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PROPER CONTRACTIONS AND INVARIANT SUBSPACES

C. S. KUBRUSLY and N. LEVAN

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ABSTRACT. Let T be a contraction and A the strong limit of $\{T^{*n}T^n\}_{n\geq 1}$. We prove the following theorem: if a hyponormal contraction T does not have a nontrivial invariant subspace, then T is either a proper contraction of class $\mathcal{C}_{00}$ or a nonstrict proper contraction of class $\mathcal{C}_{10}$ for which A is a completely nonprojective nonstrict proper contraction. Moreover, its self-commutator $[T^*,T]$ is a strict contraction.

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1. Introduction. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T^*\| = \|T\|$. The null space (kernel) of T, which is the subspace of $\mathcal{H}$, will be denoted by $\mathcal{N}(T)$. A contraction is an operator T such that $\|T\| \leq 1$ (i.e., $\|Tx\| \leq \|x\|$ for every x in $\mathcal{H}$ or, equivalently, $T^*T \leq I$). A strict contraction is an operator T such that $\|T\| < 1$ (i.e., $\sup_{0\neq x}(\|Tx\|/\|x\|) < 1$; equivalently, $T^*T \prec I$, which means that $T^*T \leq \gamma I$ for some $\gamma \in (0,1)$). An isometry is a contraction for which $\|Tx\| = \|x\|$ for every x in $\mathcal{H}$ (i.e., $T^*T = I$ so that $\|T\| = 1$).

We summarize below some well-known results on contractions that will be applied throughout (cf. [16, page 40], [5, 9, 10, 11, 13], and [8, Chapter 3]). If T is a contraction, then $T^{*n}T^n \xrightarrow{s} A$. That is, the sequence $\{T^{*n}T^n\}_{n\geq 1}$ of operators on $\mathcal{H}$ converges strongly to an operator A on $\mathcal{H}$, which means that $\|(T^{*n}T^n - A)x\| \to 0$ for every x in $\mathcal{H}$. Moreover, A is a nonnegative contraction (i.e., $O \leq A \leq I$), $\|A\| = 1$ whenever $A \neq O$, $T^{*n}AT^n = A$ for every integer $n \geq 1$ (so that T is an isometry if and only if $A = I$), $\|T^n x\| \to \|A^{1/2}x\|$ for every x in $\mathcal{H}$, and the null spaces of A and $I - A$, viz. $\mathcal{N}(A) = \{x \in \mathcal{H} : Ax = 0\}$ and $\mathcal{N}(I-A) = \{x \in \mathcal{H} : Ax = x\}$, are given by

$$\begin{aligned} \mathcal{N}(A) &= \{x \in \mathcal{H} : T^n x \longrightarrow 0\}, \\ \mathcal{N}(I-A) &= \{x \in \mathcal{H} : \|T^n x\| = \|x\| \ \forall n \geq 1\} \\ &= \{x \in \mathcal{H} : \|Ax\| = \|x\|\}. \end{aligned} \tag{1.1}$$

Recall that T is a contraction if and only if T^* is. Thus $T^nT^{*n} \xrightarrow{s} A_*$, where $O \leq A_* \leq I$, $\|A_*\| = 1$ whenever $A_* \neq O$, $T^nA_*T^{*n} = A_*$ for every $n \geq 1$ (so that T is a co-isometry—i.e., T^* is an isometry—if and only if $A_* = I$), $\|T^{*n}x\| \to \|A_*^{1/2}x\|$ for every x in $\mathcal{H}$, and

$$\begin{aligned} \mathcal{N}(A_*) &= \{x \in \mathcal{H} : T^{*n} x \longrightarrow 0\}, \\ \mathcal{N}(I-A_*) &= \{x \in \mathcal{H} : \|T^{*n} x\| = \|x\| \ \forall n \geq 1\} \\ &= \{x \in \mathcal{H} : \|A_* x\| = \|x\|\}. \end{aligned} \tag{1.2}$$

