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MATHEMATICAL MORPHOLOGY AND POSET GEOMETRY

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Abstract. The aim of this paper is to characterize morphological convex geometries (resp., antimatroids). We define these two structures by using closure operators, and kernel operators. We show that these convex geometries are equivalent to poset geometries.

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1. Introduction.

Convex geometries are particular closure operators. These objects link the theory of convex sets to combinatorial theory. More precisely poset geometries are basic structures, because any convex geometry can be generated from them [6].

Mathematical morphology, introduced by Serra [7] is a very important tool in image processing and pattern recognition. The framework of mathematical morphology consists in erosions and dilations (resp., Galois functions) which result in morphological closures and kernels. These particular operators are well adapted to algorithmic computation [5].

Theorem 4.2 proves that poset geometries and morphological geometries defined below are equivalent. The convexity is very important in mathematical morphology [5]. So this article tries to demonstrate the relation between combinatorial convexity, mathematical morphology, and image processing.

2. Basic concepts.

In this section, we introduce the definitions that will be needed. We will use x for $\{x\}$, and $\mathbf{P}(\mathbf{S})$ will be the power set of $\mathbf{S}$. Throughout (until Section 6) we will suppose that all sets are finite.

2.1. Convex geometries.

Let $\mathbf{S}$ be a set, consider the family $\mathscr{C}$ of subsets of $\mathbf{S}$ with the following properties:

$$\varnothing \in \mathscr{C}, \qquad S \in \mathscr{C}, \tag{2.1}$$

$$A, B \in \mathscr{C} \quad \text{implies} \quad A \cap B \in \mathscr{C}. \tag{2.2}$$

This family gives rise to the closure operator

$$\phi(X) = \cap\{A \in \mathscr{C},\ X \subseteq A\}. \tag{2.3}$$

Conversely, every closure operator defines a family $\mathscr{C}'$ with the properties (2.1) and (2.2). Elements of $\mathscr{C}$, or elements defined by ϕ will be called convex.

The couple $(\mathbf{S}, \phi)$ is a *convex geometry* if ϕ verifies the anti-exchange axiom

$$\forall x, y \notin \phi(X),\ x \neq y,\ x \in \phi(X \cup y) \quad \text{implies} \quad y \notin \phi(X \cup x). \tag{2.4}$$

In the same way [4]

$$\text{If } \phi(X) \neq S,\ \exists p \in S \setminus \phi(X), \quad \phi(X \cup p) = \phi(X) \cup p. \tag{2.5}$$

2.1.1. Poset geometries. Let P be a partially ordered set and X be a subset of P, we define

$$D_p(X) = \{y \in P,\ y \leq x \text{ for some } x \in X\}, \tag{2.6}$$

(P, D_p) is a convex geometry called poset geometry. Convex geometries give rise to poset geometries which are easily characterized by the following result.

Theorem 2.1. *The convex geometry $(\mathbf{S}, \phi)$ arises from the poset geometry on a poset P if and only if*

$$\phi(A \cup B) = \phi(A) \cup \phi(B) \quad \forall A, B \subseteq S. \tag{2.7}$$

Proof. See [4]. □

2.2. Antimatroid. On the set $\mathbf{S}$ we consider the family $\mathcal{F}$ of subsets of $\mathbf{S}$.

The couple $(\mathbf{S};\mathcal{F})$ is an *antimatroid* if $\mathcal{F}$ verifies the following axioms:

(i) $\emptyset \in \mathcal{F}$, $\mathcal{F}$ is closed under union.

(ii) For $X \in \mathcal{F}$, $X \neq \emptyset$, there exists an $x \in X$ such that $X \setminus x \in \mathcal{F}$.

2.3. Morphological operators

2.3.1. Erosion. An *erosion* is defined by an operator

$$\varepsilon : P(\mathbf{S}) \longrightarrow P(\mathbf{S}) \tag{2.8}$$

with the following properties:

$$\varepsilon(\mathbf{S}) = \mathbf{S}, \qquad A, B \in P(\mathbf{S}), \quad \varepsilon(A \cap B) = \varepsilon(A) \cap \varepsilon(B). \tag{2.9}$$

2.3.2. Dilation. A *dilation* is defined by an operator

$$\delta : P(\mathbf{S}) \longrightarrow P(\mathbf{S}) \tag{2.10}$$

with the following properties:

$$\delta(\emptyset) = \emptyset, \qquad A, B \in P(\mathbf{S}), \quad \delta(A \cup B) = \delta(A) \cup \delta(B). \tag{2.11}$$

