IJMMS 29:6 (2002) 355–359
PII. S016117120200652X
http://ijmms.hindawi.com

MULTIPLIERS ON $L(S)$, $L(S)^{**}$, AND $LUC(S)^*$ FOR A LOCALLY COMPACT TOPOLOGICAL SEMIGROUP

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Received 16 January 2001

We study compact and weakly compact multipliers on $L(S)$, $L(S)^{**}$, and $LUC(S)^*$, where the latter is the dual of $LUC(S)$. We show that a left cancellative semigroup S is left amenable if and only if there is a nonzero compact (or weakly compact) multiplier on $L(S)^{**}$. We also prove that S is left amenable if and only if there is a nonzero compact (or weakly compact) multiplier on $LUC(S)^*$.

2000 Mathematics Subject Classification: 43A20.

1. Introduction. Let S be a locally compact, Hausdorff topological semigroup. Let $M(S)$ be the space of all complex Borel measures on S. It is known that $M(S) = C_0(S)^*$, therefore, $M(S)$ is a Banach space and with convolution $\mu * \nu(\psi) = \iint \psi(xy)d\mu(x)d\nu(y)$ $(\mu,\nu \in M(S),\ \psi \in C_0(\psi))$, $M(S)$ is a Banach algebra. The subalgebra $L(S)$ of $M(S)$ is defined by $L(S) = \{\mu \in M(S) \mid x \to |\mu| * \delta_x,\ x \to \delta_x * |\mu|$ from S to $M(S)$ are norm continuous$\}$ [1]. A semigroup S is called foundation if $S = \overline{\bigcup_{\mu\in L(S)} \operatorname{supp}\mu}$. A trivial example is a topological group and in this case $L(S) = L^1(G)$. Let $C_b(S)$ be the set of all bounded continuous function on S. Let $LUC(S) = \{f \in C_b(S) \mid x \to l_x f$ is norm continuous$\}$, $RUC(S) = \{f \mid f \in C_b(S),\ x \to r_x f$ is norm continuous$\}$ where $l_x f(y) = f(xy)$, $r_x f(y) = f(yx)$. When S is foundation, it is known that $L(S)$ has a bounded approximate identity [1], and therefore, the multiplier algebra of $L(S)$ is $M(S)$ [4]. Let $L(S)^*$ and $L(S)^{**}$ be the first and second duals of $L(S)$ and similarly, $M(S)^*$ and $M(S)^{**}$ be the first and second duals of $M(S)$. We also use the notation $LUC(S)^*$, $RUC(S)^*$ for the duals of $LUC(S)$, and $RUC(S)$, respectively. The subalgebras $LUC(S)$ and $RUC(S)$ are Banach C^*-subalgebras of $C_b(S)$. With Arens product, $L(S)^{**}$ and $M(S)^{**}$ are Banach algebra. Also, with the same type product $LUC(S)^*$ is a Banach algebra. In this paper, among other things, we show that when S is a left cancellative foundation semigroup, then S is left (right) amenable if and only if there is a nonzero left (right) compact or weakly compact multiplier on $L(S)^{**}$ (or $LUC(S)^*$).

2. Preliminaries. For a Banach algebra A, we denote by A^* and A^{**} the first and second dual of A, respectively. On A^{**} we define the first Arens product by

$$\langle mn, f\rangle = \langle m, nf\rangle, \qquad \langle nf, a\rangle = \langle n, fa\rangle, \qquad \langle fa, b\rangle = f(ab) \tag{2.1}$$

$(m,n \in A^{**};\ f \in A^*;\ a,b \in A)$. With this product A^{**} is a Banach algebra. We can also define a similar product on $LUC(S)^*$ such that $\langle mn, f\rangle = \langle m, nf\rangle$, $nf(x) = n(l_x f)$, $l_x f(y) = f(xy)$ $(m,n \in LUC(S)^*;\ f \in LUC(S);\ x,y \in S)$. Clearly, $LUC(S)^*$ is a Banach algebra. A linear map on a Banach algebra A is called a multiplier if