An operator T on $\mathcal{H}$ is uniformly stable if the power sequence $\{T^n\}_{n\geq 1}$ converges uniformly to the null operator (i.e., $\|T^n\| \to 0$). It is strongly stable if $\{T^n\}_{n\geq 1}$ converges strongly to the null operator (i.e., $\|T^n x\| \to 0$ for every x in $\mathcal{H}$), and weakly stable if $\{T^n\}_{n\geq 1}$ converges weakly to the null operator (i.e., $\langle T^n x; y\rangle \to 0$ for every $x, y \in \mathcal{H}$ or, equivalently, $\langle T^n x; x\rangle \to 0$ for every $x \in \mathcal{H}$). It is clear that uniform stability implies strong stability, which implies weak stability. The converses fail (a unilateral shift is a weakly stable isometry and its adjoint is a strongly stable co-isometry) but hold for compact operators. T is uniformly stable if and only if T^* is uniformly stable, and T is weakly stable if and only if T^* is weakly stable. However, strong convergence is not preserved under the adjoint operation so that strong stability for T does not imply strong stability for T^* (and vice versa). If T is a strongly stable contraction (i.e., if $\mathcal{N}(A) = \mathcal{H}$, which means that $A = O$), then it is usual to say that T is a $\mathscr{C}_{0\cdot}$-contraction. If T^* is a strongly stable contraction (i.e., if $\mathcal{N}(A_*) = \mathcal{H}$, which means that $A_* = O$), then T is a $\mathscr{C}_{\cdot 0}$-contraction. On the other extreme, if a contraction T is such that $T^n x \nrightarrow 0$ for every nonzero vector x in $\mathcal{H}$ (i.e., if $\mathcal{N}(A) = \{0\}$), then it is said to be a $\mathscr{C}_{1\cdot}$-contraction. Dually, if a contraction T is such that $T^{*n} x \nrightarrow 0$ for every nonzero vector x in $\mathcal{H}$ (i.e., if $\mathcal{N}(A_*) = \{0\}$), then it is a $\mathscr{C}_{\cdot 1}$-contraction. These are the Nagy-Foiaş classes of contractions (see [16, page 72]). All combinations are possible leading to classes $\mathscr{C}_{00}$, $\mathscr{C}_{01}$, $\mathscr{C}_{10}$, and $\mathscr{C}_{11}$. In particular, T and T^* are both strongly stable contractions if and only if T is of class $\mathscr{C}_{00}$. Generally,

$$\begin{aligned} T \in \mathscr{C}_{00} &\Longleftrightarrow A = A_* = O, \\ T \in \mathscr{C}_{01} &\Longleftrightarrow A = O,\ \mathcal{N}(A_*) = \{0\}, \\ T \in \mathscr{C}_{10} &\Longleftrightarrow \mathcal{N}(A) = \{0\},\ A_* = O, \\ T \in \mathscr{C}_{11} &\Longleftrightarrow \mathcal{N}(A) = \mathcal{N}(A_*) = \{0\}. \end{aligned} \tag{1.3}$$

If T is a strict contraction, then it is uniformly stable, and hence of class $\mathscr{C}_{00}$. Thus, a contraction not in $\mathscr{C}_{00}$ is necessarily nonstrict (i.e., if $T \notin \mathscr{C}_{00}$, then $\|T\| = 1$). In particular, contractions in $\mathscr{C}_{1\cdot}$ or in $\mathscr{C}_{\cdot 1}$ are nonstrict.

2. Proper contractions. An operator T is a proper contraction if $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$ or, equivalently, if $T^*T < I$. The terms "strict" and "proper" contractions are sometimes interchanged in current literature. We adopt the terminology of [7, page 82] for strict contraction. Obviously, every strict contraction is a proper contraction, every proper contraction is a contraction, and the converses fail: any isometry is a contraction but not a proper contraction, and the diagonal operator $T = \operatorname{diag}\{(k+1)(k+2)^{-1}\}_{k=0}^{\infty}$ is a proper contraction on ℓ_+^2 but not a strict contraction. Thus, proper contractions comprise a class of operators that is properly included in the class of all contractions and properly includes the class of all strict contractions. If T is a proper contraction, then so is T^*T (reason: ST is a proper contraction whenever S is a contraction and T is a proper contraction). Thus, the point spectrum $\sigma_P(T^*T)$ lies in the open unit disc. If, in addition, T is compact, then so is T^*T and hence its spectrum $\sigma(T^*T)$, which is always closed, also lies in the open unit disc (for $\sigma(K)\backslash\{0\} = \sigma_P(K)\backslash\{0\}$ whenever K is compact). This implies that the spectral radius $r(T^*T)$ is less than one. Therefore, $\|T\|^2 = r(T^*T) < 1$.

