A GENERALIZATION OF KY FAN'S INEQUALITY

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ABSTRACT. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means. Let $F(x)$ be a C^1 function, $y = y(x)$ an implicit decreasing function defined by $f(x,y) = 0$ and $0 < m < M \le m'$, $n \ge 2$, $x_i \in [m,M]$, $y_i \in [m',M']$. Then for $-1 \le r \le 1$, if $f'_x/f'_y \le 1$, $|(F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y})))/(F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x})))| < (\max_{m' \le \xi \le M'}|F'(\xi)|)/(\min_{m \le \eta \le M}|F'(\eta)|) \cdot M/m'$. A similar result exists for $f'_x/f'_y \ge 1$. By specifying $f(x,y)$ and $F(x)$, we get various generalizations of Ky Fan's inequality. We also present some results on the comparison of $P_{n,s}^\alpha(\mathbf{y}) - P_{n,r}^\alpha(\mathbf{y})$ and $P_{n,s}^\alpha(\mathbf{x}) - P_{n,r}^\alpha(\mathbf{x})$ for $s \ge r$, $\alpha \in \mathbb{R}$.

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1. Introduction. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means defined by $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^n \omega_i x_i^r)^{1/r}$, where w_i, $1 \le i \le n$ are positive real numbers with $\sum_{i=1}^n w_i = 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Here we denote $P_{n,0}(\mathbf{x})$ as $\lim_{r\to 0^+} P_{n,r}(\mathbf{x})$. Let $f(x,y)$ be a real function, we write $f(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ if for all $i = 1, \ldots, n$, $f(x_i, y_i) = 0$.

In this paper, we always assume $x_1 \le x_2 \le \cdots \le x_n$, $f(\mathbf{x},\mathbf{y}) = 0$ and denote $x_1 = m$, $x_n = M$, $y_1 = M'$, $y_n = m'$. We also write $A_n = P_{n,1}(\mathbf{x})$, $G_n = P_{n,0}(\mathbf{x})$, $H_n = P_{n,-1}(\mathbf{x})$, $A_n' = P_{n,1}(\mathbf{y})$, $G_n' = P_{n,0}(\mathbf{y})$, $H_n' = P_{n,-1}(\mathbf{y})$.

The following inequality, originally due to Ky Fan, was first published in the monograph *Inequalities* by Beckenbach and Bellman [6, page 5].

THEOREM 1.1. *For $f(x,y) = x + y - 1$, $x_i \in [0, 1/2]$,*

$$\frac{A_n'}{G_n'} \le \frac{A_n}{G_n} \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Ky Fan's inequality has evoked the interest of several mathematicians and many papers appeared providing new proofs, generalizations and sharpenings of (1.1). We refer the reader to the survey article [3] and the references therein.

Under the same condition of Theorem 1.1, the following additive analogue of (1.1) was proved by Alzer [1].

THEOREM 1.2. *For $f(x,y) = x + y - 1$, $x_i \in [0, 1/2]$,*

$$A_n' - G_n' \le A_n - G_n \tag{1.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Refinements of (1.1) and (1.2), were obtained by Alzer (see [4, 5]) in the following two theorems, respectively.

THEOREM 1.3. *Let $x_i \in (0, 1/2]$ $(i = 1, 2, \ldots, n;\ n \geq 2)$ and $m < M$, then*

$$\frac{m}{1-m} < \frac{A'_n - G'_n}{A_n - G_n} < \frac{M}{1-M}. \tag{1.3}$$

THEOREM 1.4. *Let $x_i \in [a, b]$ $(i = 1, 2, \ldots, n;\ 0 < a < b < 1)$, then*

$$\left(\frac{A_n}{G_n}\right)^{(a/(1-a))^2} < \frac{A'_n}{G'_n} < \left(\frac{A_n}{G_n}\right)^{(b/(1-b))^2}. \tag{1.4}$$

Recently, Mercer obtained the following generalized Ky Fan's inequality [7].

