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A STABILITY THEORY FOR PERTURBED DIFFERENTIAL EQUATIONS

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ABSTRACT. The problem of determining the behavior of the solutions of a perturbed differential equation with respect to the solutions of the original unperturbed differential equation is studied. The general differential equation considered is

$$X' = f(t,X)$$

and the associated perturbed differential equation is

$$Y' = f(t,Y) + g(t,Y).$$

The approach used is to examine the difference between the respective solutions $F(t,t_o,x_o)$ and $G(t,t_o,y_o)$ of these two differential equations. Definitions paralleling the usual concepts of stability, asymptotic stability, eventual stability, exponential stability and instability are introduced for the difference $G(t,t_o,y_o) - F(t,t_o,x_o)$ in the case where the initial values y_o and x_o are sufficiently close. The principal mathematical technique employed is a new modification of Liapunov's Direct Method which is applied to the difference of the two solutions. Each of the various stability-type properties considered is then shown to be guaranteed by the existence of a Liapunov-type function with appropriate properties.

KEY WORDS AND PHRASES. Liapunov functions, asymptotic behavior of solutions, asymptotic equivalence.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 34D10, 34D20.

1. INTRODUCTION.

One of the paramount uses of stability theory is to determine which stability properties of a particular system of differential equations are preserved under small perturbations. This problem has been studied in numerous ways. The method introduced in the present paper, however, is felt to be an essentially new approach for dealing with this situation. It is similar to some work done by Lakshmikantham [3] in a different context. Instead of considering explicitly which stability properties are preserved under perturbations, a theory based on the actual behavior of the solutions of the perturbed differential equation with respect to those of the original differential equation is developed. The results so obtained are given in terms of the existence of an extended form of Liapunov function.

We note that a similar concept, that of asymptotic equivalence, was introduced by Brauer [1], though the general approach both he and his successors used is different from the one employed here. Moreover, the notion of asymptotic equivalence is merely one of the numerous possibilities which can be considered in terms of the present approach.

Some analogous results for the discrete case involving the behavior of solutions of difference equations have previously been done by the author [2].

2. DEFINITIONS AND BASIC CONCEPTS.

We will consider the differential equation

$$X'(t) = f(t,X(t)). \tag{2.1}$$

$f(t,X)$ here represents a function with values in E^m, an arbitrary m-dimensional vector space, and defined on some region D in $I \times E^m$ which contains the axis $\{X = 0, t \in I\}$, where I is the set of non-negative real numbers. For simplicity, we may take for D the semi-infinite cylinder

$$D = D_{t_0 R} = \{(t,X) \in I \times E^m : t \geq t_0 \geq 0, \ \|X\| \leq R\} .$$

Here, $\|X\|$ denotes any m-dimensional norm of the vector X. We note that in most cases, the upper bound R will be taken to be finite. The sole exception to this convention would occur when we are dealing with the case of instability for the solutions of the differential equations when the solutions become unbounded and hence the region must accomodate them.

In addition, the differential equation (2.1) will be subject to the initial condition $X(t_0) = x_0$. Moreover, we will consider only those equations for which the solution is uniquely determined by the initial point and this unique solution to the differential equation (2.1) satisfying the initial condition will be denoted by $F(t,t_0,x_0)$.

In addition to the differential equation (2.1), we also consider the associated perturbed differential equation

$$Y'(t) = f(t,Y(t)) + g(t,Y(t)) , \tag{2.2}$$

where $g(t,Y)$ is also a function from $D_{t_0 R}$ into E^m. If the additional term $g(t,Y)$ is small in some sense, it is reasonable to expect that the behavior of the solutions of the perturbed equation will be similar to that for the solutions of the original equation, provided that the initial values for the two equations are sufficiently close. In this regard, the assumed unique solution to the perturbed differential equation (2.2) satisfying the initial condition $Y(t_0) = y_0$ will be denoted by $G(t,t_0,y_0)$.