Conclusion. *The concepts of proper and strict contraction coincide for compact operators.*

Proper contractions have been investigated in connection with unitary dilations (*the minimal unitary dilation of a proper contraction is a bilateral shift whose multiplicity does not exceed the dimension of* $\mathcal{H}$—see [16, page 91]), and also with strong stability of contractive semigroups (cf. [1]). They were further investigated in [15] by considering different topologies in $\mathcal{H}$. Here are three basic properties of proper contractions that will be needed in the sequel.

Proposition 2.1. *T is a proper contraction if and only if T^* is a proper contraction.*

Proof. Recall that $\|T^*x\|^2 = \langle T^*x;T^*x\rangle = \langle TT^*x;x\rangle \le \|TT^*x\|\|x\|$ for every x in $\mathcal{H}$, for all operators T on $\mathcal{H}$. Take an arbitrary nonzero vector x in $\mathcal{H}$. If $T^*x = 0$, then $\|T^*x\| < \|x\|$ trivially. On the other hand, if $T^*x \ne 0$ and T is a proper contraction, then $\|TT^*x\| < \|T^*x\| \ne 0$ so that $\|T^*x\|^2 < \|T^*x\|\|x\|$, and hence $\|T^*x\| < \|x\|$. That is, T^* is a proper contraction. Dually, since $T^{**} = T$, it follows that T is a proper contraction whenever T^* is. □

If S is a contraction and T is a proper contraction, then ST is a proper contraction (as we have already seen above) and so is S^*T^* by Proposition 2.1. Another application of Proposition 2.1 ensures that $TS = (S^*T^*)^*$ is still a proper contraction. Summing up: *left or right product of a contraction and a proper contraction is again a proper contraction.*

Proposition 2.2. *Every proper contraction is weakly stable.*

Proof. If $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$, then T is completely nonisometric (i.e., there is no nonzero reducing subspace $\mathcal{M}$ for T such that $\|T^nx\| = \|x\|$ for every $x \in \mathcal{M}$ and every $n \ge 1$), and therefore completely nonunitary. But a completely nonunitary contraction is weakly stable. In fact, the Foguel decomposition for contractions says that every contraction is the direct sum of a weakly stable contraction and a unitary operator (cf. [6, page 55] or [8, page 106]). □

The converse of Proposition 2.2 fails: shifts are weakly stable isometries. However, as it was raised in [1], *a proper contraction is not necessarily strongly stable.* Indeed, if T is the weighted unilateral shift $T = \text{shift}\{(k+1)^{1/2}(k+2)^{-1}(k+3)^{1/2}\}_{k=0}^{\infty}$ on ℓ_+^2, which is a proper contraction because $(k+1)(k+2)^{-2}(k+3) < 1$ for every $k \ge 0$, then A is the diagonal operator $A = \text{diag}\{(k+1)(k+2)^{-1}\}_{k=0}^{\infty} \ne O$ (cf. [10] or [8, pages 51, 52]) so that T is not strongly stable. As a matter of fact, $\mathcal{N}(A) = \{0\}$ and (as it is readily verified) $A_* = O$. Hence T is a proper contraction of class $\mathcal{C}_{10}$. The converse is much simpler: *strongly stable contractions are not necessarily proper contractions.* For instance, a backward unilateral shift S_+^* is a strongly stable co-isometry (in fact, an operator is a strongly stable co-isometry if and only if it is a backward unilateral shift). Thus S_+^* is a strongly stable contraction but not a proper contraction (it is a nonproper contraction of class $\mathcal{C}_{01}$). Actually, even *a $\mathcal{C}_{00}$-contraction is not necessarily a proper contraction.* For example, the weighted bilateral shift $T = \text{shift}\{(|k|+1)^{-1}\}_{k=-\infty}^{\infty}$ on ℓ^2 is a contraction of class $\mathcal{C}_{00}$ (reason: $\prod_{k=0}^{n}(|k|+1)^{-1} = (n!)^{-1} \to 0$ as $n \to \infty$, which means that both products $\prod_{k=0}^{\infty}(|k|+1)^{-1}$ and $\prod_{k=-\infty}^{0}(|k|+1)^{-1}$ diverge to

