Internat. J. Math. & Math. Sci.
Vol. 2 #3 (1979) 473-480

A POINTWISE CONTRACTION CRITERIA FOR THE EXISTENCE OF FIXED POINTS

V. M. SEHGAL

Department of Mathematics
University of Wyoming
Laramie, Wyoming 82071
U.S.A.

(Received May 7, 1979)

ABSTRACT. Let S be a subset of a metric space (X,d) and $T: S \to X$ be a mapping. In this paper, we define the notion of lower directional increment $QT(x,y]$ of T at $x \in S$ in the direction of $y \in X$ and give sufficient conditions for T to have a fixed point when $QT(x,Tx] < 1$ for each $x \in S$. The results herein generalize the recent theorems of Clarke (Canad. Math. Bull. Vol. 21(1), 1978, 7-11) and also improve considerably some earlier results.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. *Primary* 47H10, *secondary* 54H25.

INTRODUCTION.

In a recent paper [2], Clarke introduced the notion of lower derivative $\underline{D}T(x,Tx)$ for a mapping $T: X \to X$ on a metric space X and obtained sufficient conditions for a continuous mapping T to have a fixed point in X when $\underline{D}T(x,Tx) < 1$ for each $x \in X$. However, in order that $\underline{D}T(x,y)$ be finite, it is necessary that (x,y) (to be defined later) contain points arbitrary close to x whenever $x \neq y$. The purpose of this paper is (a) to remove the above restriction by introducing the notion of lower directional increment (see below), (b) to consider mappings that are not necessarily continuous and are defined on a subset S of X with values in X. As a consequence of our main result, we obtain the results contained in [2] and also some other results (see [3] and [5]).

1. PRELIMINARIES.

Throughout this paper, let (X,d) denote a complete metric space and S a nonempty subset of X. A function $\phi: S \to R^+$ (nonnegative reals) is lower semicontinuous (l.s.c.) on S iff for each $x_o \in S$ $\{x \in S: \phi(x) > r\}$ is open for each real r. It is easy to verify that given a function $\phi: S \to R^+$, the function ϕ induces a partial order $\leq$ in S given by

$$x \leq y \text{ in } S \text{ iff } d(x,y) \leq \phi(y) - \phi(x). \tag{1.1}$$

The following Lemma is well-known (see Brondsted [1] or Kasahara [4]).

LEMMA 1. Let S be a closed subset of X and $\phi: S \to R^+$ be a l.s.c. function on S. Then there is an element $u \in S$ which is minimal with respect to partial order (1.1) in S.

As a consequence of Lemma 1, we have

LEMMA 2. Let S be a closed subset of X and $\phi: S \to R^+$ be a l.s.c. function on S. Then for each ε with $0 < \varepsilon < 1$, there exists a $u = u(\varepsilon) \in S$ such that

$$\phi(u) \leq \phi(x) + \varepsilon d(x,u), \tag{1.2}$$

for each $x \in S$.

PROOF. The proof is immediate by Lemma 1 (if we replace the metric d by d_ε, $d_\varepsilon = \varepsilon\cdot d$).

LEMMA 3. Let S be a closed subset of X and $\phi: S \to R^+$ be a l.s.c. on S. If for a sequence $\{x_n\} \subseteq S$ with a cluster point x_o,

$$\phi(x_n) \leq \phi(x) + \frac{1}{n}\, d(x,x_n),$$

for each $x \in S$, then $\phi(x_o) \leq \phi(x)$ for each $x \in S$.

PROOF. The proof is immediate since for any l.s.c. f, $d(y_n,y) \to 0$ implies that $f(y) \leq \underline{\lim}_{n\to\infty} f(y_n)$.

2. MAIN RESULTS.

Let S be a subset of X. For an $x \in S$ and $y \in X$ with $x \neq y$, define

$$(x,y] = \{z \in X: z \neq x \text{ and } d(x,z) + d(z,y) = d(x,y)\}$$

note that $y \in (x,y]$.

Let $T: S \to X$ be a mapping. For $x \in S$ and $y \in X$, define the lower directional increment $QT(x,y]$ of T at x in the direction y as

$$\begin{aligned} QT(x,y] &= 0, \quad \text{if } x = y, \\ &= \inf\{\frac{d(Tx,Tz)}{d(x,z)}: z \in (x,y] \cap S\}, \quad \text{if } (x,y] \cap S \neq \phi, \\ &= \infty, \quad \text{if } (x,y] \cap S = \phi. \end{aligned}$$

For the convenience of the notation, we shall denote $\rho(x,y) = \dfrac{d(Tx,Ty)}{d(x,y)}$ if $x \neq y$.

