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A NOTE ON LOCAL ASYMPTOTIC BEHAVIOR FOR BROWNIAN MOTION IN BANACH SPACES

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ABSTRACT. In this paper we obtain an integral characterization of a two-sided upper function for Brownian motion in a real separable Banach space. This characterization generalizes that of Jain and Taylor [2] where $B = \mathbb{R}^n$. The integral test obtained involves the index of a mean zero Gaussian measure on the Banach space, which is due to Kuelbs [3]. The special case that when B is itself a real separable Hilbert space is also illustrated.

KEY WORDS AND PHRASES. Gaussian measures on B-spaces, abstract Wiener spaces, covariance operators, Brownian motion in B-space, upper and lower functions, integral test.

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1. INTRODUCTION

Let B be a real separable Banach space with norm $||\cdot||$ and let B* be the topological dual of B. If μ is a mean zero Gaussian measure on B then it is well known from [1] that B contains a Hilbert space H_μ with norm $||\cdot||_\mu$ such that $||\cdot||$ is a measurable norm on H_μ in the sense of [1]. As a consequence, the B norm $||\cdot||$ is weaker than $||\cdot||_\mu$. Thus through a restriction map we have the relation that $B^* \subseteq H^* \equiv H \subseteq B$. Furthermore, it is also shown in [1] that μ is the extension of the canonical normal distribution on H_μ to B and we shall say that μ is generated by H_μ. If K denotes the unit ball of H_μ in the norm $||\cdot||_\mu$, let $\Gamma = \sup_{x \in K} ||x||$. The definition of index of μ, n_1, is due to Kuelbs [3], where

$$n_1 = \sup \{k: \exists f_1, \ldots, f_k \in B^*;\ f_1, \ldots, f_k \text{ orthogonal in } H_\mu;$$

$$||f_j||_{B^*} = 1 \text{ and } ||f_j||_\mu = \Gamma \ (1 \le j \le k)\}.$$

It is known from [3] that n_1 exists and is finite and if B itself is a Hilbert space then n_1 on B equals the multiplicity of the maximal eigenvalue of the covariance operator for μ. Let $\{W(t): 0 \le t < \infty\}$ denote μ-Brownian motion in B. Let Φ_ε denote the class of functions from $(0,\varepsilon)$ to $[0,\infty)$ such that $\phi(t)\uparrow\infty$ as $t\downarrow 0$ and $t^{1/2}\phi(t)\downarrow 0$ as $t\downarrow 0$.

DEFINITION 1. A function $\phi \in \Phi_\varepsilon$ is called an upper function for $\{W(t): t > 0\}$ with respect to the norm $||\cdot||$, if given $t > 0$, there exists a $\delta > 0$ such that $P(||W(t+v) - W(t-u)|| < 2^{1/2}(u+v)^{1/2}\phi(u+v)\,\Gamma$ for all $u, v \ge 0$ with $0 < u+v < \delta) = 1$. In this case, we say $\phi \in U$. $\phi \in \Phi_\varepsilon$ is called a lower function for $\{W(t): t \ge 0\}$ with respect to the norm $||\cdot||$, denoted by $\phi \in L$, if $\phi \notin U$.