0—see [3, page 181]) but not a proper contraction because $(|k|+1)^{-1} = 1$ for $k = 0$. It is worth noticing that the weighted bilateral shift $T = \text{shift}\{1-(|k|+2)^{-2}\}_{k=-\infty}^{\infty}$ on ℓ^2 is a proper contraction of class $\mathscr{C}_{11}$. Indeed, $0 < 1-(|k|+2)^{-2} < 1$ for each integer k, and both products $\prod_{k=0}^{\infty}(1-(|k|+2)^{-2})$ and $\prod_{k=-\infty}^{0}(1-(|k|+2)^{-2})$ do not diverge to 0 (cf. [3, page 181] again)—these products converge once the series $\sum_{k=0}^{\infty}(|k|+2)^{-2}$ converges.

Proposition 2.3. *If T is a proper contraction, then A is a proper contraction.*

Proof. Let T be a proper contraction and take an arbitrary nonzero vector x in $\mathscr{H}$. If $T^m x = 0$ for some $m \geq 1$, then $T^n x = 0$ for every integer $n \geq m$. If $T^n x \neq 0$ for every integer $n \geq 1$, then $\|T^{n+1}x\| = \|TT^n x\| < \|T^n x\| < \|x\|$ so that $\{\|T^n x\|\}_{n\geq 1}$ is a strictly decreasing sequence of positive numbers. In the former case T is trivially strongly stable so that $A = O$ is a trivial proper contraction. In the latter case $\{\|T^n x\|\}_{n\geq 1}$ converges in the real line to $\|A^{1/2}x\|$ so that $\|A^{1/2}x\| < \|x\|$. Thus $\|Ax\| \leq \|A^{1/2}x\| < \|x\|$. □

A backward unilateral shift shows that the converse of Proposition 2.3 does not hold true as well (i.e., *there exist nonproper contractions T for which A is a proper contraction*).

3. Invariant subspaces.

A subspace $\mathscr{M}$ of $\mathscr{H}$ is a closed linear manifold of $\mathscr{H}$. $\mathscr{M}$ is nontrivial if $\{0\} \neq \mathscr{M} \neq \mathscr{H}$. If T is an operator on $\mathscr{H}$ and $T(\mathscr{M}) \subseteq \mathscr{M}$, then $\mathscr{M}$ is invariant for T (or $\mathscr{M}$ is T-invariant). If $\mathscr{M}$ is a nontrivial invariant subspace for T, then its orthogonal complement $\mathscr{M}^\perp$ is a nontrivial invariant subspace for T^*. If $\mathscr{M}$ is invariant for both T and T^* (equivalently, if both $\mathscr{M}$ and $\mathscr{M}^\perp$ are T-invariant), then $\mathscr{M}$ reduces T. A classical open question in operator theory is: *does a contraction not in $\mathscr{C}_{00}$ have a nontrivial invariant subspace?* Although this is still an unsolved problem we know that the following result holds true.

Lemma 3.1. *If a contraction has no nontrivial invariant subspace, then it is either a $\mathscr{C}_{00}$, a $\mathscr{C}_{01}$, or a $\mathscr{C}_{10}$-contraction.*

Proof. See, for instance, [8, page 71]. □

The class of contractions T for which A is a projection was investigated in [4, 10]. It coincides with the class of all contractions T that commute with A; that is, $A = A^2$ if and only if $AT = TA$ (cf. [4]). Equivalently, $\mathscr{N}(A-A^2) = \mathscr{H}$ if and only if $\mathscr{N}(AT-TA) = \mathscr{H}$. The next proposition extends this equivalence.