The present investigation will be carried out by using a certain class of continuous real scalar functions $V(t,X)$ also defined on D_{t_0R} and satisfying the requirement $V(t,0) = 0$ for all $t \geq t_0$. The following additional properties will all be required. Let M_0 represent the class of all real-valued monotone increasing functions, $a(r)$, defined and positive for $r \geq 0$ and such that $a(0) = 0$. In terms of this, a real scalar function $V(t,X)$ is said to be positive definite if there exists a function $a(r)$ of class M_0 such that

$$V(t,X) \geq a(\| X \|)$$

for all $t \geq t_0$. A real scalar function $V(t,X)$ is said to be positive semi-definite if $V(t,X) \geq 0$ for all $t \geq t_0$. Entirely similar definitions hold for such functions being either negative definite or negative semi-definite.

Moreover, corresponding to a function $V(t,X)$, we define its total derivative with respect to the differential equation (2.1) as

$$\begin{aligned} V'(t) &= \frac{\partial}{\partial t} V(t,X) + \nabla V(t,X) \circ X'(t) \\ &= \frac{\partial}{\partial t} V(t,X) + \nabla V(t,X) \circ f(t,X), \end{aligned}$$

where

$$\nabla V(t,X) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \right) V(t,X).$$

Here, $x_1, \ldots, x_m$ denote the components of X. $V'(t,X)$ is obviously a measure of the growth or decay of the function $V(t,X)$ with regard to increasing t along the trajectories represented by the solutions of the differential equation (2.1). It should be noted that, in general, this can be calculated without direct knowledge of the actual solutions.

We now introduce the types of possible behavior for the solutions of the perturbed differential equation (2.2) which will be of interest to us in the sequel.

DEFINITION 1: The solutions of the perturbed differential equation (2.2) are said to be <u>stable with respect to the unperturbed differential equation</u>

(2.1) if, for all $\varepsilon > 0$ and for all $t_o \varepsilon I$, there exists a $\delta(\varepsilon, t_o) > 0$ such that $\| y_o - x_o \| < \delta$ implies that

$$\| G(t,t_o,y_o) - F(t,t_o,x_o) \| < \varepsilon$$

for all $t \geq t_o$, for every solution $G(t,t_o,y_o)$ of the perturbed equation (2.2).

DEFINITION 2: The solutions of the perturbed differential equation (2.2) are said to be <u>asymptotically stable with respect to the unperturbed differential equation (2.1)</u> if they are stable with respect to the equation (2.1) and if, for all $t_o \varepsilon I$, there exists a $\delta_o(t_o) > 0$ such that $\| y_o - x_o \| < \delta_o$ implies that

$$\| G(t,t_o,y_o) - F(t,t_o,x_o) \| \to 0$$

as $t \to \infty$ for every solution $G(t,t_o,y_o)$ of the perturbed equation (2.2).

The above two definitions are equivalent to the statement that all solutions of the perturbed differential equation which start sufficiently close to the initial value of the unperturbed solution respectively remain close to it or eventually approach it. The latter case is essentially the same concept as Brauer's asymptotic equivalence.

The next definition expresses an intermediate type of behavior whereby the perturbed solutions initially may diverge from the unperturbed solution, but eventually become arbitrarily close to the latter. The concept is somewhat similar to that introduced by LaSalle and Rath [4].

DEFINITION 3. The solutions of the perturbed differential equation (2.2) are said to be <u>eventually stable with respect to the unperturbed differential equation (2.1)</u> if, for every $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_o) > 0$ such that, for any x_o, $\| y_o - x_o \| < \delta$ implies that

$$\| G(t,t_o,y_o) - F(t,t_o,x_o) \| < \varepsilon$$

for all $t \geq T$, for some $T \geq t_o$, for every solution $G(t,t_o,y_o)$ of the perturbed equation (2.2).

Finally, we give two further definitions of modes of behavior which will be considered.

DEFINITION 4: The solutions of the perturbed differential equation (2.2) are said to be <u>exponentially stable with respect to the unperturbed differential equation (2.1)</u> if there exist positive numbers a and B and a $\delta_0 > 0$ such that $t_0 \in I$, $\|y_0 - x_0\| < \delta_0$ imply that

$$\|G(t,t_0,y_0) - F(t,t_0,x_0)\| \leq B\|y_0 - x_0\| \; e^{-a(t-t_0)}$$

for all $t \geq t_0$, for every solution $G(t,t_0,y_0)$ of equation (2.2).