In the case that $B = \mathbb{R}^d$, a d-dimensional Euclidean space, Jain and Taylor [2] have shown that $\phi\in\Phi_\varepsilon$ is an upper function for d-dimension standard Brownian motion $\{W(t): 0 \leq t < \infty\}$ with respect to the Euclidean norm $||\cdot||_2$ if and only if $\int_{0+} \frac{[\phi(t)]^{d+2}}{t} e^{-\phi^2(t)/2}\, dt < \infty$. This integral test for two-sided growth in $\mathbb{R}^d$ is the same as that for one-sided growth in $\mathbb{R}^{d+2}$. In the case that B is an infinite dimensional real separable Banach space, Kuelbs [3] has shown that $\phi(t)$, a nonnegative, non-decreasing continuous function defined for large values of t, is a one-sided upper function for μ-Brownian motion $\{W(t): 0 \leq t < \infty\}$ with respect to some equivalent norm $||\cdot||_1$ on B if and only if $\int^\infty \frac{[\phi(t)]^{n_1}}{t} e^{-\phi^2(t)/2}\, dt < \infty$ where the ϕ is called one-sided upper function with respect to $||\cdot||_1$ if P $(||W(t)||_1 > t^{1/2}\phi(t)\Gamma$ for only a bounded set of t's) = 1. Based on the results of Jain-Taylor and Kuelbs, it is very natural to conjecture that $\phi\in\Phi_\varepsilon$ is in U (Definition 1) for $\{W(t): 0 \leq t < \infty\}$ with respect to some equivalent norm $||\cdot||_1$ on B if and only if $\int_{0+} \frac{[\phi(t)]^{n_1+2}}{t} e^{-\phi 2(t)/2}\, dt < \infty$. The main purpose of this note is to verify this conjecture. Throughout this note c will stand for a positive number whose value may change from line to line. The notation $a(h) \sim b(h)$ means $\lim_{h\to 0} \frac{a(h)}{b(h)} = 1$.

2. MAIN RESULTS

The following useful estimates have been used repeatedly in [3] and they can be verified by the argument similar to that in d-dimensional Euclidean space $\mathbb{R}^d$ [4, p. 222].

LEMMA 2. Let $\{W(t): 0 \leq t < \infty$ be Brownian motion in a real separable Banach space B having norm $||\cdot||$. Then for all λ, $h > 0$

$$P\left(\sup_{0 \leq t_1 < t_2 \leq h} ||W(t_2) - W(t_1)|| > \lambda h^{1/2}\right) \leq \tag{1}$$

$$2P\left(\sup_{0 \leq t \leq h} ||W(t)|| > \lambda h^{1/2}\right) \leq 4P\left(||W(h)|| > \lambda h^{1/2}\right).$$

We have the following integral test for a two-sided upper function for $\{W(t)\colon 0 \le t < \infty\}$:

THEOREM 3. Let $\{W(t)\colon 0 \le t < \infty\}$ be μ-Brownian motion in a real separable Banach space B having norm $||\cdot||$, and assume $\phi \in \Phi_\varepsilon$. Let n_1 denote the index of μ. Then there is an equivalent norm $||\cdot||_1$ on B such that $\sup_{x\in K} ||x||_1 = \Gamma$ and $\phi \in U$ with respect to $||\cdot||_1$ if and only if

$$\int_{0+} \frac{[\phi(t)]^{n_1+2}}{t} e^{-\phi^2(t)/2}\, dt < \infty . \tag{2}$$

PROOF. The construction of the equivalent norm $||\cdot||_1$ is due to [3], which is defined to be

$$||x||_1 = \max\{\Gamma||\pi x||_\mu,\ ||Qx||\},$$

where

$$\pi(x) = \sum_{j=1}^{n_1} e_j(x)e_j \quad (x\in B)$$

and

$Q(x) = x - \pi(x)$, $e_j(\cdot)$ denotes the linear function $f_j(\cdot)/\Gamma$.

Consider the sequence $a_m = e^{-m/\log m}$, $m \ge 2$. Then $\frac{a_m}{a_{m+1}} \sim 1 + (\log m)^{-1}$ as $m\to\infty$. Let $u_{n,i} = \frac{i}{\log n} a_n$; $v_{n,i} = (1 - \frac{i}{\log n})\, a_n$, $0 \le i \le \log n$.

Note that for each i, $u_{n,i} + v_{n,i} = a_n$. If $u,v \ge 0$ are sufficiently small we can choose an n sufficiently large such that $a_{n+1} \le u+v < a_n$. Assume that the integral (2) diverges. Define

$$E_{n,i} = \{\omega\colon\ ||W(t+v_{n,i}) - W(t-u_{n,i})||_1 > 2^{1/2}a_n^{1/2}\phi(a_n)\Gamma\} \text{ and}$$

$$F_{n,i} = \{\omega\colon\ ||W(t+v_{n,i}) - W(t-u_{n,i})||_\mu > 2^{1/2}a_n^{1/2}\phi(a_n)\}.$$

Since

$\{\pi W(t): 0 \leq t < \infty\}$ is standard n_1 dimensional Brownian motion in $\pi B = \pi H_\mu$, divergence of the integral (2) implies that

$\sum_{n=2}^{\infty} \sum_{i=0}^{\log n} P(F_{n,i}) = \infty$ and infinitely many of $F_{n,i}$ occur.