REMARK. It may be noted that if $QT(x,y]$ is finite and $x \neq y$, then there is a sequence $\{z_n\} \subseteq (x,y] \cap S$ such that $\rho(x,z_n) \to QT(x,y)$.

The following is the main result of this paper and is related to the lines of argument in [2].

THEOREM 1. Let S be a closed subset of X and $T: S \to X$ be a mapping satisfying the following conditions:

The mapping $\phi: S \to R^+$ defined by $\phi(x) = d(x,Tx)$ is l.s.c. on S, (2.1)

For each $x \in S$, $QT(x,Tx] < 1$, (2.2)

If $\alpha = \sup\{QT(x,Tx]: x \in S\}$ then either (a) $\alpha < 1$ or (b) if $\alpha = 1$ then any sequence $\{x_n\} \subseteq S$ for which $QT(x_n,Tx_n] \to 1$ implies that the sequence $\{x_n\}$ has a cluster point. (2.3)

Then T has a fixed point in S.

PROOF. It follows by Lemma 2, that for each positive integer n, there is a $u_n \in S$ such that

$$\phi(u_n) \le \phi(x) + \frac{1}{n} d(x,u_n), \tag{2.4}$$

for each $x \in S$. We assert that if $\alpha < 1$ then $u_n = Tu_n$ for some n and if $\alpha = 1$, then $QT(u_n,Tu_n] \to 1$. Suppose $u_n \ne Tu_n$ for any n. Then by the remark, for each fixed n, there exists sequence $\{z_m\} \subseteq (u_n,Tu_n] \cap S$ such that

$$\rho(u_n,z_m) \to QT(u_n,Tu_n] \tag{2.5}$$

as $m \to \infty$. It now follows by (2.4) that for each m,

$$\phi(u_n) \le \phi(z_m) + \frac{1}{n} d(u_n,z_m) \le d(z_m,Tu_n) + d(Tu_n,Tz_m) + \frac{1}{n} d(u_n,z_m). \tag{2.6}$$

Since for each m, $d(u_n,z_m) + d(z_m,Tu_n) = \phi(u_n)$. We have for each m,

$$(1 - \frac{1}{n}) \le \rho(u_n,z_m).$$

Therefore, as $m \to \infty$, it follows by (2.5) and (2.2) that for each fixed n,

$$(1 - \frac{1}{n}) \le QT(u_n,Tu_n] < 1.$$

Consequently, if $u_n \ne Tu_n$ for any n, then $QT(u_n,Tu_n) \to 1$. Therefore, if (2.3a) holds, then $u_n = Tu_n$ for some n and the theorem is established in this case, otherwise by (2.3b), the sequence $\{u_n\}$ has a cluster point $u \in S$.

It follows by Lemma 3, that

$$\phi(u) \leq \phi(x), \tag{2.7}$$

for each $x \in S$. We assert that $Tu = u$. Suppose $Tu \neq u$. Then again by the remark, there is a sequence $z_n \in (u,Tu] \cap S$ such that as $n \to \infty$,

$$\rho(u,z_n) \to QT(u,Tu] \tag{2.8}$$

However, by (2.7) and the relation $d(u,z_n) + d(z_n,Tu) = \phi(u)$, we have for each n,

$$d(u,z_n) + d(z_n,Tu) = \phi(u) \leq \phi(z_n) \leq d(z_n,Tu) + d(Tu,Tz_n).$$

This implies that $\rho(u,z_n) \geq 1$ for each n and hence by (2.8) $QT(u,Tu] \geq 1$. This contradicts (2.2). Thus $u = Tu$.

3. SOME APPLICATIONS.

For a mapping $T\colon X \to X$, Clarke [2] defined lower derivative $\underline{D}T(x,y)$ of T at x in the direction of y as

$$\begin{aligned}
\underline{D}T(x,y) &= 0, \quad \text{if } x = y,\\
&= \varliminf_{\substack{z\to x\\ z \in (x,y)}} \rho(x,z), \quad \text{if } (x,y) = (x,y]\setminus\{y\} \neq \phi,\\
&= \infty, \quad \text{if } (x,y) = \phi,
\end{aligned}$$

where $\varliminf_{\substack{z\to x\\ z \in (x,y)}} \rho(x,z) = \lim_{\varepsilon\to 0} [\inf_{\substack{z \in (x,y)\\ d(z,x) < \varepsilon}} \rho(x,z)]$.

Since for any $x,y \in X$, $QT(x,y] \leq \underline{D}T(x,y)$, the following results in [2] are special cases of Theorem 1.

