IJMMS 29:2 (2002) 115–119
PII. S0161171202003393
http://ijmms.hindawi.com

PYTHAGOREAN IDENTITY FOR POLYHARMONIC POLYNOMIALS

ALLAN FRYANT and MURALI KRISHNA VEMURI

Received 13 May 1999

Polyharmonic polynomials in n variables are shown to satisfy a Pythagorean identity on the unit hypersphere. Application is made to establish the convergence of series of polyharmonic polynomials.

2000 Mathematics Subject Classification: 31B99.

1. Introduction. Let L_n^k denote the vector space of real homogeneous polynomial solutions of degree k of Laplace's equation

$$\Delta u = 0, \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1.2}$$

Such polynomials are called spherical harmonics. As shown in [9, pages 140–141],

$$\dim L_n^k = d_n^k = (n+k-2)\frac{(n+2k-3)!}{k!(n-2)!}. \tag{1.3}$$

Suppose that $\{y_j^k(x)\}_{j=1}^{d_n^k}$ is an orthonormal basis for L_n^k, where orthonormality is with respect to the inner product

$$\langle f,g\rangle = \int_{\sum_1} f(x)g(x)dx \tag{1.4}$$

on the unit sphere $\sum_1 : x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. It is well known (cf. [9, page 144]) that for all $s \in \sum_1$,

$$\sum_{j=1}^{d_n^k} [y_k^j(s)]^2 = \omega_n d_n^k, \tag{1.5}$$

where ω_n is the surface area of the unit sphere $\sum_1$ in $\mathbb{R}^n$. We call (1.5) the Pythagorean identity for spherical harmonics, since it generalizes the Pythagorean theorem

$$\sin^2\theta + \cos^2\theta = 1. \tag{1.6}$$

Solutions of partial differential equation

$$\Delta^m u = 0, \tag{1.7}$$

where Δ is the Laplacian (1.2) and m is a positive integer, are called polyharmonic functions. In the case $m = 2$, such functions are called biharmonic and are used to model the bending of thin plates (for a brief history of this application, see [7, pages 416 and 432–443]).

We show here that homogeneous polyharmonic polynomials satisfy a Pythagorean identity on $\sum_1$ and use this identity to establish the convergence of polyharmonic polynomial series.

2. Pythagorean identity. Let J_n^k denote the vector space of real homogeneous polynomial solutions of the partial differential equation (1.7). Since Δ^m is a homogeneous differential operator of order $2m$, using a standard argument (cf. [5, Theorem 1]) we find that

$$\dim J_n^K = b_n^k = \binom{n-1+k}{k} - \binom{n-1+k-2m}{k-2m}. \tag{2.1}$$

In the vector space J_n^k, we introduce the Calderón inner product [1]

$$(p,q) = p\left(\frac{\partial}{\partial x}\right)q(x), \tag{2.2}$$

where

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right), \qquad p\left(\frac{\partial}{\partial x}\right) = p\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right). \tag{2.3}$$

Theorem 2.1. *Suppose that $\{Q_k^j(x)\}_{j=1}^{b_n^k}$ is an orthonormal basis for the vector space J_n^k of homogeneous polyharmonic polynomials of degree k, where orthonormality is with respect to the inner product (2.2). Then for all $s = (s_1, s_2, \dots, s_n) \in \sum_1$, the unit sphere in $\mathbb{R}^n$,*

$$\sum_{j=1}^{b_n^k} [Q_k^j(s)]^2 = \gamma_n^k, \tag{2.4}$$

where γ_n^k is a constant depending only on n and k.

Proof. A modification in the argument used for spherical harmonics suffices: fix a point $y \in \mathbb{R}^n$ and consider the linear functional $L : J_n^k \to \mathbb{R}$ defined by

$$L(p) = p(y). \tag{2.5}$$

Since J_n^K is a finite-dimensional inner product space, there exists a unique $Z_y \in J_n^k$ such that

$$L(p) = (p(x), Z_y(x)), \tag{2.6}$$

for all $p \in J_n^k$ (i.e., all finite-dimensional inner product spaces are self-dual). Further, since $\{Q_k^j(x)\}_{j=1}^{b_n^k}$ is an orthonormal basis for J_n^k,