Erosion and dilation are linked by the *morphological duality.*

Theorem 2.2. *Let $\mathbf{S}$ be a set, a monotone mapping δ is a dilation if and only if there exists another operator ε such that*

$$\delta(X) \subseteq Y \Longleftrightarrow X \subseteq \varepsilon(Y). \tag{2.12}$$

The operator ε is unique and monotone and given by

$$\varepsilon(X) = \cup\{B, B \in P(S),\ \delta(B) \subseteq X\}, \tag{2.13}$$

we have

$$\delta(X) = \cap\{B, B \in P(S),\ X \subseteq \varepsilon(B)\}. \tag{2.14}$$

Proof. See [7]. □

Remark 2.3. Morphological duality in [5] is called *adjuction* and *Galois function* in [1].

The operator $\phi = \varepsilon \circ \delta$ defines a closure called *morphological closure* and $\phi^* = \delta \circ \varepsilon$ defines a kernel called *morphological kernel*.

2.4. Structuring function and dilation. Let $\mathbf{S}$ be a set, we define a *structuring function* under $\mathbf{P}(\mathbf{S})$ as any mapping δ from $\mathbf{S}$ into $\mathbf{P}(\mathbf{S})$. Let δ be a structuring function, consider $X \in \mathbf{P}(\mathbf{S})$ its image under δ, defined by

$$\delta(X) = \bigcup_{x \in X} \delta(x). \tag{2.15}$$

Thus δ defines a dilation from $\mathbf{P}(\mathbf{S})$ into $\mathbf{P}(\mathbf{S})$. Conversely, any dilation induces a structuring function by (2.15) symbolized again by δ.

3. Convex geometries, antimatroid, and duality. In the literature antimatroids are defined as (i) and (ii) above, and rarely with kernel operators.

On the set $\mathbf{S}$ we consider a family $\mathcal{F}$ of subsets of $\mathbf{S}$ verifying

$$\emptyset \in \mathcal{F}, \qquad \mathbf{S} \in \mathcal{F}, \tag{3.1}$$

$$A, B \in \mathcal{F} \quad \text{implies} \quad A \cup B \in \mathcal{F}. \tag{3.2}$$

This family gives rise to the kernel operator

$$\phi^*(X) = \cup\{A \in \mathcal{F},\ A \subseteq X\}. \tag{3.3}$$

Conversely, every kernel operator results in the family $\mathcal{F}$ with the properties (3.1) and (3.2).

Lemma 3.1. *The pair $(\mathbf{S}, \phi^*)$ is an antimatroid if ϕ^* verifies the following axiom:*

$$\textit{For } \phi^*(X) \neq \emptyset,\ \exists\, p \in \phi^*(X); \quad \phi^*(X \setminus p) = \phi^*(X) \setminus p. \tag{3.4}$$

Proof. It is sufficient to verify that (ii) is equivalent to (3.4). We suppose (ii) is true.

Since $\phi^*(X \setminus p) \subseteq X \setminus p$ so $p \notin \phi^*(X \setminus p)$. From monotonicity, $X \setminus p \subset X$ implies $\phi^*(X \setminus p) \subset \phi^*(X)$. So $\phi^*(X \setminus p) \subseteq \phi^*(X) \setminus p$.

Conversely, $\phi^*(X) \setminus p \subseteq X \setminus p$ implies $\phi^*(\phi^*(X) \setminus p) \subseteq \phi^*(X \setminus p)$. From (ii), $\phi^*(X) \setminus p \subseteq \phi^*(X \setminus p)$, consequently, $\phi^*(X) \setminus p = \phi^*(X \setminus p)$.

The converse is obvious. □

We have defined convex geometries (resp., antimatroids) thanks to closure operators (resp., kernel operators). These concepts are dual. Indeed, in Boolean lattices, the complementation $X \to X^c$ induces a duality that produces a correspondence between any closure ϕ from $\mathbf{P}(\mathbf{S})$ into $\mathbf{P}(\mathbf{S})$, and its dual ϕ^*, as defined by

$$\phi^*(X) = \phi^c(X^c) \tag{3.5}$$

with ϕ^* as a kernel. So suppose that $(\mathbf{S};\phi)$ is a convex geometry; since ϕ is closed under intersection, ϕ^* is closed under union, and ϕ verifies (2.5). We have

$$\begin{aligned}\phi^*(X)\setminus p &= \phi^c(X^c)\setminus p = \phi^c(X^c)\cap p^c = [\phi(X^c)\cup p]^c \\ &= [\phi(X^c\cup p)]^c = [\phi[(X\cap p^c)^c]]^c = [\phi((X\setminus p)^c)]^c \\ &= \phi^*(X\setminus p).\end{aligned} \tag{3.6}$$

So we have a new proof of the well-known result due to Bjorner [8].