DEFINITION 5: The solutions of the perturbed differential equation (2.2) are said to be <u>unstable with respect to the unperturbed differential equation (2.1)</u> if, for every $\varepsilon > 0$ and every $t_0 \in I$, there exists some y_0 with $\|y_0 - x_0\| < \varepsilon$ and such that

$$\|G(t_1,t_0,y_0) - F(t_1,t_0,x_0)\| \geq \varepsilon$$

for some $t_1 > t_0$.

The above definition requires that for each solution of the unperturbed equation (2.1), a solution of the perturbed equation (2.2) can be found which starts arbitrarily close to the unperturbed solution and which eventually diverges from it.

We note that all of these definitions are independent of the behavior of the solutions of the unperturbed equation. In fact, we specifically indicate that the equilibria of the original differential equations may be stable, asymptotically stable or even unstable. This is illustrated by the following:

EXAMPLE 1: Consider the unperturbed differential equation

$$X' = -aX,$$

with $a > 0$, whose asymptotically stable solution is given by

$$F(t,t_o,x_o) = x_o e^{-a(t-t_o)} .$$

Further, consider the associated perturbed equation

$$Y' = -(a+b)Y,$$

whose solution is given by

$$G(t,t_o,y_o) = y_o e^{-(a+b)(t-t_o)} .$$

As a consequence,

$$G(t,t_o,y_o) - F(t,t_o,x_o) = e^{-a(t-t_o)} [y_o e^{-b(t-t_o)} - x_o] .$$

If $b > 0$, this difference approaches 0 as $t \to \infty$ and thus the perturbed solutions are asymptotically, and in fact exponentially, stable with respect to the unperturbed equation. On the other hand, if $b < 0$, then the perturbed solutions are unstable with respect to the unperturbed equation.

EXAMPLE 2: Consider the equation

$$X' = f(t),$$

where $f(t)$ is any function which is defined and non-integrable on $[t_o , \infty)$. The unstable solution to this equation is given by

$$F(t,t_o,x_o) = x_o + \int_{t_o}^{t} f(s)\, ds .$$

Further, consider the associated perturbed equation

$$Y' = f(t) + g(t,Y),$$

where

$$\| g(t,Y)\| \leq a \| h(t)\|$$

for some sufficiently small positive constant a and for some function $h(t)$ which is integrable on $[t_o , \infty)$. The solution is given by

$$G(t,t_o,y_o) = y_o + \int_{t_o}^{t} f(s)\, ds + \int_{t_o}^{t} g(s,Y(s))\, ds,$$

and hence

$$\| G(t,t_o,y_o) - F(t,t_o,x_o)\| \leq \| y_o - x_o\| + a \int_{t_o}^{t} \| h(s)\| ds,$$

which can be made arbitrarily small. Therefore, the perturbed solutions are stable with respect to the unperturbed differential equation.

3. PRINCIPAL RESULTS.

We now present several theorems which supply sufficient conditions for the above types of behavior to hold in terms of the existence of continuous real scalar, Liapunov-type, functions $V(t,X)$.

THEOREM 1. If there exists a function $V(t,X)$ on D_{t_oR} such that

a) $V(t,X)$ is positive definite

b) $V(t,X)$ is continuous for $X = 0$

c) $V'(t,Y(t) - X(t))$ is negative semi-definite,

then the solutions of the perturbed differential equation (2.2) are stable with respect to the unperturbed differential equation (2.1), provided that for all $t \geq t_o$,

$$\|G(t,t_o,y_o) - F(t,t_o,x_o)\| \leq R.$$

PROOF: Since $V(t,X)$ is positive definite, there is a function $a(r)$ of class M_o such that

$$V(t,X) \geq a(\|X\|).$$

Now, given any ε, choose y_o sufficiently close to x_o so that

$$\|y_o - x_o\| < \varepsilon \qquad \text{and} \qquad V(t_o,y_o - x_o) < a(\varepsilon).$$

It then follows that

$$\|G(t,t_o,y_o) - F(t,t_o,x_o)\| < \varepsilon$$

for all $t \geq t_o$; for, if not, there would be some $t_1 > t_o$ such that

$$\|G(t_1,t_o,y_o) - F(t_1,t_o,x_o)\| \geq \varepsilon .$$

This, however, would imply that

$$\begin{aligned} V(t_1, G(t_1,t_o,y_o) - F(t_1,t_o,x_o)) &\geq a(\| G(t_1,t_o,y_o) - F(t_1,t_o,x_o)\|) \\ &\geq a(\varepsilon) \\ &> V(t_o, y_o - x_o) \\ &\geq V(t_1, G(t_1,t_o,y_o) - F(t_1,t_o,x_o)) , \end{aligned}$$

which is a contradiction.