$$Z_y(x) = \sum_{j=1}^{b_n^k} (Z_y(x), Q_k^j(x))Q_k^j(x). \tag{2.7}$$

But, by the defining property of Z_y,

$$(Z_y(x), Q_k^j(x)) = Q_k^j(y). \tag{2.8}$$

Hence

$$Z_y(x) = \sum_{j=1}^{b_n^k} Q_k^j(y) Q_k^j(x). \tag{2.9}$$

Since the choice of $y \in \mathbb{R}^n$ was arbitrary, $Z_y(x)$ is a function of the two variables $x, y \in \mathbb{R}^n$; thus, we write

$$Z(x,y) = Z_y(x) = \sum_{j=1}^{b_n^k} Q_k^j(x) Q_k^j(y). \tag{2.10}$$

The Calderón inner product (2.2) is invariant with respect to rotations; that is, if $O : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation, then $(f(x), g(Ox)) = (f(O^{-1}x), g(x))$. Suppose $p(x) \in J_n^k$. Then

$$(p(x), Z(Ox, Oy)) = (p(O^{-1}x), Z(x, Oy)) = (q(x), Z(x, Oy)), \tag{2.11}$$

where $q(x) = p(O^{-1}x)$. Since rotations are invariant transformations for the Laplacian, it follows that $q(x) \in J_n^k$. Thus, by the defining property of $Z(x,y)$,

$$(q(x), Z(x, Oy)) = q(Oy). \tag{2.12}$$

But $q(Oy) = p(O^{-1}Oy) = p(y)$. Thus, we have shown that

$$(p(x), Z(Ox, Oy)) = p(y). \tag{2.13}$$

From the uniqueness of the representation of linear functionals, it follows that

$$Z(Ox, Oy) = Z(x, y), \tag{2.14}$$

for all $x, y \in \mathbb{R}^n$. In particular,

$$Z(Ox, Ox) = Z(x, x), \tag{2.15}$$

for every rotation O. Since every point on the unit sphere $\sum_1$ is the image under rotation for some fixed point on $\sum_1$, the equality (2.15) implies that $Z(x,x)$ is constant on $\sum_1$. That is,

$$\sum_{j=1}^{b_n^k} Q_k^j(s) Q_k^j(s) = C, \tag{2.16}$$

a constant, for all $s \in \sum_1$. □

3. Polyharmonic polynomial series. Pythagorean identities have been used to establish the convergence of series of spherical harmonics [4], as well as series of orthonormal homogeneous polynomials in several real variables in general [3]. We obtain here convergence for series of polyharmonic polynomials.

Theorem 3.1. *Suppose that* $\{Q_k^j(x)\}_{j=1}^{b_n^k}$ *are sets of orthonormal polyharmonic polynomials in* $\mathbb{R}^n$ *of degree* k, $k = 0,1,2,\ldots$. *Then the series*

$$\sum_{k=0}^{\infty}\sum_{j=1}^{b_n^k} a_{kj}Q_k^j(x) \tag{3.1}$$

converges absolutely and uniformly on compact subsets of the open ball $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} < R$, *where*

$$R^{-1} = \limsup_{k\to\infty}\left(\sqrt{\gamma_n^k}\|a_k\|\right)^{1/k}, \qquad \|a_k\| = \left(\sum_{j=1}^{b_n^k} a_{kj}^2\right)^{1/2}, \tag{3.2}$$

and γ_n^k *is the Pythagorean constant appearing in (2.4).*

Proof. Since each of the polynomials Q_k^j is homogeneous of degree k, we have $Q_k^j(x) = r^k Q_k^j(x/r)$, where $r = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Thus

$$\begin{aligned}\left|\sum_{k=0}^{\infty}\sum_{j=1}^{b_n^k} a_{kj}Q_k^j(x)\right| &= \left|\sum_{k=0}^{\infty} r^k\sum_{j=1}^{b_n^k} a_{kj}Q_k^j\left(\frac{x}{r}\right)\right| \\ &\le \sum_{k=0}^{\infty} r^k\left|\sum_{j=1}^{b_n^k} a_{kj}Q_k^j\left(\frac{x}{r}\right)\right|,\end{aligned} \tag{3.3}$$

by the Cauchy-Schwarz inequality

$$\left|\sum_{k=0}^{\infty}\sum_{j=1}^{b_n^k} a_{kj}Q_k^j(x)\right| \le \sum_{k=0}^{\infty} r^k\left(\sum_{j=1}^{b_n^k} a_{kj}^2\right)^{1/2}\left(\sum_{j=1}^{b_n^k} Q_k^j\left(\frac{x}{r}\right)\right)^{1/2}. \tag{3.4}$$