THEOREM 1.5. *For $f(x, y) = x^p + y^p - 1$, $p \geq 1$, $n \geq 2$, $x_i \in [0, 2^{-(1/p)}]$,*

$$P_{n,1}(\mathbf{x})P_{n,0}(\mathbf{y}) \geq P_{n,1}(\mathbf{y})P_{n,0}(\mathbf{x}) \tag{1.5}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Our main goal in this paper is to present a theorem which provides essentially a unified treatment of Theorems 1.1, 1.2, 1.3, 1.4, and 1.5 and also gives new extensions of Ky Fan's inequality. In Section 3, applications to Ky Fan's inequality will be given by specifying the functions $f(x, y)$, $F(x)$.

More generally, we can talk about the comparison of $P_{n,s}^{\alpha}(\mathbf{y}) - P_{n,r}^{\alpha}(\mathbf{y})$ and $P_{n,s}^{\alpha}(\mathbf{x}) - P_{n,r}^{\alpha}(\mathbf{x})$ for real α. The case of $A_n'^{\alpha} - G_n'^{\alpha}$ and $A_n^{\alpha} - G_n^{\alpha}$ was discussed in [5] and we will give some results related to the general case in Section 4.

2. The main theorem

THEOREM 2.1. *Let $F(x)$ be a C^1 function, $y = y(x)$ an implicit decreasing function defined by $f(x, y) = 0$ and $0 < m < M \leq m'$, $n \geq 2$. Then for $-1 \leq r \leq 1$, if $f_x/f_y \leq 1$,*

$$\left| \frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x}))} \right| < \frac{\max_{m' \leq \xi \leq M'} |F'(\xi)|}{\min_{m \leq \eta \leq M} |F'(\eta)|} \cdot \frac{M}{m'}. \tag{2.1}$$

If $f'_x/f'_y \geq 1$,

$$\frac{\min_{m' \leq \xi \leq M'} |F'(\xi)|}{\max_{m \leq \eta \leq M} |F'(\eta)|} \cdot \frac{m}{M'} < \left| \frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - (P_{n,r}(\mathbf{x}))} \right| \tag{2.2}$$

provided the denominators on both sides are nonzero.

PROOF. Since the proofs of (2.1) and (2.2) are very similar, we only prove (2.1) for $r \neq 0$ here, the case $r = 0$ is also similar. We will consider the case $F(x) = x$ first. We define for $1 \leq i \leq n - 1$ and $0 < x < x_{i+1}$:

$$\begin{aligned}
\mathbf{x}_i &= (x, \ldots, x, x_{i+1}, \ldots, x_n), \\
\mathbf{y}_i &= (y, \ldots, y, y_{i+1}, \ldots, y_n), \\
D(\mathbf{x}_i) &= x_n(P_{n,1}(\mathbf{x}_i) - P_{n,r}(\mathbf{x}_i)) - y_n(P_{n,1}(\mathbf{y}_i) - P_{n,r}(\mathbf{y}_i)), \\
g(\mathbf{x}_i) &= x_n P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1} + y_n P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1},
\end{aligned} \tag{2.3}$$

and $D_i(x) = D(\mathbf{x}_i)$, $g_i(x) = g(\mathbf{x}_i)$.

We need to show that $D_1(x_1) > 0$. Differentiation of $D_1(x_1)$ yields

$$\Omega_i^{-1} D_i'(x) = x_n \left(1 - P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1}\right) + y_n \frac{f_x'}{f_y'} \left(1 - P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1}\right)$$

$$\leq x_n \left(1 - P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1}\right) + y_n \left(1 - P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1}\right) \qquad (2.4)$$

$$= x_n + y_n - g_i(x),$$

where $\Omega_i = \sum_{j=1}^i \omega_i$.

Consider

$$g_i'(x) = -(1-r) \sum_{j=i+1}^n \omega_j \left[\left(\frac{P_{n,r}(\mathbf{x}_i)}{x}\right)^{1-2r} \cdot \frac{x_n x_j^r}{x^{r+1}} - \frac{f_x'}{f_y'} \cdot \left(\frac{P_{n,r}(\mathbf{y}_i)}{y}\right)^{1-2r} \cdot \frac{y_n y_j^r}{y^{r+1}} \right]$$

$$\leq -(1-r) \sum_{j=i+1}^N \omega_j \left[\left(\frac{P_{n,r}(\mathbf{x}_i)}{x}\right)^{1-2r} \cdot \frac{x_n x_j^r}{x^{r+1}} - \left(\frac{P_{n,r}(\mathbf{y}_i)}{y}\right)^{1-2r} \cdot \frac{y_n y_j^r}{y^{r+1}} \right] < 0. \qquad (2.5)$$

