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DIFFERENTIALS OF THE 2ND KIND ON A PRODUCT SURFACE

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ABSTRACT. This paper deals with the problems of representing an arbitrary double differential of the second kind, defined on a surface which is the topological product of two curves, in terms of the products of simple differentials of the second kind on the two curves. The curves are assumed to be non-singular and irreducible in a complex projective 2-space.

KEY WORDS AND PHRASES. *Algebraic surface, product surface, algebraic curves, double differentials of the second kind, simple differentials of the second kind, Abelian differentials, genus, exterior product, polar curves, resultant.*

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1. INTRODUCTION.

Our purpose in this paper is to prove a theorm, stated by Lefschetz, concerning double differentials of the 2nd kind on an algebraic surface. We will employ the Picard-Lefschetz technique as described in Lefschetz [1].

This amounts to reducing the initial differential to a new one of the same kind, but in a more suitable form for our purposes. A precise statement will be given below. In all that follows, our ground field, denoted by K, will be the complex numbers. Also, K(x, y, ...) or K[x, y, ...] will indicate the field of rational functions or ring of polynomials, respectively, in the complex variables x, y, ... with coefficients in K.

Suppose C_1 and C_2 are two distinct, irreducible, non-singular algebraic curves, each in a complex projective 2-space. The product surface Φ, i.e., the topological product $C_1 \times C_2$, is then a non-singular surface in projective 4-space. If C_1 and C_2 have affine equations $\phi(\alpha, \beta) = 0$ and $\psi(\alpha^*, \beta^*) = 0$, respective, then an affine model, S, of Φ may be thought of as the surface of pairs of points $(\alpha, \beta; \alpha^*, \beta^*)$ in complex 4-space. We will deal primarily with an affine model in this paper. Our objective is to prove the following: "Every double differential of the 2nd kind on Φ is reducible, modulo $d\omega$, (where ω is a simple differential on Φ), to a linear combination of products $\omega \wedge \omega^*$ of simple differentials of the 2nd kind, where ω belongs to C_1 and ω^* belongs to C_2 and neither are derived." See Lefschetz [2].

We point out that the last above statement can also be realized by using modern techniques of algebraic topology. In particular, see page 80 of Hodge and Atiyah's paper "Integrals of the 2nd Kind on an Algebraic Variety" appearing in the Annals of Mathematics, Vol. 62, 1955. However, we feel that the classical approach used here has merit in itself since technique ranks equally with results in algebraic geometry.

SECTION 1.

Let $R = \dfrac{P(\alpha, \beta, \alpha^*, \beta^*)}{Q(\alpha, \beta, \alpha^*, \beta^*)}$

be a rational function such that for fixed α and β or fixed α^* and β^*, Q is not in the ideal generated by ϕ or ψ. We assume that the double differential $\omega_2 = Rd\alpha^*d\alpha$ is of the 2nd kind on S. Here we employ the Lefschetz definition [3], i.e., if $Q = Q_1Q_2 \ldots Q_k$, where the Q's are distinct, irreducible polynomials, then ω_2 is of the 2nd kind relative to Q_h provided there exists a simple differential ω on S such that $\omega_2 - d\omega$ is regular at the points of intersection of $Q_h = 0$ with S. Such points generally form a curve on S, called a polar curve of ω_2, and we designate it by ζ_h. The simple differential depends on the choice of h and we say that ω_2 is of the 2nd kind on S provided such an ω exists for each polar curve of ω_2. As an analogy to Lefschetz's plane section, we employ the curve C_2 and its copies on S, the latter obtained by varying α and β on C_1. First consider the intersection of ζ_h with such a "general" C_2. A suitable choice of affine co-ordinates will insure that none of the polar curves pass through a point at infinity on a general C_2. Also, a non-singular linear transformation will guarantee that both ϕ and ψ contain β^{d_1} and $(\beta^*)^{d_2}$, where d_1 and d_2 are the degrees of ϕ and ψ, respectively. Such co-ordinate changes do not effect the kind of differential involved. Treating α and β as parameters, we designate by $P_1(\alpha, \beta, \alpha_1^*, \beta_1^*)$, $P_2(\alpha, \beta, \alpha_2^*, \beta_2^*)$, ..., $P_m(\alpha, \beta, \alpha_m^*, \beta_m^*)$ the intersection points of ζ_h with a general C_2. The α_i^* and β_i^*'s will depend on the choice of α and β. Suppose ζ_h is a polar curve of order n. Then, in a neighborhood of P_i on our general C_2, the following expansion of R is valid:

