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RESEARCH NOTES

ON SLOW OSCILLATION AND NONOSCILLATION IN RETARDED EQUATIONS

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ABSTRACT. Sufficient conditions have been found to ensure that all oscillatory solutions of

$$(r(t)y'(t))' + a(t)y(t-\zeta(t)) = f(t)$$

are slowly oscillating. This behaviour is further linked to nonoscillation.

KEY WORDS AND PHRASES. Oscillation, Nonoscillation, Moderately Oscillating, Slowly Oscillating.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 34C10.

1. INTRODUCTION.

In studying the asymptotic nature of oscillatory solutions of the equation

$$(r(t)y'(t))' + a(t)y(t-\zeta(t)) = f(t), \tag{1}$$

this author in [9, Theo. 2] showed that a nontrivial oscillatory solution y(t) of (1) satisfies $\lim_{t\to\infty} y(t) = o$ if $\int^\infty |a(t)|dt < \infty$, $\int^\infty |f(t)|dt < \infty$ and $\int^\infty \frac{1}{r}dt < \infty$.

In theorem 4 in [9], it was observed that $\int^{\infty}\frac{1}{r}dt = \infty$ lead to slowly oscillating solutions which do not approach zero as $t \to \infty$.

In this work, we give sufficient conditions which cause all oscillatory solutions of the equation

$$(r(t)y'(t))' + a(t)h(y(g(t))) = f(t) \tag{2}$$

to oscillate slowly.

Even though voluminous literature exists about various types of oscillatory and nonoscillatory criteria for such equations, the asymptotic nature of oscillatory solutions of these equations has not been so extensively studied. For a good literature study of related results Graef [1] and Graef and Spike [2] have included an exhaustive reference list. The work of Travis [11] (c.f[8]) shows that common techniques found for ordinary differential equations fail on retarded differential equations even when the retardations are small. For oscillation criteria see T. Kusano and H. Onose [4] and this author [7].

2. ASSUMPTIONS AND DEFINITIONS

The entire study in this work is subject to the following assumptions:

(i) $a(t)$, $r(t)$, $f(t)$, $h(t)$ and $g(t)$ are $C^0(R)$, where R is the real line,

(ii) $r(t) > o$, $g(t) > o$, $g'(t) > o$;

(iii) $g(t) \leq t$ and $g(t) \to \infty$ as $t \to \infty$;

(iv) h is odd, sign $(h(t)) = \text{sign}(t)$ and there exists positive constants α,m such that $o < \alpha \leq \frac{h(t)}{t} \leq m$ on some positive half line.

In order to reckon the half line we shall assume that it would mean for $t \geq t_0$ for some positive t_0. t_0 will be referred to without further mention. All functions considered are real valued.

We call a function $H(t) \varepsilon C^0[t_0,\infty)$ oscillatory if $H(t)$ has arbitrarily large zeros in $[t_0,\infty)$; otherwise we call it nonoscillatory. In order to be more precise we shall use the term "solution" only for those nontrivial solutions (of equations under consideration) which can be extended continuously for $t \geq t_0$.

We define a function $H_1(t) \varepsilon C^0[t_0,\infty)$ to be slowly oscillating if $H_1(t)$ is oscillatory and the set: $Z_{H_1} = \{y_0 - x_0 : y_0 > x_0 ;$ y_0 and x_0 are consecutive zeros of $H_1(t)$, $x_0 > t_0\}$ is unbounded. If Z_{H_1} is bounded, H_1 is called "moderately oscillating".

3. ON SLOW OSCILLATION

Theorem 4 in [9] states that moderately oscillatory solutions of (1) approach zero if $\int^\infty |a(t)|dt < \infty$ and $\int^\infty |f(t)|dt < \infty$. Our next Theorem gives sufficient conditions for all oscillatory solutions of (2) to be slowly oscillating.

THEOREM (1). Suppose $a(t) = a_1(t) + a_2(t)$ where $a_1(t) > o$ and $\frac{a_2(t)}{a_1(t)}$ is bounded for large t. Further suppose that $\int^\infty |f(t)|dt < \infty$ and $\liminf_{t\to\infty} \frac{|f(t)|}{a_1(t)} > o$. Then all oscillatory solutions $y(t)$ of equation (2) satisfy

$$\limsup_{t\to\infty} |y(t)| > o \tag{3}$$

and

$$y(t) \text{ is slowly oscillating.} \tag{4}$$

PROOF. Let $y(t)$ be an oscillatory solution of (2). Rewriting (2) we have

$$\frac{(ry')'}{a_1(t)} + h(y(g(t))) + \frac{a_2(t)}{a_1(t)} h(y(g(t))) = \frac{f(t)}{a_1(t)} \quad . \tag{5}$$

Suppose to the contrary that $\lim_{t\to\infty} y(t) = o$. Since $\frac{a_2(t)}{a_1(t)}$ is bounded and $h(x) \to o$ as $x \to o$ we get that