The last inequality holds, since when $-1 \leq r \leq 1/2$, $k = i+1, \ldots, n$, we have

$$\left(\frac{P_{n,r}(\mathbf{x}_i)}{x}\right)^{1-2r} \geq \left(\frac{P_{n,r}(\mathbf{y}_i)}{y}\right)^{1-2r}, \qquad \frac{x_k}{x} > \frac{y_k}{y},$$

$$\frac{x_n}{y_n} \cdot \left(\frac{x_j}{y_j}\right)^r \geq \left(\frac{x_j}{y_j}\right)^{1+r} > \left(\frac{x}{y}\right)^{1+r}, \qquad (2.6)$$

when $1/2 \leq r \leq 1$, we have

$$\left(\frac{P_{n,r}(\mathbf{x}_i)}{x}\right)^{1-2r} \geq \left(\frac{x_n}{x}\right)^{1-2r}, \qquad \left(\frac{P_{n,r}(\mathbf{y}_i)}{y}\right)^{1-2r} \leq \left(\frac{y_n}{y}\right)^{1-2r},$$

$$\left(\frac{x_n}{y_n}\right)^{2-2r} \cdot \left(\frac{x_j}{y_j}\right)^r \geq \left(\frac{x_j}{y_j}\right)^{2-2r} \cdot \left(\frac{x_j}{y_j}\right)^r = \left(\frac{x_j}{y_j}\right)^{2-r} > \left(\frac{x}{y}\right)^{2-r}. \qquad (2.7)$$

Thus $g_i(x) > g_i(x_{i+1}) = g_{i+1}(x_{i+1}) \geq g_{i+1}(x_{i+2}) \geq \cdots \geq g_{n-1}(x_{n-1}) \geq g_{n-1}(x_n) = x_n + y_n$, which implies $D_i'(x) < 0$ for all $x \in (0, x_{i+1})$, so

$$D_1(x_1) \geq D_1(x_2) = D_2(x_2) \geq D_2(x_3) \geq \cdots \geq D_{n-1}(x_{n-1}) \geq D_{n-1}(x_n) = 0. \qquad (2.8)$$

Since D_i is strictly decreasing, we conclude from $m < M$ that $D_1(x_1) > 0$.

Next, for arbitrary F, by using the mean value theorem, (2.1) is equivalent to

$$\frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x}))} = \frac{F'(\xi)}{F'(\eta)} \cdot \frac{P_{n,1}(\mathbf{y}) - P_{n,r}(\mathbf{y})}{P_{n,1}(\mathbf{x}) - P_{n,r}(\mathbf{x})}, \qquad (2.9)$$

where $m' \leq \xi \leq M'$, $m \leq \eta \leq M$. Taking absolute value and applying the result for $F(x) = x$, we get the desired inequality (2.1). This completes the proof. $\qquad \square$

3. Consequences of Theorem 2.1. In this section, by choosing different functions $f(x, y)$, $F(x)$, we will give several results of generalized Ky Fan's inequality of type (2.1). There are corresponding ones of type (2.2) and we leave the statements to the reader. To simplify expressions, we define

$$\Delta_{s,r,\alpha} = \frac{P_{n,s}^{\alpha}(\mathbf{y}) - P_{n,r}^{\alpha}(\mathbf{y})}{P_{n,s}^{\alpha}(\mathbf{x}) - P_{n,r}^{\alpha}(\mathbf{x})} \tag{3.1}$$

with

$$\Delta_{s,r,0} = \frac{\left(\ln\left(P_{n,s}(\mathbf{y})/P_{n,r}(\mathbf{y})\right)\right)}{\left(\ln\left(P_{n,s}(\mathbf{x})/P_{n,r}(\mathbf{x})\right)\right)}. \tag{3.2}$$

Also in order to include the case of equality for various inequalities in our discussion, we define $0/0 = 1$ from now on.

As a generalization of Theorem 1.3, we have the following corollary.