Theorem 3.2. *The couple $(\mathbf{S},\phi)$ is a convex geometry if and only if $(\mathbf{S},\phi^*)$ is an antimatroid.*

4. Poset geometries and mathematical morphology.

The following theorem gives a characterization of convex geometries in the particular case of morphological closure operators (these convex geometries will be called morphological geometries). We show that these morphological geometries are equivalent to poset geometries.

We say that $(\mathbf{S};\delta)$ is separated in a primary sense, if δ verifies the following two properties:

(a) For any family $(x_i)_{i\in I}$ of elements of $\mathbf{S}$ and for any element $x \in S$ verifying $\delta(x) \subseteq \bigcup_{i\in I}\delta(x_i)$, there exists $j \in I$ such that $\delta(x) \subseteq \delta(x_j)$.

(b) $\delta(x) = \delta(y)$ is equivalent to $x = y$ for any $x, y \in S$.

We will call *morphological geometry* (resp., *morphological poset*) any convex geometry (resp., poset geometry) $(\mathbf{S};\phi)$ such that $\phi = \varepsilon\circ\delta$.

For any dilation we can canonically associate a binary relation defined by xRy is equivalent to $x \in \delta(y)$ for $x, y \in S$. Another interesting binary relation is given by $xR'y$ is equivalent to $\delta(x) \subseteq \delta(y)$, for $x, y \in S$. We are now going to clarify the relations between R and R'.

Lemma 4.1. *Let δ be a dilation on S, R its binary relation canonically associated with it. The following two assertions are equivalent:*

(1) *R is reflexive and transitive.*

(2) *xRy is equivalent to $\delta(x) \subseteq \delta(y)$.*

Proof. The proof that (2) implies (1) is trivial.

Supposing R reflexive and transitive. Consequently, $\delta(x) \subseteq \delta(y)$ and $x \in \delta(x)$ imply that $x \in \delta(y)$ which is equivalent to xRy. So $\delta(x) \subseteq \delta(y)$ implies xRy. Supposing xRy. For any $z \in \delta(x)$, we have zRy by transitivity, that implies $z \in \delta(y)$, consequently $\delta(x) \subseteq \delta(y)$. □

Theorem 4.2. *Let $\mathbf{S}$ be a set and $\phi = \varepsilon\circ\delta$, a morphological closure. The following three assertions are equivalent:*

(i) *δ separates $\mathbf{S}$ in a primary sense.*

(ii) *$(\mathbf{S};\phi)$ is a morphological poset.*

(iii) *$(\mathbf{S};\phi)$ is a poset geometry.*

Proof. Suppose that δ separates $\mathbf{S}$ a primary sense. The relation xRy is equivalent to $\delta(x) \subseteq \delta(y)$ is an order relation. Let $\phi(X) = \varepsilon\circ\delta(X)$, it is easy to verify that

$$\phi(X) = \{y \in \mathbf{S},\ \delta(y) \subseteq \delta(X)\}. \tag{4.1}$$

We have two cases.

• There exists $x \in X$ such that $\delta(x) \subseteq \delta(y)$, which is equivalent to xRy.

• There exists a family (x_i), $i \in \{1,2,\dots,n\}$ $(n > 1)$ of elements of X such that $\delta(y) \subseteq \delta(x_1) \cup \delta(x_2) \cup \cdots \cup \delta(x_n)$, but from our hypothesis δ separates S a primary sense, consequently $\delta(y) \subseteq \delta(x_1)$ or $\delta(y) \subseteq \delta(x_2)$ or $\cdots$ or $\delta(y) \subseteq \delta(x_n)$, so yRx_1 or yRx_2 or $\cdots$ or yRx_n, which leads to

$$\phi(X) = \{y \in S,\ \delta(y) \subseteq \delta(X)\} = \{y \in S,\ yRx \text{ for } x \in X\}. \tag{4.2}$$

This set is an ideal of $\mathbf{S}$. So any closed set from ϕ gives rise to an ideal of S. Under the hypothesis (i), R is an order relation, it is easy to show that any ideal of S gives rise to a closed set. The couple $(\mathbf{S};\phi)$ is a poset geometry.

Suppose that $(\mathbf{S};\phi)$ is a morphological poset. We have $u, v \notin \phi(X)$, $v \neq u$ and $u \in \phi(X \cup v)$ involves $v \notin \phi(X \cup u)$, we have $v \notin \{y, \delta(y) \subseteq \delta(X) \cup \delta(u)\}$ but $u \in \{y, \delta(y) \subseteq \delta(X) \cup \delta(v)\}$. So the condition $\delta(u) = \delta(v)$ is equivalent to $u = v$, which means that the axiom (b) is verified.