Proposition 3.2. *$\mathscr{N}(A-A^2)$ is the largest subspace of $\mathscr{H}$ that is included in $\mathscr{N}(AT-TA)$ and is T-invariant.*

Proof. See [10] (or [8, page 52]). □

We will say that A is completely nonprojective if $Ax \neq A^2x$ for every nonzero x in $\mathscr{H}$ (i.e., if $\mathscr{N}(A-A^2) = \{0\}$). Since $\mathscr{N}(A-A^2)$ reduces the selfadjoint operator A, this means that no nonzero direct summand of A is a projection. If A is completely nonprojective, then T is a $\mathscr{C}_{1\cdot}$-contraction (for $\mathscr{N}(A) \subseteq \mathscr{N}(A-A^2)$).

Lemma 3.3. *If a contraction T has no nontrivial invariant subspace, then either T is strongly stable or A is a completely nonprojective nonstrict proper contraction.*

Proof. Suppose that T is a contraction without a nontrivial invariant subspace. Since $\mathcal{N}(A-A^2)$ is an invariant subspace for T (by Proposition 3.2), it follows that either $\mathcal{N}(A-A^2)=\mathcal{H}$ or $\mathcal{N}(A-A^2)=\{0\}$. In the former case A is a projection (i.e., $A=A^2$). However, as it was shown in [10], if A is a projection then T is the direct sum of a strongly stable contraction G, a unilateral shift S_+, and a unitary operator U, where any of the direct summands of the decomposition

$$T=G\oplus S_+\oplus U \tag{3.1}$$

may be missing (see also [8, page 83]). But T has no nontrivial invariant subspace so that $T=G$. That is, T is a strongly stable contraction, for S_+ and U clearly have nontrivial invariant subspaces (isometries have nontrivial invariant subspaces). In the latter case A is a completely nonprojective proper contraction. Indeed, $\{x\in\mathcal{H}:\|Ax\|=\|x\|\}=\mathcal{N}(I-A)\subseteq\mathcal{N}(A-A^2)=\{0\}$. Finally, the contraction A is not strict (i.e., $\|A\|=1$) whenever T is not strongly stable (i.e., whenever $A\neq O$). □

Another classical open question in operator theory is: *does a hyponormal operator have a nontrivial invariant subspace?* Recall that an operator T on $\mathcal{H}$ is hyponormal if $TT^*\leq T^*T$ (equivalently, if $\|T^*x\|\leq\|Tx\|$ for every x in $\mathcal{H}$), and T is cohyponormal if T^* is hyponormal. Here is a consequence of Lemmas 3.1 and 3.3 for hyponormal contractions. It uses the fact that a cohyponormal contraction T is such that A is a projection. This implies that a completely nonunitary cohyponormal contraction is strongly stable (cf. [9, 12, 14]).

Theorem 3.4. *If a hyponormal contraction T has no nontrivial invariant subspace, then it is either a $\mathscr{C}_{00}$-contraction or a $\mathscr{C}_{10}$-contraction for which A is a completely nonprojective nonstrict proper contraction.*

Proof. If T has no nontrivial invariant subspace, then T^* has no nontrivial invariant subspace. If T is a contraction, then Lemmas 3.1 and 3.3 ensure that either $A=A_*=O$, $A=O$ and A_* is a completely nonprojective nonstrict proper contraction, or A is a completely nonprojective nonstrict proper contraction and $A_*=O$. However, if T is hyponormal, then A_* is a projection [9] so that $A_*=O$ (see also [8, page 78]). □

Can the conclusion in Theorem 3.4 be sharpened to $T\in\mathscr{C}_{00}$? In other words, *does a hyponormal contraction not in $\mathscr{C}_{00}$ have a nontrivial invariant subspace?* The question has an affirmative answer if we replace "$\mathscr{C}_{00}$-contraction" with "proper contraction." That is, *if a hyponormal contraction is not a proper contraction, then it has a nontrivial invariant subspace*. This will be proved in Theorem 3.6 below, but first we consider the following auxiliary result. Let D denote the self-commutator of T; that is,

$$D=[T^*,T]=T^*T-TT^*. \tag{3.2}$$

Thus, a hyponormal is precisely an operator T for which D is nonnegative (i.e., $D\geq O$).