$$R = \frac{R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^n} + \frac{R_{-n+1}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^{n-1}} + \ldots + \frac{R_{-1}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)} + \text{higher powers of } (\alpha^* - \alpha_i^*). \tag{1.1}$$

We use $(\alpha^* - \alpha_i^*)$ as parameter and all numerators are rational in their variables.

LEMMA 1. There exists a simple differential ω_{1h} on S such that $\omega_2 + d\omega_{1h}$ is regular on ζ_h.

PROOF. Define $H(\alpha, \beta, \alpha^*, \beta^*) = R \prod_{i=1}^{m} (\alpha^* - \alpha_i^*)^n$. We first will obtain an expression for $R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)$. Let E_1 be the resultant of Q_h and ψ with respect to β^* and E_2 the resultant of E_1 and ϕ with respect to β. E_2 may be considered as a polynomial in α^* with polynomial coefficients in α and as such, has the α_i^*'s as its only zeroes. For α finite, this implies that the elementary symmetric functions of the α_i^*'s are polynomials in α. Thus $\prod_{i=1}^{m} (\alpha^* - \alpha_i^*)^n$ is a polynomial in α and α^*. For simplicity, set $\prod_{i=1}^{m} (\alpha^* - \alpha_i^*)^n = \chi_h^{\ n}(\alpha, \alpha^*)$, where $\chi_h^{\ n}$ may be thought of as the projection of ζ_h on the (α, α^*)-plane. It is evident that $H(\alpha, \beta, \alpha^*, \beta^*)$ is of order zero on ζ_h. Also, $H(\alpha, \beta, \alpha_i^*, \beta_i^*) = \prod_{\substack{j=1 \\ j\neq i}}^{m} (\alpha_i^* - \alpha_j^*)^n \cdot R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)$. Hence

$$R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*) = \frac{H(\alpha, \beta, \alpha_i^*, \beta_i^*)}{\prod_{\substack{j=1 \\ j\neq i}}^{m} (\alpha_i^* - \alpha_j^*)^n} \tag{1.2}$$

Next we set

$$V_n = \frac{1}{n-1} \left[\sum_{i=1}^{m} \frac{R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^{n-1}} \right] . \tag{1.3}$$

Using R_{-n} from (1.2), we see that V_n not only has ζ_h as a polar curve but can have poles

i) at points of intersection of ζ_h with ζ_k, $k \neq h$, for then H has poles and their occurrence depends on α and β;

ii) at points where α is such that $\alpha_i^* = \alpha_j^*$, $j \neq i$;

iii) if α or β is infinite.

We can now show that $\omega_2 + d[V_n d\alpha] = R d\alpha^* d\alpha + d[V_n d\alpha]$ behaves like R on ζ_h but with n replaced by (n - 1). First

$$\frac{R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^n} d\alpha^* d\alpha + d\left[\frac{1}{n-1} \sum_{i=1}^{m} \frac{R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^{n-1}} d\alpha\right]$$
$$= \frac{R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^n} d\alpha^* d\alpha + \frac{1}{n-1}\left[\sum_{i=1}^{m} d\left(\frac{R_{-n}}{(\alpha^* - \alpha_i^*)^{n-1}}\right) da\right]. \tag{1.4}$$

A typical element of the last sum can be written as

$$\frac{1}{(\alpha^* - \alpha_i^*)^{n-1}} \left[\frac{\partial R_{-n}}{\partial \alpha} d\alpha + \frac{\partial R_{-n}}{\partial \beta} d\beta\right] - (n-1) \frac{R_{-n}}{(\alpha^* - \alpha_i^*)^n} d\alpha^*.$$