Moreover, we have $\phi(X) = \{y \in S,\ \delta(y) \subseteq \delta(X)\} = \{y \in S,\ \delta(y) \subseteq \bigcup_{x \in X} \delta(x)\}$ which implies that for any $y \in \mathbf{S}$, $\delta(y) \subseteq \delta(X)$ there exists $x \in X$ such that $\delta(y) \subseteq \delta(x)$. Consequently δ separates S in a primary sense.

If $(\mathbf{S},\phi)$ is a morphological poset, a fortiori it is a poset geometry.

Suppose now that $(\mathbf{S},\phi)$ is a poset geometry. We have $\phi(X) = \{y \in \mathbf{S},\ y \leq x$ for $x \in X\}$. From Lemma 4.1, we can associate a dilation to this relation δ, so $\phi(X) = \{y \in \mathbf{S},\ y \leq x \text{ for } x \in X\} = \{y \in \mathbf{S},\ \delta(y) \subseteq \delta(x) \text{ for } x \in X\} = \{y \in \mathbf{S},\ \delta(y) \subseteq \delta(X)\}$ which is a morphological poset. □

5. Poset geometry, mathematical morphology, and topology. We are now going to extend this result to the case where $\mathbf{S}$ is an infinite set. Before we will give some definitions and remarks.

A topological space is an *Alexandroff space* if the intersection of any family of open sets is open (resp., the union of any family of closed sets is closed) [3].

A topological space $\mathbf{S}$ is a T_0-*space* whenever, for x and y are distinct elements of $\mathbf{S}$ if $x \in \phi(y)$ then $y \notin \phi(x)$.

Let $\mathbf{S}$ be an infinite set, we now rewrite the axioms of closure operators [2]. The family of subsets $\mathscr{C}$ of $\mathbf{S}$, with the following properties:

(i) $\emptyset \in \mathscr{C}$, $S \in \mathscr{C}$,

(ii) $\mathscr{C}$ is preserved under intersection,

(iii) $\mathscr{C}$ is preserved by nested union,

is equivalent to give a closure operator ϕ with the following condition called *algebraic condition*:

$$\forall x \in \phi(X) \text{ there exists a finite set } F \text{ include in } X \text{ such that } x \in \phi(F). \tag{5.1}$$

Families satisfying (iii) are the inductive systems. This axiom is equivalent to the algebraic condition [2].

The morphological operator can be easily extended to the infinite case [5]. From these remarks, we have the following extended result.

Theorem 5.1. *Let* $\mathbf{S}$ *be an infinite set and* $\phi = \varepsilon \circ \delta$*, a morphological closure. The following four assertions are equivalent:*

(i) δ *separates* S *in a primary sense.*

(ii) $(\mathbf{S}; \phi)$ *is a morphological poset.*

(iii) $(\mathbf{S}; \phi)$ *is a poset geometry.*

(iv) $(\mathbf{S}; \phi)$ *is a* T_0*-Alexandroff space.*

Proof. It is easy to adapt the proof of Theorem 4.2 to prove the equivalences between (ii), (iii), and (iv). So we will just prove the equivalence between (i) and (ii).

Suppose that $(\mathbf{S}; \phi)$ is a T_0-Alexandroff space, $x, y \in S$, $x \neq y$ with $x, y \notin \phi(X)$, and $x \in \phi(X \cup \{y\})$, implies that $x \in \phi(y)$, but $(\mathbf{S}; \phi)$ is a T_0-space, so we have $y \notin \phi(X \cup x)$. Consequently,

$$\forall x,\ y \notin \phi(X),\ x \neq y,\ x \in \phi(X \cup y) \quad \text{implies} \quad y \notin \phi(X \cup x). \tag{5.2}$$

The anti-exchange axiom is verified.

Suppose that $x \in \phi(X)$, $(S; \phi)$ being an Alexandroff space there exists $y \in X$ such that $x \in \phi(y)$. Consequently, for all $x \in \phi(X)$, there exists $F \subseteq X$, F being a finite set such that $x \in \phi(F)$. So $(\mathbf{S}; \phi)$ is a poset geometry.

Suppose now that $(\mathbf{S}; \phi)$ is a poset geometry. For all $x \in \phi(X)$, there exists $F \subseteq X$, F, being a finite set such that $x \in \phi(F)$. So for all $x \in \phi(X)$ implies that $x \in \phi(y)$ for some $y \in X$. It is easy to deduce that $(\mathbf{S}; \phi)$ is an Alexandroff space.