$(1 + a_2(t)/a_1(t)) h(y(g(t))) \to o$ as $t \to \infty$. Since $\liminf_{t\to\infty} (|f(t)| / a_1(t)) > o$, (t) reveals that $y'(t)$ assumes a constant sign making $y(t)$ nonoscillatory. This contradiction shows that

$$\limsup_{t\to\infty} |y(t)| > o .$$

Now

$$\int_T^\infty |a(t)|dt \leq \int_T^\infty a_1(t)dt + \int_T^\infty |a_2(t)|dt$$

$$= \int_T^\infty a_1(t)dt + \int_T^\infty \frac{|a_2(t)|}{a_1(t)} a_1(t)dt$$

$$< \infty ,$$

since $\frac{a_2(t)}{a_1(t)}$ is bounded as $t \to \infty$ and $\int_T^\infty |f(t)|dt < \infty$. By Theorem 4 of this author [9], $y(t)$ is slowly oscillating. The proof is now complete.

EXAMPLE (1). Consider the equation.

$$y''(t) + \frac{5}{4t^2} y(t) = \frac{1}{t^{3/2}} , \quad t > o . \tag{6}$$

Here $a_1(t) \equiv a(t) = \frac{5}{4t^2}$. All conditions of Theorem 3 are satisfied. Hence all oscillatory solutions of (6) are slowly oscillating and satisfy $\limsup_{t\to\infty} |y(t)| > o$. In fact $y(t) = \sqrt{t}\ (1 + 2 \sin(\ln t))$ is one such solution.

EXAMPLE (2). Let $y(t)$ be a solution of

$$y''(t) + \frac{1 + 2 \sin t}{t^3} y(t - \pi) = \frac{2}{t^3} + \frac{\cos t}{t^4} , \quad t > \pi . \tag{7}$$

Taking $a_1(t) = 1/t^3$, $a_2(t) = 2 \sin t / t^3$ we find that all conditions of this theorem are satisfied. Hence either $y(t)$ is nonoscillatory or slowly oscillating with no limit at ∞ .

Example 1 suggests the following theorem:

THEOREM (2). Suppose $a(t) = a_1(t) + a_2(t)$ where $a_1(t) > o$ and $a_2(t) / a_1(t)$ is bounded for large t . Further suppose that

$\int^{\infty} |f(t)|dt < \infty$ and $\frac{|f(t)|}{a_1(t)} \to \infty$ as $t \to \infty$. Let $y(t)$ be an oscillatory solution of (2) . Then $y(t)$ is slowly oscillating and $\limsup_{t\to\infty} |y(t)| = \infty$.

PROOF. We only need to show that $\limsup_{t\to\infty} |y(t)| = \infty$. Suppose to the contrary that $y(t)$ is bounded. Then (5) immediately reveals that $(ry')'/a_1(t) \to \infty$ as $t \to \infty$.
Since $a_1(t) > o$, $|y'(t)| > o$ eventually. This forces $y(t)$ to be nonoscillatory, a contradiction. This proves the theorem.

EXAMPLE (3). The equation

$$y''(t) + \left(\frac{5}{4t^2} + \frac{6\sin(\ell nt)}{t^4}\right) y(t) = \frac{1}{t^{3/2}} + \frac{6}{t^{7/2}}\left(\sin(\ell n\ t) + 2\sin^2(\ell n\ t)\right), \quad (8)$$

$t > o$,

satisfies all conditions of Theorem 2 by choosing a_1 and a_2 as $a_1(t) = \frac{5}{4t^2}$ and $a_2(t) = \frac{6\sin(\ell nt)}{t^4}$. This equation has

$$y(t) = \sqrt{t}\ (1 + 2\sin(\ell nt))$$

as an oscillatory solution satisfying the conclusion of this theorem.

Our next theorem gives conditions for boundedness of all moderately oscillating solutions of equation 2. It is well known that if $a(t) = a_1(t) + a_2(t)$ where $a_1(t) \to L > o$ as $t \to \infty$; $a_1(t)$ is of bounded variation and $\int^{\infty} |a\ (t)|dt < \infty$ then all solutions of

$$y''(t) + a(t)y(t) = o \quad (9)$$

are bounded with bounded derivatives. See Cesari [3, p.85]. It is not true for retarded equations

$$y''(t) + a(t)h(y(g(t))) = o \quad (10)$$

as the following example shows.

EXAMPLE (4). The equation

$$y'' + 2e^{\pi/2}y(t - \pi/2) = o \tag{11}$$

has $y(t) = e^t \sin t$ as a solution. However our next theorem gives the partial result. We prove it in more generality for equation 2.

