wang [7] also worked on Mues' conjecture and proved the following theorem.

**Theorem 1.3.** *There exists a positive number $K=K(f)$ such that for every positive integer $k\geq K$*

$$\sum_{a\neq\infty}\delta\left(a;f^{(k)}\right)\leq 1. \tag{1.4}$$

We see that in Theorem 1.3 the set of exceptional integers $k$ is different for different function $f$. In this paper, we show that if $f$ is a square or a higher power of a meromorphic function, then the relation (1.1) holds for any integer $k \geq 2$. This result follows as a consequence of the following theorem because such a function has no simple zero.

**Theorem 1.4.** *If $N(r,\alpha;f \mid= 1) = S(r,f)$ for some $\alpha \neq \infty$, then for $k \geq 2$*

$$\sum_{a\neq\infty} \delta(a;f^{(k)}) \leq 1. \tag{1.5}$$

## 2. Lemmas.

In this section, we state two lemmas which will be needed in the proof of Theorem 1.4.

**Lemma 2.1** (see [2]). *Let $A > 1$, then there exists a set $M(A)$ of upper logarithmic density at most $\min\{(2e^{A-1}-1)^{-1},(1+e(A-1)\exp(e(1-A)))\}$ such that for $k = 1,2,3,\dots$*

$$\limsup_{r\to\infty,\ r\notin M(A)} \frac{T(r,f)}{T(r,f^{(k)})} \leq 3eA. \tag{2.1}$$

**Lemma 2.2** (see [5]). *For any integer $k(\geq 0)$ and any positive number $\varepsilon(>0)$, we get*

$$(k-2)\bar{N}(r,f)+N(r,0;f) \leq 2\bar{N}(r,0;f)+N(r,0;f^{(k)})+\varepsilon T(r,f)+S(r,f). \tag{2.2}$$

## 3. Proof of Theorem 1.4.

Without loss of generality, we may choose $\alpha = 0$. Let $g = f-\alpha$. Then $f^{(k)} = g^{(k)}$ and

$$N(r,0;g \mid= 1) = N(r,\alpha;f \mid= 1) = S(r,f) = S(r,g). \tag{3.1}$$

Applying the second fundamental theorem to $f^{(k)}$, we get for any $q$ finite distinct complex numbers $a_1,a_2,\dots,a_q$

$$\begin{aligned} &m(r,f^{(k)})+\sum_{j=1}^{q} m(r,a_j;f^{(k)}) \\ &\quad\leq 2T(r,f^{(k)})-N(r,0;f^{(k+1)})-2N(r,f^{(k)})+N(r,f^{(k+1)})+S(r,f^{(k)}), \end{aligned} \tag{3.2}$$

that is,

$$\sum_{j=1}^{q} m(r,a_j;f^{(k)}) \leq T(r,f^{(k)})+\bar{N}(r,f)-N(r,0;f^{(k+1)})+S(r,f^{(k)}). \tag{3.3}$$

By Lemma 2.2 and from (3.3) we get

$$\begin{aligned} \sum_{j=1}^{q} m(r,a_j;f^{(k)}) &\leq T(r,f^{(k)})+\bar{N}(r,f)+2\bar{N}(r,0;f)-N(r,0;f) \\ &\quad -(k-1)\bar{N}(r,f)+\varepsilon T(r,f)+S(r,f)+S(r,f^{(k)}). \end{aligned} \tag{3.4}$$

Since $2\bar{N}(r,0;f)-N(r,0;f)\le N(r,0;f\mid=1)=S(r,f)$ and $k\ge 2$, we get from (3.4)

$$\sum_{j=1}^{q} m(r,a_j;f^{(k)})\le T(r,f^{(k)})+\varepsilon T(r,f)+S(r,f)+S(r,f^{(k)}). \tag{3.5}$$

Let $E$ be the exceptional set arising out of Lemma 2.2, the second fundamental theorem, and the condition $N(r,0;f\mid=1)=S(r,f)$. We choose a sequence of positive numbers $\{r_n\}$ tending to infinity such that $r_n\notin E\cup M(A)$. Then from (3.5) we get, for $r=r_n$ in view of Lemma 2.1,

$$\sum_{j=1}^{q} m(r_n,a_j;f^{(k)})\le T(r_n,f^{(k)})+3eA\varepsilon T(r_n,f^{(k)})+o\{T(r_n,f^{(k)})\}, \tag{3.6}$$

which gives

$$\sum_{j=1}^{q}\delta(a_j;f^{(k)})\le 1+3eA\varepsilon. \tag{3.7}$$

Since $\varepsilon(>0)$ is arbitrary and $q$ is an arbitrary positive number, we get from (3.7)

$$\sum_{a\ne\infty}\delta(a;f^{(k)})\le 1. \tag{3.8}$$

This proves the theorem. □

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