COROLLARY 3.1. *Let* $f(x, y) = ax^p + by^p - 1$, $0 < a \le b$, $p \ge 1$, $0 < m < M \le (a+b)^{-(1/p)}$, $n \ge 2$. *For* $\alpha \le 1$, *let* $F(x) = \ln x$ *if* $\alpha = 0$, *or* $F(x) = x^{\alpha}$, *otherwise. Then for* $-1 \le r < 1$

$$\Delta_{1,r,\alpha} < \left(\frac{M}{m'}\right)^{2-\alpha}. \tag{3.3}$$

PROOF. This follows from $f'_x/f'_y \le 1$, $\max_{m' \le \xi \le M'} |F'(\xi)| / \min_{m \le \eta \le M} |F'(\eta)| \le (M/m')^{1-\alpha}$. $\qquad\square$

We remark here in Corollary 3.1, the case $\alpha = 0$ gives $P_{n,1}(\mathbf{y})/P_{n,r}(\mathbf{y}) < (P_{n,1}(\mathbf{x})/P_{n,r}(\mathbf{x}))^{(M/m')^2}$, which partially generalizes Theorem 1.4. Also for the case $\alpha = 0$, by only assuming $x_i \in [0, (a+b)^{-(1/p)}]$, we get $P_{n,1}(\mathbf{y})/P_{n,r}(\mathbf{y}) \le P_{n,1}(\mathbf{x})/P_{n,r}(\mathbf{x})$ for $-1 \le r \le 1$ with the equality holding if and only if $x_1 = \cdots = x_n$. This is a generalization of Theorem 1.5.

As a generalization of Theorem 1.2, we have the following corollary.

COROLLARY 3.2. *Let* $f(x, y) = ax^p + by^p - 1$, $0 < a \le b$, $p \ge 1$, $x_i \in [0, (a+b)^{-1/p}]$. *For* $\alpha \le 1$, *let* $F(x) = \ln x$ *if* $\alpha = 0$ *or* $F(x) = x^{\alpha}$, *otherwise. Then for* $-1 \le r \le p$

$$0 < \Delta_{1,r,\alpha} \le 1 \tag{3.4}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

PROOF. The first inequality is trivial, and the second inequality for the case $-1 \le r \le 1$ follows from (3.3) by noticing $M/m' \le 1$. For $1 \le r \le p$, we will prove the case $\alpha = 1$ and the general case follows from using the mean value theorem. We define for $1 \le i \le n-1$ and $x_{n-i} < x \le M$

$$\begin{align} \mathbf{x_i} &= (x_1, \ldots, x_{n-i}, x, \ldots, x), \\ \mathbf{y_i} &= (y_1, \ldots, y_{n-i}, y, \ldots, y), \\ E(\mathbf{x}_i) &= P_{n,r}(\mathbf{x}_i) - A_n(\mathbf{x}_i) - P_{n,r}(\mathbf{y}_i) + A_n(\mathbf{y}_i), \end{align} \tag{3.5}$$

and $E_i(x) = E(\mathbf{x}_i)$.

We need to show that $E_1(x_n) \geq 0$, notice first for $x_1 \leq \cdots \leq x_n$

$$P_{n,r}(\mathbf{x})^{1-r} \cdot x_n^{r-1} + P_{n,r}(\mathbf{y})^{1-r} \cdot y_n^{r-1} \geq 2 \left(\frac{P_{n,r}(\mathbf{x})P_{n,r}(\mathbf{y})}{x_n y_n} \right)^{(1-r)/2}, \qquad (3.6)$$

$$\frac{P_{n,r}(\mathbf{x})P_{n,r}(\mathbf{y})}{x_n y_n} = \left[\sum_{i=1}^{n} \omega_i^2 \left(\frac{x_i y_i}{x_n y_n} \right)^r + \sum_{1 \leq i < j \leq n} \omega_i \omega_j \left(\left(\frac{x_i y_j}{x_n y_n} \right)^r + \left(\frac{x_j y_i}{x_n y_n} \right)^r \right) \right]^{1/r}. \qquad (3.7)$$

Since the function $x[(1/b)(1-ax^p)]^{1/p}$ is increasing for $0 \leq x^p \leq 1/2a$, we have $(x_i y_i/x_n y_n)^r \leq 1$ for all i.