$T(xy) = T(x)y = xT(y)$ $(x,y \in A)$. The left (right) multiplier on $L(S)^{**}$ is defined by $l_m(n) = mn$, $(l_m(n) = nm)$. In general, $LUC(S)$ and $RUC(S)$ are different subalgebras of $C_b(S)$ and $LUC(S) = RUC(S)$ if and only if $LUC(S)$ (resp., $RUC(S)$) is right (resp., left) introverted, (see [2, Theorem 4.4.5]). For example, if S is a compact semitopological semigroup or a totally bounded topological group, then $LUC(S) = RUC(S)$ [2].

The semigroup S is called left amenable if there is a positive functional m on $LUC(S)$ such that $m(l_a f) = m(f)$, $\|m\| = 1$ for all $f \in LUC(S)$, $a \in S$. Such m is called a left invariant mean on $LUC(S)$ [7].

Let A be a Banach algebra and B a closed subalgebra of A and $i : B \to A$ the inclusion mapping, then $\pi : A^* \to B^*$ is the restriction mapping which is norm decreasing and onto (by the Hahn-Banach theorem). Following Ghahramani and Lau [3], we have the following lemma (see [3, Lemmas 1.1, 1.2, 1.4, Proposition 1.3]).

Lemma 2.1. (a) *Let $f \in A^*$, $b \in B$. Then $b\pi(f) = \pi(i(b)f)$.*

(b) *The mapping $\pi^* : B^{**} \to A^{**}$ is a homeomorphism whenever B^{**} has the weak*-topology and $\pi^*(B^{**})$ has the relative weak*-topology.*

Lemma 2.2. *Let B be a closed ideal in A, $n \in A^{**}$. If (a_α) is a bounded net in A such that $a_\alpha \to n$, then $i(b)a_\alpha \xrightarrow{\omega^*} \pi^*(b)n$ $(b \in B)$.*

Proposition 2.3. *Let B be a right (or left) ideal of A. Then $\pi^*(B^{**})$ is a right (resp., left) ideal of A^{**}.*

Lemma 2.4. *Let A be a commutative Banach algebra. Then any weak*-closed right ideal in A^{**} is an ideal. If $X = \operatorname{spec} A$, then $h(n) = \langle n, \delta_x \rangle$ is a multiplicative on A^{**}, where $\delta_x(\psi) = \langle x, \psi \rangle$.*

3. Multipliers on $LUC(S)^*$ and $L(S)^{**}$.

First we prove a theorem which is new even for topological groups.

Theorem 3.1. *Let S be a right cancellative topological semigroup with identity e. Then the following are equivalent:*

(a) *S is left amenable.*

(b) *There is a nonzero compact (or weakly compact) right multiplier on $LUC(S)^*$.*

Proof. (a)⇒(b). Let S be left amenable and m be a left invariant mean on $LUC(S)$. Then $\langle nm, f \rangle = \langle n, mf \rangle$, $mf(x) = m(l_x f) = m(f)$ $(f \in LUC(S)^*,\ f \in LUC(S))$. Therefore, $\langle nm, f \rangle = \langle n, m(f) \rangle = m(f)\langle n, 1 \rangle$, that is, $nm = \langle n, 1 \rangle m$. Thus $l_m(n) = \langle n, 1 \rangle m$ is a rank one operator and hence compact.