It should be noted that the above theorem, as well as the ones which follow, depends strongly on the condition that, for all $t \geq t_o$,

$$\| G(t,t_o,y_o) - F(t,t_o,x_o) \| \leq R. \tag{3.1}$$

This condition guarantees that both the function $V'(t,Y-X)$ remains well-defined and that the difference of the two solutions remains on $D_{t_o R}$. The following result gives one fairly simple set of criteria for the functions $f(t,X)$ and $g(t,Y)$ which insures that this holds.

THEOREM 2: If $f(t,X)$ satisfies a generalized Lipschitz condition

$$\| f(t,X_1) - f(t,X_2)\| \leq L(t) \| X_1 - X_2 \| ,$$

where $L(t)$ is integrable on $[t_o, \infty)$ and if $g(t,Y)$ satisfies

$$\| g(t,Y)\| \leq a \| h(t) \|$$

for some sufficiently small positive constant a and for some function $h(t)$ which is integrable on $[t_o, \infty)$, then if y_o is chosen sufficiently close to x_o, condition (3.1) holds for all $t \geq t_o$.

PROOF: We have

$$F(t,t_o,x_o) = x_o + \int_{t_o}^{t} f(s,X(s))\, ds$$

and

$$G(t,t_o,y_o) = y_o + \int_{t_o}^{t} f(s,Y(s))\, ds + \int_{t_o}^{t} g(s,Y(s))\, ds.$$

As a consequence,

$$\begin{aligned}
&\| G(t,t_0,y_0) - F(t,t_0,x_0)\| \\
&\quad \leq \|y_0 - x_0\| + \int_{t_0}^{t} \|f(s,Y) - f(s,X)\| \, ds + \int_{t_0}^{t} \| g(s,Y)\| \, ds \\
&\quad \leq \| y_0 - x_0\| + \int_{t_0}^{t} L(s) \| G(s,t_0,y_0) - F(s,t_0,x_0)\| \, ds + a\int_{t_0}^{t} \| h(s)\| \, ds \\
&\quad \leq \| y_0 - x_0\| + a\int_{t_0}^{\infty} \| h(s)\| \, ds + \int_{t_0}^{t} L(s) \| G(s,t_0,y_0) - F(s,t_0,x_0)\| \, ds \\
&\quad = A + \int_{t_0}^{t} L(s) \| G(s,t_0,y_0) - F(s,t_0,x_0)\| \, ds,
\end{aligned}$$

where, we observe, the quantity A can be made arbitrarily small. We now apply the following form of Gronwall's Inequality to the above relation:

If $Z(t) > 0$ and $P(t) \leq Q(t) + \int_{t_0}^{t} Z(s)\, P(s)\, ds,$

then

$$P(t) \leq Q(t) + \int_{t_0}^{t} Q(s)\, Z(s) \exp\left[\int_{s}^{t} Z(u)\, du\right] ds.$$

We therefore obtain

$$\begin{aligned}
&\| G(t,t_0,y_0) - F(t,t_0,x_0)\| \\
&\quad \leq A + A \int_{t_0}^{t} L(s) \exp\left[\int_{s}^{t} L(u)\, du\right] ds \\
&\quad = A \left\{ 1 + \int_{t_0}^{t} L(s) \exp\left[K(t) - K(s)\right] ds \right\} \\
&\quad = A \left\{ 1 + e^{K(t)} \int_{t_0}^{t} L(s)\, e^{-K(s)} \, ds \right\} \\
&\quad = A \left\{ 1 - e^{K(t)} \left(e^{-K(t)} - e^{-K(t_0)}\right) \right\} \\
&\quad = A\, e^{K(t) - K(t_0)} \\
&\quad \leq A \exp\left[\int_{t_0}^{t} L(s)\, ds\right] \\
&\quad = AC,
\end{aligned}$$

which can be made smaller than any given R by choosing the constant a sufficiently small and by choosing y_o sufficiently close to x_o. We note that in the above, K(t) represents an indefinite integral of L(t).

We now turn to a result giving sufficient conditions for asymptotic behavior for the two solutions.