Now for fixed $i \leq j$, define

$$h(x) = 2x_j^r y_j^r - y^r x_j^r - x^r y_j^r, \qquad (3.8)$$

then $h'(x_i) \leq r x_i^{p-1} y_i^{r-p} x_j^r - r x_i^{r-1} y_j^r \leq 0$ since $r \leq p$ and $x_i/y_i \leq 1 \leq y_j/x_j$. Thus $h(x_i) = 2x_j^r y_j^r - y_i^r x_j^r - x_i^r y_j^r \geq h(x_j) = 0$, which implies

$$\left(\frac{x_i y_j}{x_n y_n} \right)^r + \left(\frac{x_j y_i}{x_n y_n} \right)^r \leq \left(\frac{x_i y_j}{x_j y_j} \right)^r + \left(\frac{x_j y_i}{x_j y_j} \right)^r = \left(\frac{x_i}{x_j} \right)^r + \left(\frac{y_i}{y_j} \right)^r \leq 2. \qquad (3.9)$$

Back to (3.7), we have

$$\frac{P_{n,r}(\mathbf{x})P_{n,r}(\mathbf{y})}{x_n y_n} = \left(\left(\sum_{i=1}^{n} \omega_i \frac{x_i^r}{x_n^r} \right) \left(\sum_{i=1}^{n} \omega_i \frac{y_i^r}{y_n^r} \right) \right)^{1/r} \leq \left(\sum_{i=1}^{n} \omega_i^2 + \sum_{i<j} 2\omega_i \omega_j \right)^{1/r} = 1. \qquad (3.10)$$

In particular this gives (where $\Omega_i^{-1} = \sum_{k=n-i+1}^{n} \omega_k$)

$$\Omega_i^{-1} E_i'(x) = P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1} - 1 + \frac{a}{b} \cdot \frac{x^{p-1}}{y^{p-1}} (P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1} - 1)$$

$$\geq P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1} - 1 + P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1} - 1 \geq 0.$$

Thus we deduce

$$E_1(x_n) \geq E_1(x_{n-1}) = E_2(x_{n-1}) \geq \cdots \geq E_{n-1}(x_2) \geq E_{n-1}(x_1) = 0. \qquad (3.11)$$

A close look of the proof tells us the equality holds in (3.4) if and only if $x_1 = \cdots = x_n$ and the proof is completed. $\qquad \square$

As a special case of the above corollary, we have $A_n' - H_n' \leq A_n - H_n$ for generalized weighted means, a proof of this for the special case where $\omega_1 = \cdots = \omega_n$ was given in [2].

We remark here if $0 < b < a$, then in general $P_{n,1}(\mathbf{x}) - P_{n,r}(\mathbf{x})$ and $P_{n,1}(\mathbf{y}) - P_{n,r}(\mathbf{y})$ are not comparable. For example, if we let $a = 2$, $b = 1$, $n = 2$, $\omega_1 = \omega_2$, then when $x_1 = 1/3$, $x_2 = 1/27$, $A_2 - G_2 = A_2' - G_2'$; when $x_1 = 1/3$, $x_2 = 0$, $A_2 - G_2 > A_2' - G_2'$ and when $x_1 = 1/3$, $x_2 = 1/4$, $A_2 - G_2 < A_2' - G_2'$.

The classical case of Ky Fan's inequality corresponds to the choice of $f(x,y) = x + y - 1$, where $f_x'/f_y' = 1$. In this case both inequalities (2.1) and (2.2) hold and combinations of previous results yield the following corollary.

COROLLARY 3.3. *For* $f(x,y) = x + y - 1$, $0 < m < M \leq 1/2$, $n \geq 2$ *then for* $-1 \leq r \leq 1$, $\alpha \leq 1$

$$\left(\frac{m}{1-m}\right)^{2-\alpha} < \Delta_{1,r,\alpha} < \left(\frac{M}{1-M}\right)^{2-\alpha}. \tag{3.12}$$

4. The comparison of $P_{n,s}^{\alpha}(\mathbf{y}) - P_{n,r}^{\alpha}(\mathbf{y})$ **and** $P_{n,s}^{\alpha}(\mathbf{x}) - P_{n,r}^{\alpha}(\mathbf{x})$. In this section, fixing $f(x,y) = x + y - 1$, $x_i \in [0,1/2]$, we give some results relating the comparison of $P_{n,s}^{\alpha}(\mathbf{y}) - P_{n,r}^{\alpha}(\mathbf{y})$ and $P_{n,s}^{\alpha}(\mathbf{x}) - P_{n,r}^{\alpha}(\mathbf{x})$, where $s > r$, $\alpha \in \mathbb{R}$.