(b)⇒(a). Let T be a nonzero weakly compact right multiplier on $LUC(S)^*$. Then $T(m) = T(m\delta_e) = mT(\delta_e) = l_{T(\delta_e)} m$. So, $T = l_n$ where $n = T(\delta_e)$. Note that $\delta_e \in LUC(S)^*$ and $\delta_e(f) = f(e)$ $(f \in LUC(S))$. Now, let $A = \{\delta_x n \mid x \in S\} = \{\delta_x T(\delta_e) \mid x \in S\} = \{T(\delta_x) \mid x \in S\}$ which is weakly compact. By Krein-Smulian's theorem $K = \overline{\text{co}}^{\omega} A$ is weakly compact [2]. Now, we show that if $k \neq k' \in K$, then $\|\delta_x k_1\| \le \|k_1\|$. On the other hand, if we define

$$g(y) = \begin{cases} f(t), & y = tx, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

then g is well defined and belongs to $\beta(S)$ (the space of bounded functions on S), then $\delta_x g(t) = \delta_x(l_t g) = g(tx) = r_x g(t) = f(t)$. Let $\bar{k}_1$ be the extension of k_1 to $\beta(S)$ (by the Hahn-Banach theorem). Then

$$\begin{aligned} ||k_1|| = ||\bar{k}|| &\le \sup\{|\langle \bar{k}_1, f\rangle| \mid f \in \beta(S)\} \\ &= \sup\{|\langle \bar{k}_1, \delta_x g\rangle| \mid g \in \beta(S)\} \\ &= \sup\{|\langle \delta_x \bar{k}_1, g\rangle| \mid g \in \beta(S)\} \\ &= ||\delta_x \bar{k}_1|| \\ &= ||\delta_x k_1||. \end{aligned} \tag{3.2}$$

It follows that $\|\delta_x k_1\| = \|k_1\| \ne 0$. Now, we show that if $k, k' \in \text{co}(A)$, and $k \ne k'$, then a similar argument shows that $\|\delta_x(k-k')\| \ne 0$. Finally, we show that $0 \notin \{\delta_x(k-k') \mid x \in S\}$ since, by a completely similar argument, we have $\|\delta_{x_\alpha}(k-k')\| = \|k-k'\| \ne 0$. Therefore, $0 \notin \{\delta_x(k-k') \mid x \in S\}^-$. Hence, by Ryll-Nardzewski fixed point theorem [2], there exists a point $q \in K$ such that $\delta_x q = q$. It follows that $\delta_x|q| = |\delta_x q| = |q|$, and $\|q\| = \|n\| \ne 0$. Now, if we take $m = |q|/\|q\|$, then clearly $\delta_x m = m$, so, $m(f) = \delta_x m(f) = \delta_x(mf) = mf(x) = m({}_x f)$. Therefore, m is a left invariant mean on $LUC(S)$, that is, S is left amenable.

For a foundation semigroup S, let $i : LUC(S) \to L(S)^*$ be such that $\langle i(f), \mu\rangle = \langle \mu, f\rangle$ ($f \in LUC(S)$, $\mu \in L(S)$) is an embedding and $\pi = i^* : L(S)^{**} \to LUC(S)^*$ is onto. It is clear from the proof of [3, Lemma 2.2] for topological groups that $\pi(E) = \delta_e$ where E is a right identity, π is a homomorphism and $FG = F\pi(G)$. Also we have the following proposition which is similar to [6, Theorem 2.3]. □

We prove the following proposition for foundation semigroups with identity e.

Proposition 3.2. *Let E be a right identity in $L(S)^{**}$. Then π is an isometric isomorphism of $EL(S)^{**}$ onto $LUC(S)^*$.*

Proof. Let I be the identity operator on $L(S)^{**}$. Then

$$L(S)^{**} = EL(S)^{**} + (I-E)L(S)^{**}. \tag{3.3}$$

Now, if $m \in L(S)^{**}$, then $\pi((I-E)m) = \pi(m) - \pi(E)\pi(m) = \pi(m) - \delta_e\pi(m) = \pi(m) - \pi(m) = 0$. Thus $(I-E)m \in \ker\pi$. On the other hand, if $m \in \ker\pi$, then $Em = E\pi(m) = 0$. So $m = m - Em = (I-E)m \in (I-E)L(S)^{**}$. Thus,

