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AN EXAMPLE OF A BLOCH FUNCTION

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ABSTRACT. A Bloch function is exhibited which has radial limits of modulus one almost everywhere but fails to belong to H^p, for each $0 < p \leq \infty$.

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1. INTRODUCTION.

The purpose of this note is to give an example which seems to be useful in settling several questions about Bloch functions.

Let E be the subset of the complex plane $\mathbb{C}$ consisting of the closed unit disc together with the Gaussian integers $\mathbb{Z}^2$. Let G be the complement of E in in $\mathbb{C}$. Let $g : D \to G$ be the analytic universal covering map of G given by the uniformization theorem (D denotes the unit disc).

PROPOSITION. The function g is an unbounded Bloch function with the properties

(i) g has a radial limit $g(e^{i\theta})$ at almost every point $e^{i\theta}$ of the unit circle.

(ii) the function $g(e^{i\theta})$ is of modulus one almost everywhere on the unit circle,

(iii) g is the reciprocal of a singular inner function, and so g does not belong to any H^p class.

Bloch functions on the unit disc may be defined as those analytic functions f on D for which the radii of the schlicht discs in the range of f are bounded above. The Bloch functions are somewhat analagous to functions in the disc algebra--Bloch functions can be characterized (see [1]) as those analytic functions which are uniformly continuous when D is given the hyperbolic metric and $\mathbb{C}$ the Euclidean metric. Since Bloch functions may be characterized (see [1]) as those analytic functions f on D for which the quantity $|f'(z)|\ (1 - |z|^2)$ is bounded for $z \in D$, it follows that the modulus of a Bloch function grows rather slowly--at most as fast as $\log(1/(1 - |z|))$. Because functions in the disc algebra and bounded functions have good boundary behaviour, it is natural to ask about boundary values of Bloch functions--in particular about radial boundary values. (It is shown in [4] that a Bloch function has a radial limit at a point of the unit circle if and only if it has a non-tangential limit there.)

In [5], Pommerenke gave an example of a Bloch function with radial limits almost nowhere. The example given here is constructed in a similar way, but it contrasts with Pommerenke's in that it shows that Bloch functions which have radial limits almost everywhere need not be particularly well-behaved.

The example answers a question posed by Joseph Cima (private communication). He asked whether a Bloch function which has radial limits almost everywhere and has the additional property that the boundary function belongs to L^p need be in H^p. The function g provides a negative answer to this question since $g(e^{i\theta}) \in L^\infty$ while $g \notin H^p$ for any $0 < p \leq \infty$. In fact g does not belong to the class N^+ (see [2] p. 25) which contains H^p for every p.

PROOF. It is evident that g is an unbounded Bloch function. Also, to verify properties (i), (ii) and (iii), it is clearly sufficient to verify (iii).

To establish (iii), consider the analytic function $f = 1/g$ on D. The function f is bounded (by 1) and is the universal covering map : $D \to D - K$, where K is the countable set

$$\{0\} \cup \{1/(m+in) \mid m,n \in \mathbb{Z}, |m+in| > 1\} .$$

Being a bounded analytic function, f has radial limits almost everywhere on the unit circle. It is easy to see from the properties of covering maps that these radial limits are either of modulus 1 or else belong to K. To complete the proof that f is a singular inner function, it is only necessary to show that the radial limit $f(e^{i\theta})$ belongs to K on a subset of the unit circle of measure zero.

But, for each $k \in K$ it is true that the set of $e^{i\theta}$ for which $f(e^{i\theta}) = k$ has measure zero (see [2] p. 17). Since K is countable, it follows that the set of $e^{i\theta}$ for which $f(e^{i\theta})$ belongs to K also has measure zero. The proof is now complete.

The example may also be viewed as elucidating the almost total lack of relationships between the class $\mathcal{B}$ of Bloch functions on D and the subclasses H^p and N^+ of the Nevanlinna class N (see [2]). The only containment which holds between $\mathcal{B}$ and the other classes is the relation $H^\infty \subseteq \mathcal{B}$. It is known that $H^p \not\subseteq \mathcal{B}$ for any $0 < p < \infty$ and that $\mathcal{B} \not\subseteq N$. The example g given above belongs to $\mathcal{B} \cap N$ but not to N^+. The fact $\mathcal{B} \not\subseteq N$ is shown by the example of Pommerenke's [5] mentioned above.

Finally, the example given here can be modified to show that there is no $\delta > 0$ such that an analytic function $f : D \to \mathbb{C}$ satisfying

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = 1$$

almost everywhere on the unit circle must have a disc of radius δ in its range. (Merely replace $\mathbb{Z}^2$ by $\delta\mathbb{Z}^2$ in the construction of g). This answers a question raised by J.S. Hwang. By contrast, he showed (see [3]) that a singular inner function (for example) must have a (Schlicht) disc of radius at least $2B/e$ in its range, where B denotes Bloch's constant.

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