Our first result is the following lemma.

LEMMA 4.1. *Given* $s > r$, *if for* $\alpha_0 \in R$, *we have* $\Delta_{s,r,\alpha_0} \leq (\geq)1$ *with equality holding if and only if* $x_1 = \cdots = x_n$, *then for all* $\alpha \leq (\geq)\alpha_0$, $\Delta_{s,r,\alpha} \leq (\geq)1$ *with equality holding if and only if* $x_1 = \cdots = x_n$.

PROOF. Let $i = s, r$, $\mathbf{v} = \mathbf{x}, \mathbf{y}$, we can assume $P_{n,i}(\mathbf{v}) \neq 0$. If $\alpha_0 \neq 0$, write $P_{n,s}^{\alpha}(\mathbf{v}) - P_{n,r}^{\alpha}(\mathbf{v}) = (P_{n,s}^{\alpha_0}(\mathbf{v}))^{\alpha/\alpha_0} - (P_{n,r}^{\alpha_0}(\mathbf{v}))^{\alpha/\alpha_0} = (\alpha/\alpha_0)\xi^{\alpha-\alpha_0}(P_{n,s}^{\alpha_0}(\mathbf{v}) - P_{n,r}^{\alpha_0}(\mathbf{v}))$ with $P_{n,r}(\mathbf{v}) < \xi < P_{n,s}(\mathbf{v})$, where when $\alpha = 0$, we define $(P_{n,i}^{\alpha_0}(\mathbf{v}))^{0/\alpha_0} = \ln P_{n,i}^{\alpha_0}(\mathbf{v})$. By taking the quotient, we get the desired result. If $\alpha_0 = 0$, we write $P_{n,i}^{\alpha}(\mathbf{v}) = e^{\alpha \ln P_{n,i}(\mathbf{v})}$ and proceed similarly. $\square$

For any $s \geq r$, the above lemma enables us to define a number $\sup(\alpha)_{s,r}$ such that $\Delta_{s,r,\alpha} \leq 1$ holds for all $\alpha < \sup(\alpha)_{s,r}$. A special case of this, $\sup(\alpha)_{1,0} = 1$ was determined in [5].

The inequality $\Delta_{s,r,\alpha} \geq 1$ seems unusual but indeed it can happen, even for the case $r = 1$. Indeed we have the following theorem.

THEOREM 4.2. $\Delta_{s,1,\alpha} \geq 1$ *for* $\alpha \geq s \geq 2$; $\Delta_{s,1,\alpha} \leq 1$ *for* $1 < s \leq 2$, $\alpha \leq s$; $\Delta_{1,r,\alpha} \leq 1$ *for* $\alpha \leq r < 0$, *in all cases the equality holds if and only if* $x_1 = \cdots = x_n$.

PROOF. From Lemma 4.1, it suffices to prove the theorem for $\alpha = s$ or r. In this case, for $x \in [0,1/2]$, consider the function $f(x) = x^t - (1-x)^t$ with $f''(x) = t(t-1)(x^{t-2} - (1-x)^{t-2})$. By considering the sign $f''(x)$ for various t and using corresponding Jensen's inequality: $\sum \omega_i f(x_i) \geq (\leq) f(\sum \omega_i x_i)$, the above assertions follow. $\square$

In Theorem 4.2, by restricting $x_i \in [m, M]$, $0 < m < M \leq 1/2$, we will get results similar to Corollary 3.3 and we will leave the statements to the reader.

We have omitted the case $0 < r < 1$ for Theorem 4.2 since we have a stronger result as Corollary 3.2. We point out an interesting phenomena here that when $s = 2$, $\Delta_{2,1,\alpha} \geq (\leq)1$ for $\alpha \geq (\leq)2$. We also remark here the proof of (1.1) follows by applying Jensen's inequality to the function $\ln x - \ln(1-x)$ for $x \in [0,1/2]$.