$$\ker\pi = (I-E)L(S)^{**}. \tag{3.4}$$

So, we have

$$L(S)^{**} = EL(S)^{**} + \ker\pi. \tag{3.5}$$

It follows that π is injective from $EL(S)^{**}$ onto $L(S)^{**}/\ker\pi$, therefore π is injective from $EL(S)^{**}$ onto $LUC(S)^*$, and so π is an algebra isomorphism. We also have $\|Em\| = \|E\pi(m)\| \le \|E\|\|\pi(m)\| = \|\pi(m)\| \le \|m\|$, since π is a quotient map. Thus $\|\pi(Em)\| \le \|\pi\|\|Em\| \le \|Em\| \le \|\pi(m)\|$. So $\|\pi(Em)\| = \|\pi(m)\| = \|Em\|$, that is, π is an isometry. □

Now, we have another main theorem.

Theorem 3.3. *Let S be a right cancellative locally compact foundation semigroup with identity e. Then the following are equivalent:*

(a) *S is left amenable.*

(b) *There is a nonzero compact (or weakly compact) right multiplier on $L(S)^{**}$.*

Proof. (a)⇒(b). The proof of this part exactly reads the same line of the proof of (a)⇒(b) of Theorem 3.1, so it is omitted.

(b)⇒(a). Let T be a nonzero weakly compact right multiplier on $L(S)^{**}$, so $T = l_n$ for some $n \in L(S)^{**}$. Now l_{En} is also a nonzero right multiplier on $EL(S)^{**}$ where E is a right identity of $L(S)^{**}$ with norm 1, since $l_{En}(Em) = EmEn = Emn$. Now by Proposition 3.2, $\pi(EL(S)^{**}) = (LUC(S))^*$ isometrically isomorphic. If we define $l'_n = l_{En} \circ \pi$, then l'_n is a nonzero right multiplier on $LUC(S)^*$. Therefore, S is left amenable. □

In [3, Theorem 2.1] it was also shown that a locally compact group G is amenable if and only if there is a nonzero compact (weakly compact) right multiplier on $M(G)^{**}$. But we are not able to extend this result to $M(S)^{**}$.

Proposition 3.4. *A right multiplier $l_n(m) = mn$ $(m \in LUC(S)^*)$ is compact if and only if the restriction of l_n to $M(S)$ is compact.*

Note 3.5. It is clear that $M(S) \subseteq LUC(S)^*$ since, if $\mu \in M(S)$, then we can take $\langle \mu, f\rangle = \int_S f\, d\mu$ $(f \in LUC(S))$.

Proof. Let l_n be compact, then clearly the restriction of l_n to $M(S)$ is compact. Conversely, let $l_n : M(S) \to LUC(S)^*$ be compact, where $l_n(\mu) = \mu n$ $(\mu \in M(S))$. Let $m \in LUC(S)^*$ with $\|m\| \leq 1$. Since, the linear span of δ_x's is weak*-dense in $LUC(S)^*$, there is a net $\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}\delta_{x_{\alpha,i}}$ such that $\mu_\alpha \to m$ in weak*-topology. By compactness of l_n, there is a subnet $(\mu_{\alpha(\beta)})$ such that $(\mu_{\alpha(\beta)}n)$ converges in norm.

Now, we have $mn = \omega^* - \lim \mu_{\alpha(\beta)}n$. Thus $mn = \lim \mu_{\alpha(\beta)}n$ with norm topology. It follows that

$$\{mn \mid \|m\| \leq 1\} \subseteq \{\mu n \mid \mu \in L(S),\ \|\mu\| \leq 1\}. \tag{3.6}$$

Thus, l_n is compact. □

Theorem 3.6. *Let S be a right cancellative semigroup with identity e and l_n a right multiplier on $LUC(S)^*$. Then l_n can be written as a linear combination of four compact right multiplier l_{n_i} $(i = 1,2,3,4)$, $n_i \geq 0$, $n_i \in LUC(S)^*$.*

