THE CAYLEY TRANSFORM OF BANACH ALGEBRAS

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The main result of Haynes (1991) is that a square matrix is convergent ($\lim_{n\to\infty} D^n = 0$) if and only if it is the Cayley transform $C_A = (I-A)^{-1}(I+A)$ of a stable matrix A. In this note, we show, with a simple proof, that the above is true in a much more general setting of complex Banach algebras.

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A square matrix D is called convergent if $\lim_{n\to\infty} D^n = 0$, or equivalently, $\lim_{n\to\infty} \|D^n\| = 0$, where $\|\cdot\|$ denotes the matrix norm. A square matrix is called stable if its characteristic values all have negative real parts. The above definitions can be generalized to elements in general complex Banach algebras as follows: let $\mathcal{A}$ be a complex Banach algebra with norm $\|\cdot\|$, and let $\sigma(T)$ be the spectrum of T and $r(T)$ the spectral radius of T for any $T \in \mathcal{A}$. An element $D \in \mathcal{A}$ is called convergent if $\lim_{n\to\infty} \|D^n\| = 0$ and an element $A \in \mathcal{A}$ is called stable if $\sigma(A)$ is contained in the open left half of the complex plane. Inspired by the work of Stein and Taussky [3, 4] and Haynes [2] shows that a square matrix is convergent if and only if it is the Cayley transform of some stable matrix. In this note, we show that this holds for any complex Banach algebra and our proof is much simpler than that of [2].

To this end, we need the following elementary lemma.

LEMMA 1. *Let $D \in \mathcal{A}$, then D is convergent if and only if $r(D) < 1$.*

PROOF. ($\Rightarrow$). First, we show that if $T \in \mathcal{A}$ is convergent then $I - T$ is invertible. Suppose T is convergent, choose a fixed n large enough such that $\|T^n\| < 1$. Then $I - T^n$ is invertible. Since $I - T^n = (I-T)\sum_{k=0}^{n-1} T^k$, it follows that $I - T$ is invertible.

Now, note that if D is convergent then so is $(1/t)D$ for any complex number t such that $|t| \geq 1$. Thus $I - (1/t)D$ is invertible. Therefore, $tI - D$ is invertible. This implies that $r(D) < 1$.

($\Leftarrow$). Suppose $r(D) < 1$. Then choose a fixed $\epsilon > 0$ such that $r(D) + \epsilon < 1$. For n large enough, we have $\|D^n\|^{1/n} < r(D) + \epsilon$. Thus, $\|D^n\| < (r(D) + \epsilon)^n$. Therefore, D is convergent. □

For any stable element $A \in \mathcal{A}$, $I - A$ is invertible. In this case, we define the Cayley transform of A as $C_A = (I-A)^{-1}(I+A)$.

THEOREM 2. *For any complex Banach algebra $\mathcal{A}$, an element $D \in \mathcal{A}$ is convergent if and only if $D = C_A$ for some stable $A \in \mathcal{A}$.*

PROOF. ($\Leftarrow$). If $D = C_A$ for some stable $A \in \mathcal{A}$, let $f(z) = (1+z)/(1-z)$. Then $f(z)$ is analytic in the open left half of the complex plane. In fact, $f(z)$ is a *Möbius*

transform that maps the open left half of the complex plane onto the open unit disk and $f(A) = C_A$. Since $f(z)$ is analytic in a neighborhood of $\sigma(A)$. By the Spectral Mapping theorem (see [1, page 208]), $\sigma(f(A)) = f(\sigma(A))$. Since $f(\sigma(A))$ is contained in the open unit disk, $\sigma(D) = \sigma(f(A))$ is contained in the open unit disk. By Lemma 1, D is convergent.

($\Rightarrow$). Suppose D is convergent. By Lemma 1, $\sigma(D)$ is contained in the open unit disk. Thus $D + I$ is invertible. Let $A = (D+I)^{-1}(D-I)$. Define $g(z) = (z-1)/(z+1)$. The map $g(z)$ is analytic in the open unit disk. In fact, $g(z)$ is a *Möbius* transform that maps the open unit disk onto the open left half of the complex plane and $g(D) = A$. Similarly, by the Spectral Mapping theorem, we can obtain $\sigma(A) = \sigma(g(D)) = g(\sigma(D))$. This implies that $\sigma(A)$ is contained in the open left half of the complex plane, that is, A is stable. Solving for D from $A = (D+I)^{-1}(D-I)$, we have $D = (I-A)^{-1}(I+A)$, that is, $D = C_A$. $\qquad\square$

References

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