THEOREM 3. If there exists a function V(t,X) on D_{t_oR} such that

a) V(t,X) is bounded below

b) V'(t,Y(t) - X(t)) is negative definite,

then the solutions of the perturbed differential equation (2.2) are asymptotically stable with respect to the unperturbed differential equation (2.1) provided that condition (3.1) holds for all $t \geqslant t_o$.

PROOF. Since V'(t,Y-X) is negative definite, there exists a function a(r) of class M_o such that

$$V'(t,Y-X) \leqslant - a(\| Y - X \|).$$

Moreover, we have that

$$\begin{aligned} &V(t,G(t,t_o,y_o) - F(t,t_o,x_o)) \\ &\quad = V(t_o,y_o - x_o) + \int_{t_o}^{t} V'(s,G(s,t_o,y_o) - F(s,t_o,x_o))\, ds \\ &\quad \leqslant V(t_o,y_o - x_o) - \int_{t_o}^{t} a(\| G(s,t_o,y_o) - F(s,t_o,x_o)\|)\, ds. \end{aligned}$$

Taking the limit as $t \to \infty$ and using the fact that V(t,X) is bounded below by some B, we find that

$$\lim_{t \to \infty} \int_{t_o}^{t} a(\| G(s,t_o,y_o) - F(s,t_o,x_o)\|)\, ds \leqslant V(t_o,y_o - x_o) - B,$$

which implies that, as $t \to \infty$,

$$a(\| G(t,t_o,y_o) - F(t,t_o,x_o)\|) \to 0.$$

Therefore, since a(r) is monotonically increasing, it follows that

$$\| G(t,t_o,y_o) - F(t,t_o,x_o)\| \to 0$$

as $t \to \infty$, thus proving the theorem.

THEOREM 4. If there exists a function $V(t,X)$ on D_{t_oR} such that

a) $V(t,X)$ is positive definite

b) $V(t,X)$ is continuous as $X = 0$

c) $V'(t,Y - X) \leqslant - b(t)$, where $\int_{t_o}^{\infty} b(s)\, ds \geqslant 0$,

then the solutions of the perturbed differential equation (2.2) are eventually stable with respect to the unperturbed differential equation (2.1), provided that condition (3.1) holds for all $t \geqslant t_o$.

PROOF: Since $V(t,X)$ is positive definite, there exists a function $a(r)$ of class M_o such that

$$V(t,X) \geqslant a(\|X\|).$$

Now, suppose that the solutions of (2.2) are not eventually stable with respect to (2.1). Then, for any $\varepsilon > 0$ and any x_o, there exist sequences $\{z_k\} \to x_o$ and $\{t_k\} \to \infty$ as $k \to \infty$ such that

$$\| G(t_k,t_o,z_k) - F(t_k,t_o,x_o) \| \geqslant \varepsilon.$$

Consequently,

$$\begin{aligned} V(t_k,G(t_k,t_o,z_k) - F(t_k,t_o,x_o)) &\geqslant a(\| G(t_k,t_o,z_k) - F(t_k,t_o,x_o) \|) \\ &\geqslant a(\varepsilon) > 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} &V(t_k,G(t_k,t_o,z_k) - F(t_k,t_o,x_o)) \\ &\quad = V(t_o,z_k - x_o) + \int_{t_o}^{t_k} V'(s,G(s,t_o,z_k) - F(s,t_o,x_o))\, ds \\ &\quad \leqslant V(t_o,z_k - x_o) - \int_{t_o}^{t_k} b(s)\, ds. \end{aligned}$$

Now, letting $k \to \infty$, we have $z_k - x_o \to 0$, so that $V(t_k,z_k - x_o) \to 0$ also. However, since by assumption,

$$\int_{t_o}^{\infty} b(s)\, ds \geqslant 0,$$

we are led to a contradiction.

THEOREM 5: If there exists a function $V(t,X)$ on D_{t_0R} such that

a) $a_1 \|X\|^p \leq V(t,X) < a_2 \|X\|^p$

for some positive constants a_1 and a_2 and for some $p > 0$,

b) $V'(t,Y - X) \leq -a_3 \|Y - X\|^p$,

for some positive constant a_3,

then the solutions of the perturbed differential equation (2.2) are exponentially stable with respect to the unperturbed differential equation (2.1) provided that condition (3.1) holds for all $t \geq t_o$.