Since not both ϕ_α and ϕ_β vanish, we can assume $\phi_\beta \neq 0$ and get $d\beta = \frac{-\phi_\alpha}{\phi_\beta} d\alpha$. Then (1.4) can be written as

$$\frac{R_{-n}(\alpha, \beta, \alpha_i^*, \beta_i^*)}{(\alpha^* - \alpha_i^*)^n} d\alpha^* d\alpha +$$
$$+ \frac{1}{n-1}\left[\sum_{i=1}^{m} \{\frac{1}{(\alpha^* - \alpha_i^*)^{n-1}} \cdot \left(\frac{\partial R_{-n}}{\partial \alpha} d\alpha - \frac{\partial R_{-n}}{\partial \beta} \cdot \frac{\phi_\alpha}{\phi_\beta} d\alpha\right) - \frac{(n-1) R_{-n}}{(\alpha^* - \alpha_i^*)^n} d\alpha^*\} d\alpha\right].$$

The exterior product then gives

$$\frac{R_{-n}}{(\alpha^* - \alpha_i^*)^n} d\alpha^* d\alpha - \sum_{i=1}^{m} \frac{R_{-n}}{(\alpha^* - \alpha_i^*)^n} d\alpha^* d\alpha.$$

Thus, in a neighborhood of P_i, $R d\alpha^* d\alpha + d[V_n d\alpha]$ behaves like ω_2 but with n replaced by (n - 1). Next, we can define a V_{n-1} of the same type as V_n with (n - 1) in place of n and $R d\alpha^* d\alpha + d[(V_n + V_{n-1}) d\alpha]$ will begin, in some neighborhood of P_i, with the term $\frac{\hat{R}_{-n+2}}{(\alpha^* - \alpha_i^*)^{n-2}}$. Continuing in this manner, we will arrive at the differential $R d\alpha^* d\alpha + d[V_n + V_{n-1} + \dots + V_2) d\alpha]$, a double differential of the 2nd kind in which the corresponding n is one.

If we still denote the coefficient of $\frac{1}{\alpha^* - \alpha_i^*}$ by R_{-1}, then since ω_2 is of the 2nd kind, its residue relative to any polar curve must be derived, i.e., $R_{-1} = \frac{dT}{d\alpha}$ where $T(\alpha, \beta, \alpha_i^*, \beta_i^*)$ is a rational function on ζ_h having poles only of type i), ii), or iii), above. See Lefschetz [4]. If we set

$$\omega_{1h} = (V_2 + \dots + V_n)d\alpha + \sum_{i=1}^{m} \frac{T(d\alpha^* - \frac{d\alpha_i^*}{d\alpha} d\alpha)}{(\alpha^* - \alpha_i^*)}, \tag{1.5}$$

then we can immediately show that $Rd\alpha^* d\alpha + d\omega_{1h}$ is regular on ζ_h, as follows: From the above, it is evident that $Rd\alpha^* d\alpha + d[(V_2 + \dots + V_n)d\alpha]$ behaves like ω_2 relative to ζ_h but with the corresponding $n = 1$, so we need to look only at

$$\tau = \sum_{i=1}^{m} \frac{T(d\alpha^* - \frac{d\alpha_i^*}{d\alpha} d\alpha)}{(\alpha^* - \alpha_i^*)} .$$

Then,

$$d\tau = \left[\sum_{i=1}^{m} \frac{dT}{(\alpha^* - \alpha_i^*)}\right] d\alpha^* - \sum_{i=1}^{m} d\left[\frac{d\alpha_i^*}{d\alpha} \cdot \frac{T}{\alpha^* - \alpha_i^*}\right] d\alpha =$$

$$\left[\sum_{i=1}^{m} \frac{(\alpha^* - \alpha_i^*)dT - Td(\alpha^* - \alpha_i^*)}{(\alpha^* - \alpha_i^*)^2}\right] d\alpha^* -$$

$$\left\{\sum_{i=1}^{m} \left[\frac{d\alpha_i^*}{d\alpha} \cdot d\left(\frac{T}{(\alpha^* - \alpha_i^*)}\right) + \frac{T}{(\alpha^* - \alpha_i^*)} \cdot d\left(\frac{d\alpha_i^*}{d\alpha}\right)\right]\right\} d\alpha =$$