COROLLARY 1. Let $T\colon X \to X$ be a continuous mapping such that $\sup\{\underline{D}T(x,Tx)\colon x \in X\} < 1$. Then T has a fixed point.

COROLLARY 2. Let $T\colon X \to X$ be a continuous mapping such that

$\underline{D}T(x,Tx) < 1$ for each $x \in X$. If for any sequence $\{x_n\}$ in X with $\underline{D}T(x_n,Tx_n) \to 1$ implies that the sequence $\{x_n\}$ has a cluster point, then T has a fixed point.

The following simple examples show that both Corollaries 1 and 2 are indeed special cases of Theorem 1.

EXAMPLE 1. Let $X = \{0,1\}$ with the discrete metric and $T: X \to X$ be a constant mapping defined by $Tx = 0$ for each $x \in X$. Since $(1,T1) = \phi$, $\underline{D}T(1,T1) = \infty$, T does not satisfy the conditions of Corollary 1. However, since T is continuous and $QT(x,Tx] = 0$ for each $x \in X$, T satisfies conditions of Theorem 1 and it follows T has a fixed point.

EXAMPLE 2. Let X be the closed interval $[\frac{1}{2},3]$ with the usual metric. Let $T: X \to X$ be the mapping defined by

$$Tx = \frac{1}{x} + 1.$$

Clearly, T is continuous, strictly decreasing and for each x with $Tx \neq x$, $(x,Tx) \neq \phi$. Further, it is easy to verify that for any $x \neq z$, $\rho(x,z) = \frac{1}{xz}$ and therefore, for any $x \in X$ with $x \neq Tx$, $\underline{D}T(x,Tx) = \frac{1}{x^2}$. Consequently, if $x \leq 1$, $\underline{D}T(x,Tx) \geq 1$ and hence T does not satisfy conditions of Corollary 2. However, since for any $x \neq Tx$, $Tx \in (x,Tx]$,

$$QT(x,Tx] = \inf \{\frac{1}{xz}: z \in (x,Tx]\} \leq \frac{1}{x \cdot Tx} = \frac{1}{x+1} < 1.$$

Since X is compact, T satisfies conditions of Theorem 1. In this case $x = \frac{1+\sqrt{5}}{2}$ is the only fixed point of T in X.

For a set $S \subseteq X$, let S^o denote the interior of S and δS its boundary. A mapping $T: S \to X$ is a contraction mapping if there exists a constant $k < 1$ such that for all $x,y \in S$, $d(Tx,Ty) \leq kd(x,y)$. As another consequence of Theorem 1, we have

COROLLARY 3. Let S be a closed subset of a Banach space X and $T: S \to X$ be a contraction mapping. If $T(\delta S) \subseteq S$, then T has a fixed point.

PROOF. Since T is continuous and for any $x,z \in S$, $\rho(x,z) \leq k < 1$, it suffices to show that for any $x \in S$ with $x \neq Tx$, $(x,Tx] \cap S \neq \phi$. Now, if $x \in S^o$, then for some $\varepsilon > 0$, $S(x,\varepsilon) = \{y: \|y-x\| < \varepsilon\} \subseteq S$. Choose a λ, $0 < \lambda < 1$ such that $(1-\lambda)\|x-Tx\| < \varepsilon$. Then $z = (\lambda x + (1-\lambda)Tx) \in S(x,\varepsilon) \cap (x,Tx]$ and hence $(x,Tx] \cap S \neq \phi$. If $x \in \delta S$, then by hypothesis $Tx \in S$ and consequently $Tx \in (x,Tx] \cap S$. Thus $\alpha = \sup\{QT(x,Tx]: x \in S\} < 1$.

The result below was obtained by Su and the author [5] (see also Edelstein [3]) and is again a consequence of Theorem 1.

COROLLARY 4. Let S be a compact subset of a Banach space X and $T: S \to X$ be a mapping satisfying the condition: for all $x,y \in S$, $x \neq y$, $\|Tx-Ty\| < \|x-y\|$. If $T(\delta S) \subseteq S$, then T has a fixed point.

PROOF. As in the proof of Corollary 3, for any $x \in S$ with $x \neq Tx$, $(x,Tx] \cap S \neq \phi$. Therefore, it follows by hypothesis that for any $x \in S$, $QT(x,Tx] < 1$. Since compactness implies (2.3b), T satisfies the conditions of Theorem 1 and has a fixed point in S.

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KEY WORDS AND PHRASES. Lower derivative, Lower directional increment, Fixed point theorems