Moreover, ϕ verifies the anti-exchange axiom, from this axiom it is easy to deduce that $(\mathbf{S}; \phi)$ is a T_0-space. □

Corollary 5.2. *Let* $\mathbf{S}$ *be an infinite space and let* ϕ *be a closure operator. The couple* $(\mathbf{S}; \phi)$ *is a convex geometry if and only if for all* $X \subseteq \mathbf{S}$ $(\mathbf{S} \setminus \phi(X); \psi)$ *is a* T_0*-Alexandroff space where* $\psi(A) = \bigcup_{y \in A} \phi(\phi(X) \cup y) \cap \mathbf{S} \setminus \phi(X)$.

Proof. Suppose that $(\mathbf{S}; \phi)$ is a convex geometry.

We will just prove that $\psi(\psi(A)) = \psi(A)$. The other axioms can be easily shown.

$$\begin{aligned}
\psi(\psi(A)) &= \bigcup_{y \in A} \left(\bigcup_{z \in \phi(\phi(X) \cup y) \cap \mathbf{S} \setminus \phi(X)} \phi(\phi(X) \cup y) \cap \mathbf{S} \setminus \phi(X) \right) \\
&\subseteq \bigcup_{y \in A} \phi(\phi(X) \cup (\phi(\phi(X) \cup y) \cap \mathbf{S} \setminus \phi(X)) \cap \mathbf{S} \setminus \phi(X)) \\
&= \bigcup_{y \in A} \phi(\phi(\phi(X) \cup y) \cap \mathbf{S} \setminus \phi(X)) \\
&= \bigcup_{y \in A} \phi(\phi(X) \cup y) \cap \mathbf{S} \setminus \phi(X) = \psi(A).
\end{aligned} \tag{5.3}$$

So $\psi(\psi(A)) = \psi(A)$.

Let x, y be two distinct elements of $\mathbf{S} \setminus \phi(X)$ by application of anti-exchange axiom, we have: if $x \in \psi(y)$ then $x \notin \psi(x)$. Hence $(\mathbf{S} \setminus \phi(X); \psi)$ is a T_0-Alexandroff space.

Suppose that for all $X \subseteq \mathbf{S}$ $(\mathbf{S} \setminus \phi(X); \psi)$ is a T_0-Alexandroff space. Thanks to Theorem 3.2 it is a convex geometry and for all $X \subseteq \mathbf{S}$ the anti-exchange axiom is preserved. So $(\mathbf{S}; \phi)$ is a convex geometry. □

From Theorem 3.2 and Corollary 5.2 we have the following known result [4].

Corollary 5.3. *The couple* $(\mathbf{S};\phi)$ *is a convex geometry if and only if the relation* $x \in \phi(\phi(X)\cup y)\cap \mathbf{S}\setminus\phi(X)$ *is a partial order on* $\mathbf{S}\setminus\phi(X)$*, for all* $X\subseteq\mathbf{S}$ *.*

6. Concluding remarks.

We can define a partial order on the set of T_0-Alexandroff spaces (denoted by T_0-Alex$(CG(\mathbf{S}))$ on a convex geometry $(\mathbf{S},\phi)$ by

$$(\mathbf{S}\setminus\phi(X_1),\,\psi_1)\le(\mathbf{S}\setminus\phi(X_2),\,\psi_2)\Longleftrightarrow\mathbf{S}\setminus\phi(X_1)\subseteq\mathbf{S}\setminus\phi(X_2), \tag{6.1}$$

$$\forall x\in\mathbf{S}\setminus\phi(X_1),\quad \psi_1(x)\subseteq\psi_2(x). \tag{6.2}$$

The set T_0-Alex$(CG(\mathbf{S}))$ contains a maximal element. Indeed, let $(\mathbf{S}\setminus\phi(X_i),\psi_i)_{i\in I}$ be a chain, it is sufficient to prove that $(\bigcup_{i\in I}\mathbf{S}\setminus\phi(X_i);\psi)$ (with $\psi(A)=\bigcup_{i\in I}[\bigcup_{y\in A}\phi(\phi(X_i)\cup y)\cap(\mathbf{S}\setminus\phi(X_i))_{i\in I}])$ is a T_0-Alexandroff and to apply Zorn's lemma. This remark leads us to the following problem.

Problem 6.1. Can we classify any convex geometry $(\mathbf{S},\phi)$ from the maximal elements of T_0-Alex$(CG(\mathbf{S}))$?

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