Proposition 3.5. *If T is a hyponormal contraction, then D is a contraction whose power sequence converges strongly. If P is the strong limit of $\{D^n\}_{n\geq 1}$, then $PT=O$.*

Proof. Take an arbitrary x in $\mathcal{H}$ and an arbitrary nonnegative integer n. Suppose that T is hyponormal and let $R = D^{1/2} \geq O$ be the unique nonnegative square root of $D \geq O$. If, in addition, T is a contraction, then

$$\begin{aligned}\langle D^{n+1}x;x\rangle &= \|R^{n+1}x\|^2 = \langle DR^nx;R^nx\rangle \\ &= \|TR^nx\|^2 - \|T^*R^nx\|^2 \\ &\leq \|R^nx\|^2 - \|T^*R^nx\|^2 \leq \|R^nx\|^2 \\ &= \langle D^nx;x\rangle.\end{aligned} \tag{3.3}$$

This shows that R (and so D) is a contraction: set $n = 0$ above. It also shows that $\{D^n\}_{n\geq 1}$ is a decreasing sequence of nonnegative contractions. Since a bounded monotone sequence of selfadjoint operators converges strongly,

$$D^n \xrightarrow{s} P \geq O. \tag{3.4}$$

Indeed, the strong limit P of $\{D^n\}_{n\geq 1}$ is nonnegative, for the set of all nonnegative operators on $\mathcal{H}$ is weakly (thus strongly) closed. As a matter of fact, $P = P^2$ (the weak limit of any weakly convergent power sequence is idempotent) and so $P \geq O$ is a projection. Moreover,

$$\sum_{n=0}^{m} \|T^*R^nx\|^2 \leq \sum_{n=0}^{m} \left(\|R^nx\|^2 - \|R^{n+1}x\|^2\right) = \|x\|^2 - \|R^{m+1}x\|^2 \leq \|x\|^2 \tag{3.5}$$

for all $m \geq 0$ so that $\|T^*R^nx\| \to 0$ as $n \to \infty$. Hence

$$T^*Px = T^* \lim_n D^nx = \lim_n T^*R^{2n}x = 0 \tag{3.6}$$

for every x in $\mathcal{H}$, and therefore $PT = O$ (since P is selfadjoint). □

Theorem 3.6. *If a hyponormal contraction has no nontrivial invariant subspace, then it is a proper contraction and its self-commutator is a strict contraction.*

Proof. (a) Take an arbitrary operator T on $\mathcal{H}$ and an arbitrary x in $\mathcal{H}$. Note that

$$T^*Tx = \|T\|^2x \text{ if and only if } \|Tx\| = \|T\|\|x\|. \tag{3.7}$$

Indeed, if $T^*Tx = \|T\|^2x$, then $\|Tx\|^2 = \langle T^*Tx;x\rangle = \|T\|^2\|x\|^2$. Conversely, if $\|Tx\| = \|T\|\|x\|$, then $\langle T^*Tx;\|T\|^2x\rangle = \|T\|^4\|x\|^2$, and hence

$$\begin{aligned}\|T^*Tx - \|T\|^2x\|^2 &= \|T^*Tx\|^2 - 2\operatorname{Re}\langle T^*Tx;\|T\|^2x\rangle + \|T\|^4\|x\|^2 \\ &= \|T^*Tx\|^2 - \|T\|^4\|x\|^2 \leq \left(\|T^*T\|^2 - \|T\|^4\right)\|x\|^2 = 0.\end{aligned} \tag{3.8}$$