$$\sum_{i=1}^{m} \frac{\frac{dT}{d\alpha}}{(\alpha^* - \alpha_i^*)} d\alpha d\alpha^* - \sum_{i=1}^{m} \frac{T(d\alpha^* - d\alpha_i^*)d\alpha^*}{(\alpha^* - \alpha_i^*)^2} -$$

$$\sum_{i=1}^{m} \frac{d\alpha_i^*}{d\alpha} \left[\frac{(\alpha^* - \alpha_i^*)dT - Td(\alpha^* - \alpha_i^*)}{(\alpha^* - \alpha_i^*)^2}\right] =$$

$$\sum_{i=1}^{m} \frac{\frac{dT}{d\alpha}}{(\alpha^* - \alpha_i^*)} d\alpha d\alpha^* + \sum_{i=1}^{m} \frac{T\left(\frac{d\alpha_i^*}{d\alpha}\right)}{(\alpha^* - \alpha_i^*)^2} d\alpha d\alpha^* -$$

$$\sum_{i=1}^{m} \frac{\left(\frac{d\alpha_i^*}{d\alpha}\right) dT}{(\alpha^* - \alpha_i^*)} d\alpha d\alpha^* + \sum_{i=1}^{m} \frac{\left(\frac{d\alpha_i^*}{d\alpha}\right) T}{(\alpha^* - \alpha_i^*)^2} d\alpha^* d\alpha =$$

$$\sum_{i=1}^{m} \frac{\frac{dT}{d\alpha}}{\alpha^* - \alpha_i^*} d\alpha d\alpha^*,$$

since the third term in the last sum above is zero and the second and fourth terms are negatives of one another. However,

$$\sum_{i=1}^{m} \frac{\frac{dT}{d\alpha}}{\alpha^* - \alpha_i^*} d\alpha d\alpha^* = -\sum_{i=1}^{m} \frac{R_{-1}}{\alpha^* - \alpha_i^*} d\alpha^* d\alpha.$$

This concludes the proof of Lemma 1.

For brevity, we will write ω_{1h} in (1.5) as

$$\omega_{1h} = V(\alpha, \beta, \alpha^*, \beta^*)d\alpha - U(\alpha, \beta, \alpha^*, \beta^*)d\alpha^*,$$

where V and U are rational on S. From Lemma 1, V and U are infinite on the intersection of $\chi_h = 0$ with S, so that in addition to ζ_h, they are infinite on any other curve which projects onto $\chi_h = 0$. We will designate this residual intersection by D_h.

LEMMA 2. ω_{1h} may be replaced with a similar differential but with D_h eliminated as a polar curve (or curves).

PROOF. For almost any non-singular affine transformation $(\alpha, \beta, \alpha^*, \beta^*)$ to $(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)$, the projections of the transforms of ζ_h and D_h on the $(\tilde{\alpha}, \tilde{\alpha}^*)$-plane will have at most a finite number of points in common. Under such a transformation $R\, d\alpha^* d\alpha + d(V d\alpha - U d\alpha^*)$ becomes

$$\tilde{R}\, d\tilde{\alpha}^* d\tilde{\alpha} + d(\tilde{V}\, d\tilde{\alpha} - \tilde{U}\, d\tilde{\alpha}^*),$$

where we set

$$\tilde{V} = \frac{\tilde{A}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)}{\tilde{G}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)} \text{ and}$$

$$\tilde{U} = \frac{\tilde{B}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)}{\tilde{G}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)},$$