Notice $P_{n,s}(\mathbf{x}) - P_{n,r}(\mathbf{x}) \geq P_{n,s}(\mathbf{y}) - P_{n,r}(\mathbf{y})$ does not hold for arbitrary real numbers $s \geq r$, for otherwise we will have $P_{n,s}(\mathbf{x})/P_{n,r}(\mathbf{x}) \geq P_{n,s}(\mathbf{y})/P_{n,r}(\mathbf{y})$ which is not true in general according to a nice result by Chen and Wang [8].

THEOREM 4.3. *For arbitrary* $n, s > r$, $x_i \in (0, 1/2)$, $\Delta_{s,r,0} \leq 1$ *holds if and only if* $|r + s| \leq 3$, $2^r/r \geq 2^s/s$ *when* $r > 0$, $r2^r \leq s2^s$ *when* $s < 0$.

By using Lemma 4.1 and Theorem 4.3, we get the following theorem.

THEOREM 4.4. $\sup(\alpha)_{s,r} \geq 0$ *if* $|r+s| \leq 3$, $2^r/r \geq 2^s/s$ *when* $r > 0$, $r2^r \leq s2^s$ *when* $s < 0$. *Moreover,* $\sup(\alpha)_{1,r} = 1$ *for* $-1 \leq r \leq 1$ *and* $\sup(\alpha)_{s,1} = s$ *for* $1 < s \leq 2$.

PROOF. The first assertion follows from Theorem 4.3 and the definition for $\sup(\alpha)_{s,r}$. From Corollary 3.2, we know that $\sup(\alpha)_{1,r} \geq 1$ for $-1 \leq r \leq 1$ and when $\alpha > 1$, let $x_1 = 1/2$, $x_2 = \cdots = x_n = \epsilon$, and

$$f(\omega_1, \epsilon) = (P_{n,1}^{\alpha}(\mathbf{x}) - P_{n,r}^{\alpha}(\mathbf{x})) - (P_{n,1}^{\alpha}(\mathbf{y}) - P_{n,r}^{\alpha}(\mathbf{y})). \tag{4.1}$$

A simple calculation reveals that there exist positive real numbers δ and η such that we have $f(\omega_1, \epsilon) < 0$, if $0 < \omega_1 < \delta$ and $0 < \epsilon < \eta$ and $f(\omega_1, \epsilon) > 0$, if $1 - \delta < \omega_1 < 1$ and $0 < \epsilon < \eta$. Similar conclusion holds for $\sup(\alpha)_{s,1}$ and this completes the proof. □

REFERENCES

[1] H. Alzer, *Ungleichungen für geometrische und arithmetische Mittelwerte* [*Inequalities for geometric and arithmetic means*], Nederl. Akad. Wetensch. Indag. Math. **50** (1988), no. 4, 365–374 (German). MR 90a:26034. Zbl 0655.26010.

[2] ———, *An inequality for arithmetic and harmonic means*, Aequationes Math. **46** (1993), no. 3, 257–263. MR 94h:26021. Zbl 0785.26012.

[3] ———, *The inequality of Ky Fan and related results*, Acta Appl. Math. **38** (1995), no. 3, 305–354. MR 96a:26017. Zbl 0834.26013.

[4] ———, *On an additive analogue of Ky Fan's inequality*, Indag. Math. (N.S.) **8** (1997), no. 1, 1–6. MR 99c:26013. Zbl 0873.26009.

[5] ———, *Some inequalities for arithmetic and geometric means*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 2, 221–228. MR 2000c:26012. Zbl 0923.26013.

[6] E. F. Beckenbach and R. Bellman, *Inequalities*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 30, Springer-Verlag, Berlin, 1961. MR 28#1266. Zbl 0097.26502.

[7] A. McD. Mercer, *Bounds for A-G, A-H, G-H, and a family of inequalities of Ky Fan's type, using a general method*, J. Math. Anal. Appl. **243** (2000), no. 1, 163–173. MR 2001a:26021. Zbl 0944.26027.

[8] Z. Wang and J. Chen, *Generalization of Ky Fan inequality*, Math. Balkanica (N.S.) **5** (1991), no. 4, 373–380 (1992). MR 93g:26030. Zbl 0755.26010.

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