Put $\mathcal{M} = \{x \in \mathcal{H} : \|Tx\| = \|T\|\|x\|\} = \mathcal{N}(\|T\|^2I - T^*T)$, which is a subspace of $\mathcal{H}$. If T is hyponormal, then $\mathcal{M}$ is T-invariant. In fact, if T is hyponormal and $x \in \mathcal{M}$, then

$$\|T(Tx)\| \leq \|T\|\|Tx\| = \|\|T\|^2x\| = \|T^*Tx\| \leq \|T(Tx)\| \tag{3.9}$$

and so $Tx \in \mathcal{M}$ (see also [6, page 9]). Now let T be a hyponormal contraction. If $\|T\| < 1$, then it is trivially a proper contraction. If $\|T\| = 1$ and T has no nontrivial invariant subspace, then $\mathcal{M} = \{x \in \mathcal{H} : \|Tx\| = \|x\|\} = \{0\}$ (actually, if $\mathcal{M} = \mathcal{H}$, then T is an isometry, and isometries have invariant subspaces). Hence T is a proper contraction.

(b) Let $D \geq O$ be the self-commutator of a hyponormal contraction T and let P be the strong limit of $\{D^n\}_{n\geq 1}$ so that $PT = O$ (cf. Proposition 3.5). Suppose T has no nontrivial invariant subspace. Since $\mathcal{N}(P)$ is a nonzero invariant subspace for T whenever $PT = O$ and $T \neq O$, it follows that $\mathcal{N}(P) = \mathcal{H}$. Hence $P = O$ and so D is strongly stable ($D^n \xrightarrow{s} O$). Moreover, since $\bigvee\{T^n x\}_{n\geq 0}$ is a nonzero invariant subspace for T whenever $x \neq 0$, it follows that $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H}$ for each $x \neq 0$ (every nonzero vector in $\mathcal{H}$ is a cyclic vector for T). Thus the Berger-Shaw theorem (see, for instance, [2, page 152]) ensures that D is a trace-class operator so that D is compact (i.e., T is essentially normal). But for compact operators strong stability coincides with uniform stability, and uniform stability always means spectral radius less than one. Hence the nonnegative D is a strict contraction because it is clearly normaloid (i.e., $\|D\| = r(D) < 1$). □

Remark 3.7. According to the Berger-Shaw theorem, a hyponormal contraction without a nontrivial invariant subspace has a trace-class self-commutator D with trace-norm $\|D\|_1 \leq 1$. If $D \neq O$ is not a rank-one operator, then $\|D\| < \|D\|_1 \leq 1$. The above argument ensures the inequality $\|D\| < 1$ whenever a hyponormal contraction has no nontrivial invariant subspace, including the case of a hyponormal contraction with a rank-one self-commutator.

An operator is seminormal if it is hyponormal or cohyponormal. Recall that T^* has a nontrivial invariant subspace if and only if T has, T^* is a proper contraction if and only if T is (Proposition 2.1), and $[T, T^*] = -[T^*, T]$. Thus, the above theorem also holds for cohyponormal contractions. *If a seminormal contraction has no nontrivial invariant subspace, then it is a proper contraction and its self-commutator is a strict contraction*. This prompts the question: can we drop “hyponormal” from the theorem statement? In particular, *is it true that every nonproper contraction has a nontrivial invariant subspace*? Theorems 3.4 and 3.6 yield the following result.

Corollary 3.8. *If a hyponormal contraction T has no nontrivial invariant subspace, then it is either a proper contraction of class $\mathscr{C}_{00}$ or a nonstrict proper contraction of class $\mathscr{C}_{10}$ for which A is a completely nonprojective nonstrict proper contraction. Moreover, its self-commutator $[T^*, T]$ is a strict contraction.*

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C. S. Kubrusly: Catholic University of Rio de Janeiro, 22453-900 Rio de Janeiro, RJ, Brazil

E-mail address: carlos@ele.puc-rio.br

N. Levan: University of California at Los Angeles, Los Angeles, CA 90024-1594, USA

E-mail address: levan@ee.ucla.edu