Appealing now to the Pythagorean identity (2.4), we find that

$$\left|\sum_{k=0}^{\infty}\sum_{j=1}^{b_n^k} a_{kj}Q_k^j(x)\right| = \sum_{k=0}^{\infty} r^k\|a_k\|\sqrt{\gamma_n^k}, \tag{3.5}$$

from which the desired result is immediate. □

Let H_n^k denote the vector space of homogeneous polynomials of degree k in $\mathbb{R}^n$. Since every orthonormal basis of J_n^k be extended to an orthonormal basis of H_n^k, it follows from [2, Theorem 3] that

$$\gamma_n^k \le \frac{1}{k!}. \tag{3.6}$$

Thus,

$$R^{-1} = \limsup_{k\to\infty}\left(\sqrt{\gamma_n^k}\|a_k\|\right)^{1/2} \le \limsup_{k\to\infty}\left(\frac{\|a_k\|}{\sqrt{k!}}\right)^{1/k} = \rho^{-1}, \tag{3.7}$$

and appealing to the result of Theorem 3.1 we find that the polyharmonic polynomial series (3.1) converges absolutely and uniformly on compact subsets of the open ball $|x| < \rho$. We predict that the evaluation of the Pythagorean constant γ_n^k will show that such convergence actually obtains within a somewhat larger ball.

In [11], it was shown that, in the space of homogeneous harmonic polynomials L_n^k, the Calderón inner product (2.2) is a constant multiple of the inner product (1.4). That is,

$$(p,q) = c_n^k \langle p,q \rangle, \tag{3.8}$$

for all $p,q \in L_n^k$, where c_n^k is a constant depending only on n and k. Thus, the Pythagorean identity for spherical harmonics (1.5) is a special case ($m = 1$) of the result of Theorem 2.1.

The Pythagorean identity for spherical harmonics is also a special case of the addition formula for spherical harmonics [9, page 149] and [8, page 268]. This leads us to conjecture that the homogeneous polyharmonic polynomials satisfy a similar addition formula, from which Theorem 2.1 might follow as an immediate consequence. Such a development could include a significant generalization of the ultraspherical polynomials [6, 10].

References

[1] A. P. Calderón, *Integrales singulares y sus aplicaciones a ecuaciones diferenciales hiperbolicas [Singular Integrals and their Applications to Hyperbolic Differential Equations]*, Cursos y Seminarios de Matemática, Fasc. 3. Universidad de Buenos Aires, Buenos Aires, 1960 (Spanish).

[2] A. Fryant, *Multinomial expansions and the Pythagorean theorem*, Proc. Amer. Math. Soc. **124** (1996), no. 7, 2001–2004.

[3] A. Fryant, A. Naftalevich, and M. K. Vemuri, *Orthogonal homogeneous polynomials*, Adv. in Appl. Math. **22** (1999), no. 3, 371–379.

[4] A. Fryant and H. Shankar, *Bounds on the maximum modulus of harmonic functions*, Math. Student **55** (1987), no. 2-4, 103–116 (1989).

[5] A. Fryant and M. K. Vemuri, *Wave polynomials*, Tamkang J. Math. **28** (1997), no. 3, 205–209.

[6] A. J. Fryant, *Ultraspherical expansions and pseudo analytic functions*, Pacific J. Math. **94** (1981), no. 1, 83–105.

[7] V. Maz'ya and T. Shaposhnikova, *Jacques Hadamard, a Universal Mathematician*, History of Mathematics, vol. 14, American Mathematical Society, Rhode Island, 1998.

[8] G. Sansone, *Orthogonal Functions*, Pure and Applied Mathematics, vol. 9, Interscience Publishers, New York, 1959.

[9] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series, no. 32, Princeton University Press, New Jersey, 1971.

[10] G. Szegö, *Orthogonal Polynomials*, 3rd ed., American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, Rhode Island, 1967.

[11] M. K. Vemuri, *A simple proof of Fryant's theorem*, SIAM J. Math. Anal. **26** (1995), no. 6, 1644–1646.

Allan Fryant: 603F Simpson Street, Greensboro, NC 27401, USA

Murali Krishna Vemuri: Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA

E-mail address: mkvemuri@math.syr.edu