Proof. Let e be the identity of S. Then $T(m) = T(m\delta_e) = mT(\delta_e)l_{T(\delta_e)}(m)$. So, $T = l_n$ $(n = T(\delta_e) \in LUC(S)^*)$. Let $n = n_1^+ - n_1^- + i(n_2^+ - n_2^-)$ where n_k^+, n_k^- $(k = 1,2)$ are Hermitian. So, it suffices to show that $l_{n_k^+}$ and $l_{n_k^-}$ are compact. By Proposition 3.4 it suffices to prove that the restrictions of these operators to $M(S)$ are compact. Now since l_n is compact on $LUC(S)^*$, $\{\delta_x n \mid x \in S\}^-$ is compact. So $\{\|\delta_x n\| x \in S\}^-$ is compact. Since, $\|n^+\| \leq \|n\|$, $\{(\delta_x n)^+ \mid x \in S\}$ is compact. It follows that $\{\delta_x n^+ \mid x \in S\}^-$ is compact. Since the linear span of δ_x, s is weak* dense in $LUC(S)^*$, $\{\mu n^+ \mid \mu \in M(S),$ $\|\mu\| \leq 1\}^-$ is compact. Therefore, l_{n^+} is compact. This completes the proof. □

We denote by βS the space of all multiplicative linear functional on $LUC(S)$. We have another main theorem.

Theorem 3.7. *Let S be a finite topological semigroup. Then there exists $n \in \beta S$ such that l_n is compact. Conversely, if S is a subsemigroup of a topological group with identity, and there exists $n \in \beta S$ such that l_n is compact, then S is finite.*

Proof. Let S be finite, then by [2, Corollary 4.1.8], $AP(S) = C(S)$. Also, by [2, Proposition 4.4.8], $AP(S) = LUC(S) = RUC(S)$. Therefore, $LUC(S) = C(S)$. So βS is topologically isomorphic to S. On the other hand, since $\overline{l_s S} \subseteq S$ is compact, $l_s^* C(S)$ is compact. Hence, l_n is compact.

Conversely, let l_n be compact for some $n \in \beta S$, by Theorem 3.6, we may assume that n is positive, then $T_n(f) = nf$ $(f \in LUC(S))$ is compact. Now, let $F = \operatorname{range} T_n$. Clearly T_n is an algebra homomorphism, since, $T_n(fg) = n(fg)(x) = \langle n, l_x fg\rangle = n((l_x f)(l_x g)) = n(l_x f)n(l_x g) = T_n(f)T_n(g)$. Also T_n preserves conjugation. So, by [8, Theorem 5.3], $\|T_x f\| \geq \|f\|$ $(f \in LUC(S))$. So by open mapping theorem, T_n is a homeomorphism. Since T_n is compact, F is closed. Also, $\{T_n f \mid f \in LUC(S),\ \|T_n f\| \leq 1\} \subseteq \{T_n f \mid f \in LUC(S),\ \|f\| \leq 1\}$, so $\{T_n f \mid f \in LUC(S),\ \|T_n f\| \leq 1\}$ is compact. Therefore F is reflexive. It follows that F is finite dimensional (see [8, Exercise 2]). Let $\{m_1, m_2, \ldots, m_k\}$ be the spectrum of F and we can assume that m_i is positive. If we define $m(f) = (1/k)\sum_{i=1}^{k} m_i(T_n f)$, then clearly, $m \geq 0$, $m(1) = 1$. Also, since S is left cancellative, $l_x^*\{m_1, \ldots, m_k\} = \{m_1, \ldots, m_k\}$. Therefore, $\langle m_i, T_n l_x(f)\rangle = \langle m_i, l_x T_n(f)\rangle = \langle l_x^* m_i, T_n(f)\rangle = \langle m_j, Tn(f)\rangle$, for some $1 \leq j \leq k$. It follows that $m(l_x f) = m(f)$, that is, m is a left-invariant mean on $LUC(S)$, so by [5, Theorem 3] S is finite. □

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