$\tilde{A}$, $\tilde{B}$, and $\tilde{G}$ polynomials. Here we have made use of the transformations $\phi(\alpha, \beta)$ to $\tilde{\phi}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)$ and $\psi(\alpha^*, \beta^*)$ to $\tilde{\psi}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)$ in eliminating $d\tilde{\beta}$ and $d\tilde{\beta}^*$. We can also assume that neither $\frac{\partial\tilde{\phi}}{\partial\tilde{\beta}}$ nor $\frac{\partial\tilde{\psi}}{\partial\tilde{\beta}^*}$ are zero, due to the absence of singularities. Let the resultant of $\tilde{G}$ and $\tilde{\psi}$ with respect to $\tilde{\beta}^*$ be denoted by $\tilde{E}_1$, i.e., $I\tilde{G} + J\tilde{\psi} = \tilde{E}_1$. Further, the resultant of $\tilde{E}_1$ and $\tilde{\phi}$ with respect to $\tilde{\beta}$ gives a second resultant, say G_1, i.e., $L\tilde{E}_1 + M\tilde{\phi} = G_1$. Thus, $LI\tilde{G} + LJ\tilde{\psi} = G_1 - M\tilde{\phi}$ and in the transformed surface $LI\tilde{G} = G_1$ since $LJ\tilde{\psi}$ and $M\tilde{\phi}$ are zero. It follows that ω_{1h} can be written as

$$\omega_{1h} = \frac{\tilde{\tilde{A}}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)}{G_1(\tilde{\alpha}, \tilde{\alpha}^*)}\, d\tilde{\alpha} - \frac{\tilde{\tilde{B}}(\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}^*, \tilde{\beta}^*)}{G_1(\tilde{\alpha}, \tilde{\alpha}^*)}\, d\tilde{\alpha}^*,$$

where

$$\tilde{\tilde{A}} = LI\tilde{A}, \quad \tilde{\tilde{B}} = \cdot LI\tilde{B}$$

and both are polynomials. If we call the projection of the transform of ζ_h, in the $(\tilde{\alpha}, \tilde{\alpha}^*)$-plane, by $\tilde{\chi}_h$, then $G_1 = \tilde{\chi}_h{}^n(\tilde{\alpha}, \tilde{\alpha}^*)\tilde{\tilde{\chi}}_h(\tilde{\alpha}, \tilde{\alpha}^*)$, where $\tilde{\chi}_h{}^n$ and $\tilde{\tilde{\chi}}_h$ are relatively prime polynomials. Considering them as polynomials in $\tilde{\alpha}^*$ with coefficients in $\tilde{\alpha}$, there exist polynomials $s(\tilde{\alpha}, \tilde{\alpha}^*)$ and $t(\tilde{\alpha}, \tilde{\alpha}^*)$ such that

$$s\tilde{\chi}_h{}^n + t\tilde{\tilde{\chi}}_h = p(\tilde{\alpha}).$$

Using this last equality, we can write

$$\frac{1}{G_1} = \frac{1}{\tilde{\chi}_h{}^n\tilde{\tilde{\chi}}_h} = \frac{1}{p(\tilde{\alpha})}\left[\frac{s}{\tilde{\tilde{\chi}}_h} + \frac{t}{\tilde{\chi}_h{}^n}\right] = \frac{t}{p(\tilde{\alpha})\tilde{\chi}_h{}^n} + \frac{s}{P(\tilde{\alpha})\tilde{\tilde{\chi}}_h}.$$

Therefore

$$\omega_{1h} = [\tilde{\tilde{A}}d\tilde{\alpha} - \tilde{\tilde{B}}d\tilde{\alpha}^*] \cdot \left[\frac{t}{p(\tilde{\alpha})\tilde{\chi}_h{}^n} + \frac{s}{p(\tilde{\alpha})\tilde{\tilde{\chi}}_h}\right] =$$

$$\tilde{\tilde{A}}\left[\frac{t}{p(\tilde{\alpha})\tilde{\chi}_h{}^n} + \frac{s}{p(\tilde{\alpha})\tilde{\tilde{\chi}}_h}\right]d\tilde{\alpha} - \tilde{\tilde{B}}\left[\frac{t}{p\tilde{\chi}_h{}^n} + \frac{s}{p\tilde{\tilde{\chi}}_h}\right]d\tilde{\alpha}^* =$$

$$\frac{\tilde{\tilde{A}}td\tilde{\alpha} - \tilde{\tilde{B}}td\tilde{\alpha}^*}{p(\tilde{\alpha})\tilde{\chi}_h{}^n} + \frac{\tilde{\tilde{A}}sd\tilde{\alpha} - \tilde{\tilde{B}}sd\tilde{\alpha}^*}{p(\tilde{\alpha})\tilde{\tilde{\chi}}_h} .$$