THEOREM (3). Suppose $a(t) = a_1(t) + a_2(t)$ where $a_1(t) \to L > o$ as $t \to \infty$, $\int^\infty |a_2(t)|dt < \infty$ and $\int^\infty |f(t)|dt < \infty$. Further suppose that there exists a $\lambda > o$ such that $L + \lambda < (4-\lambda)/\alpha^2 m$ where for any oscillatory solution y of (31),
$\alpha = \sup \left\{ x_2 - x_1 : y(x_2) = y(x_1) = o , y(x) \neq o , x \varepsilon (x_1, x_2) \right\} < \infty$
then $y(t)$ is bounded.

PROOF. Let T be large enough so that for $t > T$, $a_1(t) \leq L + \lambda$. Let $T_1 > T$ be large enough so that $g(T_1) > T$ and $y(T_1) = o$,
$\alpha m \int_T^\infty |a_2(t)|dt < \lambda/2$ and $\int_T^\infty |f(t)\, dt < \lambda/2$.

Suppose to the contrary that $\limsup_{t\to\infty} |y(t)| = \infty$. Then there is a

$T_1' > T_2 > T_1$ such that $y(T_2) = o$ and $P_1 = \max \left\{ |y(t)| : T \leq t \leq T ' \right\} = |y(T '))| > \lambda$.
$y(x_1) = y(x_2) = o$. Let $M_1 = \text{Max} \left\{ |y(t)| : x_1 \leq t \leq x_2 \right\}$, where $[x_1,x_2]$ is the smallest closed interval containing T_1'. It is clear that $T_2 \leq x_1$,

$$|y(t)| \leq M_1 \tag{12}$$

and

$$|y(g(t))| \leq M_1 \tag{13}$$

for $t\varepsilon\ [x_1, x_2]$. Let $x_0 \varepsilon [x_1, x_2]$ such that $M_1 = |y(x_0)|$. Since

$$M_1 = \int_{x_1}^{x_0} y'(t)dt = - \int_{x_2}^{x_0} y'(t)dt$$

we have

$$2M_1 \leq \int_{x_1}^{x_2} |y'(t)|dt$$

$$= \int_{x_1}^{x_2} |y'(t)|^{\frac{1}{2}} |y'(t)|^{\frac{1}{2}} dt$$

$$4M_1^2 \leq \left(\int_{x_1}^{x_2} dt \right) \left(\int_{x_1}^{x_2} y'(t) \cdot y'(t) dt \right) .$$

by Schwarz's inequality. Integrating by parts and using (2) we get

$$4M_1^2 \leq (x_2 - x_1) \left[\int_{x_1}^{x_2} y(t)a(t)h(y(g(t)))dt - \int_{x_1}^{x_2} y(t)f(t)dt \right] . \tag{14}$$

From (14) we have

$$4M_1 \leq \alpha \int_{x_1}^{x_2} a(t) \frac{h(y(g(t)))}{y(g(t))} y(g(t))dt + \alpha \int_{x_1}^{x_2} |f(t)|dt$$

since $x_2 - x_1 \leq \alpha$. This yields in view of (12) and (13)

$$4 \leq \alpha m \int_{x_1}^{x_2} (a_1(t) + |a_2(t)|)dt + \frac{2}{M_1} \int_{x_1}^{x_2} |f(t)|dt \tag{15}$$

which gives

$$4 \leq \alpha^2 m(L + \lambda) + \alpha m \int_{x_1}^{x_2} |a_2(t)|dt + \frac{\alpha}{M_1} \int_{x_1}^{x_2} |f(t)|dt \tag{16}$$

or

$$4 \leq \alpha^2 m(L + \lambda) + \lambda , \tag{17}$$

since $\frac{\alpha}{M_1} < 1$, $\alpha m \int_{x_1}^{x_2} |a_2(t)|dt < \lambda/2$ and $\int_{x_1}^{x_2} |f(t)|dt \leq \lambda/2$.

This contradiction, in (17) , completes the proof.

REMARK. Coming back to example (4) we see that for the solution $y = e^t \sin t$, $\alpha = \pi$, $L = 2e^{\pi/2}, m = 1$, $f(t) \equiv o$ and $a_2(t) \equiv o$. Thus far any $\lambda > o$, $\alpha^2 m (L + \lambda) = \pi^2(2 e^{\pi/2} + \lambda) > 4 - \lambda$, satisfying the conclusion of the theorem.

EXAMPLE (5). The equation

$$y''(t) + 2\, e^{-3\pi/2} y(t - \tfrac{3\pi}{2}) = o \tag{18}$$

has $y = e^{-t} \sin t$ as a solution. Here $\alpha = \pi$

$L = e^{\frac{-3\pi}{2}} \cong e^{-4.7}$, $m = 1$.

Therefore for $\lambda = .01$

$m\alpha^2(L + \lambda) = \pi^2(e^{-4.7} + .01) < 4$, once again, satisfying this theorem.

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