PROOF: From the conditions on $V(t,X)$ and $V'(t,Y - X)$, we find

$$V'(t,Y - X) \leq -a_3 \|Y - X\|^p \leq -(a_3/a_2)\, V(t,Y - X) .$$

Therefore, upon integrating, we obtain

$$V(t,G(t,t_o,y_o) - F(t,t_o,x_o)) \leq V(t_o,y_o - x_o)\, e^{-a_4(t - t_o)} ,$$

where we have written $a_4 = a_3/a_2$. Moreover, it follows that

$$V(t,G(t,t_o,y_o) - F(t,t_o,x_o)) \geq a_1 \|G(t,t_o,y_o) - F(t,t_o,x_o)\|^p$$

and hence

$$\|G(t,t_o,y_o) - F(t,t_o,x_o)\|^p \leq (1/a_1)\, V(t_o,y_o - x_o)\, e^{-a_4(t - t_o)} .$$

As a result,

$$\|G(t,t_o,y_o) - F(t,t_o,x_o)\| \leq B \|y_o - x_o\|\, e^{-(a_4/p)(t - t_o)} ,$$

which completes the proof.

Finally, we conclude this section with a criterion for the solutions of (2.2) to be unstable with respect to equation (2.1).

THEOREM 6: Suppose there exists a real scalar function $V(t,X)$ such that

a) for each $\varepsilon > 0$ and each $t \geq t_o$ and each solution $F(t,t_o,x_o)$ of (2.1), there exists a solution $G(t,t_o,y_o)$ of (2.2) such that

$$\|G(t,t_o,y_o) - F(t,t_o,x_o)\| < \varepsilon$$

and

$$V(t, G(t, t_o, y_o) - F(t, t_o, x_o)) < 0;$$

Corresponding to each solution $F(t, t_o, x_o)$, the set of all points for which $V(t, Y - X) < 0$ is bounded by the hypersurfaces $\|Y - X\| = R$ and $V = 0$ and may consist of several component domains;

b) In at least one of the component domains D* in which $V(t, Y - X) < 0$ corresponding to each solution $F(t, t_o, x_o)$, $V(t, X)$ is bounded below;

c) In the domain D*,

$$V'(t, Y - X) \leq -a(\,|V(t, Y - X)|\,),$$

for some function $a(r)$ of class M_o,

then the solutions of the perturbed differential equation (2.2) are unstable with respect to the differential equation (2.1).

PROOF: Let $F(t, t_o, x_o)$ be any solution of (2.1) and choose any point $(t_1, y_1 - x_1)$ in D* such that

$$V(t_1, y_1 - x_1) = -b < 0,$$

where $x_1 = F(t_1, t_o, x_o)$. Consider the solution $G(t, t_1, y_1)$ of (2.2). We thus have

$$\begin{aligned} V(t, G(t, t_1, y_1) - F(t, t_1, x_1)) & \\ &= V(t_1, y_1 - x_1) + \int_{t_1}^{t} V'(s, G(s, t_1, y_1) - F(s, t_1, x_1))\, ds \\ &\leq -b - \int_{t_1}^{t} a(\,|V(s, G(s, t_1, y_1) - F(s, t_1, x_1))|\,)\, ds \\ &< -b - \int_{t_1}^{t} a(b)\, ds \\ &= -b - a(b)\,(t - t_1), \end{aligned}$$

which approaches $-\infty$ as $t \to \infty$. However, by assumption, $V(t, Y - X)$ is bounded below in D* and hence, the points $(t, G(t, t_1, y_1) - F(t, t_1, x_1))$ must leave D* as $t \to \infty$. This can only happen across the boundary $\|Y - X\| = R$, for any arbitrarily large R. Moreover, since y_1 can be chosen arbitrarily close to x_1, the solutions of (2.2) are unstable with respect to (2.1).

4. CONCLUDING REMARKS.

Subject to the usual difficulty in finding a Liapunov function for a differential equation, the approach presented in this paper should prove to be one of the most useful techniques in studying the behavior of the solutions of a perturbed differential equation.

Moreover, it is apparent that the concepts introduced here can be extended to encompass in addition all of the various refinements of the stability properties, such as uniform stability, equiasymptotic stability, uniform-asymptotic stability and so forth.

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