It is evident that $\tilde{R}d\tilde{\alpha}^*d\tilde{\alpha} + d\omega_{1h}$ is regular on the transform of ζ_h and this is true even if we suppress the last term above, i.e., $\frac{\tilde{\tilde{A}}sd\tilde{\alpha} - \tilde{\tilde{B}}sd\tilde{\alpha}^*}{p\tilde{\tilde{\chi}}_h}$. If we replace ω_{1h} with $\tilde{\omega}_{1h} = \frac{\tilde{\tilde{A}}td\tilde{\alpha} - \tilde{\tilde{B}}td\tilde{\alpha}^*}{p(\tilde{\alpha})\tilde{\chi}_h{}^n} = \frac{A_h d\tilde{\alpha} - B_h d\tilde{\alpha}^*}{p(\tilde{\alpha})\tilde{\chi}_h{}^n}$, we have $\tilde{R}d\tilde{\alpha}^*d\tilde{\alpha} + d\tilde{\omega}_{1h}$, a differential of the 2nd kind on the transformed surface, regular on ζ_h transform and the transform of D_h is no longer a polar curve of $\tilde{\omega}_{1h}$.

SECTION 2.

The above reduction and replacement can be carried out for all polar curves ζ_h of ω_2. Returning to the original notation and co-ordinates, we would then have a set of ω_{1h} such that $\omega_2 + d(\sum_h \omega_{1h})$ would be regular on the ζ_h but would have poles for certain values of α and β. Let us write $\omega_2 + d(\sum_h \omega_{1h})$ in the form

$$W_2 = \frac{A(\alpha,\ \beta,\ \alpha^*,\ \beta^*)d\alpha^*d\alpha}{B(\alpha,\ \beta,\ \alpha^*,\ \beta^*)G(\alpha)} ,$$

where A, B, and G are polynomials and A/B is regular except possibly for points of infinity.

LEMMA 3. $W_2 = \frac{Ad\alpha^*d\alpha}{BG}$ can be reduced to the form $\frac{\pi(\alpha,\ \beta,\ \alpha^*,\ \beta^*)}{G(\alpha)} d\alpha^*d\alpha$ where π is a polynomial.

PROOF. We begin by replacing the affine co-ordinates α^* and β^* with projective co-ordinates α_0^*, α_1^*, and α_2^* in both $\psi(\alpha^*, \beta^*)$ and $B(\alpha, \beta, \alpha^*, \beta^*)$ where B is to be regarded as a polynomial in α_0^*, α_1^*, α_2^* with coefficients in the field $K(\alpha, \beta)$. By Kapferer's Theorem [5], A is representable in the form $Y\psi + ZB$, Y and Z polynomials in α_0^*, α_1^*, and α_2^*. This is true since A must vanish at the simultaneous zeros of ψ and B with a multiplicity at least as great as B and any such zero of ψ and B is a finite simple point of ψ. Thus, $A = Y\psi + ZB$ and on the surface, $A = ZB$, so that $A/B = Z$, a polynomial in α_0^*, α_1^*, α_2^* with coefficients in $K(\alpha, \beta)$. Returning to affine co-ordinates, we see that the ratio A/B is a polynomial in α^*, β^* with rational coefficients in α and β so that we can write

$$\frac{A}{B} = \frac{F(\alpha, \beta, \alpha^*, \beta^*)}{D(\alpha, \beta)} ,$$

where F and D are polynomials. Let E be the resultant with respect to β of D and $\phi(\alpha, \beta)$, i.e., $E = \gamma\phi + \delta D$. Therefore,

$$\frac{F}{D} = \frac{\delta F}{\delta D} = \frac{\delta F}{E - \gamma\phi}$$

and on the surface

$$\frac{F}{D} = \frac{\delta F}{E} = \frac{\tilde{F}}{E} .$$

If $\alpha = \alpha_0$ is a zero of order r of E, then $\tilde{F}$ must be divisible by $(\alpha - \alpha_0)^r$ since A/B has no poles for α finite. Thus, $\tilde{F}/E$ is a polynomial $\pi(\alpha, \beta, \alpha^*, \beta^*)$ on S. Finally, we can write

$$W_2 = \frac{\pi d\alpha^* d\alpha}{G(\alpha)} \qquad (2.1)$$

and W_2 has poles only at the zeroes of G and possibly $\alpha = \infty$. We have arrived at our final reduced form for ω_2, i.e., $(\pi/G)d\alpha^* d\alpha$ (modulo $d\omega_1$), whose only polar curves on S are C_2 or its copies. Any double differential of the 2nd kind may, by subtraction of a suitable $d\omega_1$, be reduced to one of the form (2.1).

THEOREM. Any double differential of the 2nd kind on Φ can be reduced, modulo $d\omega_1$, to the form $\sum c_{ij}\omega_i \wedge \omega_j^*$ where ω_i and ω_j^* are simple differentials of the 2nd kind on C_1 and C_2 and no ω_i or ω_j^* is derived.

PROOF. We begin by writing ω_2 in the form (2.1), i.e.,

$$\omega_2 = \frac{\pi}{G}\, d\alpha^* d\alpha.$$

If π is of degree d in β^*, we can write

$$\pi = c_d[\alpha,\ \beta,\ \alpha^*](\beta^*)^d + c_{d-1}[\alpha,\ \beta,\ \alpha^*](\beta^*)^{d-1}$$
$$+ \ldots + c_1[\alpha,\ \beta,\ \alpha^*]\beta^* + c_0[\alpha,\ \beta,\ \alpha^*],$$

where the c's are polynomials. Also, each of the c's can be written as a polynomial in α^*. Let the degree of c_i in α^* be d_i. Then, each c_i can be written

$$c_i = b_{i,d_i}(\alpha^*)^{d_i} + b_{i,d_i-1}(\alpha^*)^{d_i-1} + \ldots + b_{i,1}\alpha^* + b_{i,0}\ ,$$

where the b's are polynomials in α and β. It is then clear that π can be written as a finite sum of terms of the form

$$\frac{P(\alpha,\ \beta)(\alpha^*)^p(\beta^*)^q}{G(\alpha)}\ ,$$

where P is a polynomial and p and q are positive integers. Consider

$$\frac{P(\alpha,\ \beta)(\alpha^*)^p(\beta^*)^q}{G(\alpha)}\, d\alpha^* d\alpha. \tag{2.2}$$

This is a double differential of the 2nd kind on S and can be written as

$$[(\alpha^*)^p(\beta^*)^q d\alpha^*]\left[\frac{P(\alpha,\ \beta)\,d\alpha}{G(\alpha)}\right]\ , \tag{2.3}$$

a product of two Abelian differentials, the first on C_2 and the second on C_1. Each of them must be of the 2nd kind on their respective curves since, if not, the double differential would have non-zero residues on S. Let the genus of C_1 be g_1 and that of C_2 be g_2. Also, let $d_{\mu_1}, d_{\mu_2}, \ldots, d_{\mu_{2g_1}}$ and $d_{\nu_1}, d_{\nu_2}, \ldots, d_{\nu_{2g_2}}$ be bases for differentials of the 2nd kind on C_1 and C_2, respectively. Then

$$(\alpha^*)^p(\beta^*)^q d\alpha^* = \sum_1^{2g_2} \tilde{c}_k d\nu_k$$

and

$$\frac{P d\alpha}{G} = \sum_1^{2g_1} c_i d\mu_i,$$

where the c's are constant. Thus,

$$\frac{P(\alpha, \beta)}{G(\alpha)} (\alpha^*)^p(\beta^*)^q d\alpha^* d\alpha =$$

$$\sum_{i=1}^{2g_1} \left[\sum_{k=1}^{2g_2} c_{ik} d\nu_k d\mu_i \right]$$

where $c_{ik} = c_i\tilde{c}_k$. Since each term in $\pi/G\ d\alpha^* d\alpha$ can be so written, the theorem follows.

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