Consequently infinitely many of $E_{n,i}$ occurr with probability one. Thus $\phi \varepsilon L$.

Now assume that the integral (2) converges. Let us choose a suitable $i \leq \log n$ such that $u_{n,i} \leq u \leq u_{n,i+1}$ and $v \leq v_{n,i-1}$

Then

$$P(||W(t+v) - W(t-u)||_1 > 2^{1/2}(u+v)^{1/2}\phi(u+v)\Gamma)$$

$$\leq P(||W(t+v) - W(t-u)||_1 > 2^{1/2}a_{n+1}^{1/2}\phi(a_{n+1})\Gamma)$$

$$\leq P(\sup_{0 \leq t_1 < t_2 \leq u_{n,i+1}+v_{n,i-1}} ||\pi(W(t_2) - W(t_1))||_\mu > 2^{1/2}a_{n+1}^{1/2}\ \phi(a_{n+1}))$$

$$+ P(\sup_{0 \leq t_1 < t_2 \leq u_{n,i+1}+v_{n,i-1}} ||Q(W(t_2) - W(t_1))|| > 2^{1/2}a_{n+1}^{1/2}\ \phi(a_{n+1})\Gamma).$$

Since $\{\pi W(t): 0 \leq t < \infty\}$ is standard n_1 dimensional Brownian motion in $\pi B = \pi H_\mu$, by the same argument as those in Theorem 3.1 [2], we conclude that the first term in the right hand side of the above inequality being zero for infinitely many n and i. As for the second term in the above inequality, we have

$$P(\sup_{0 \leq t_1 < t_2 \leq u_{n,i+1}+v_{n,i-1}} ||Q(W(t_2) - W(t_1))|| > 2^{1/2}a_{n+1}^{1/2}\ \phi(a_{n+1})\Gamma) \equiv P(A_{n,i})$$

$$\leq 4P\ (||QW(1)|| > (\frac{2a_{n+1}}{u_{n,i+1}+v_{n,i-1}})^{1/2}\ \phi\ (a_{n+1})\Gamma)$$

$$\leq\ C\ \exp\{-\varepsilon\ \frac{2a_{n+1}}{\mu_{n,i+1}+v_{n,i-1}}\ \phi^2\ (a_{n+1})\Gamma^2\},$$

Where

$\varepsilon < 1/2 \sup \int_B \Lambda_n^2(x)\mu^Q(dx)$, $\{\Lambda_j\}$ is a sequence in B* such that

$|| \Lambda_j ||_{B*} = 1$ and $||x|| = \sup_j | \Lambda_j(x) |$ for every $x \varepsilon B$.

The last inequality comes from [3, p. 253].

Since $u_{n,i+1} + v_{n,i-1} = (1 + 2/\log n)a_n \sim (1 + 3/\log n)a_{n+1}$, if

we choose δ be such that $\varepsilon = (1 + \delta)/2\Gamma$ we have

$$P(A_{n,i}) \leq C \exp \{-\delta\, \phi^2(a_{n+1})\} \exp \{-\phi^2(a_{n+1})\}$$

$$\leq C \exp \{-\phi^2(a_{n+1}) \}$$

Thus

$$\sum_{n=2}^{\infty} \sum_{i=0}^{\log n} P(A_{n,i}) < c \sum_{n=2}^{\infty} (\log n) \exp \{-\phi^2(a_{n+1})\}$$

$$< c \sum_{n=2}^{\infty} [\phi(a_{n+1})]^{n_1} \exp \{-\phi^2(a_{n+1})\}$$

$< \infty$, since the integral (2) converges

(see Lemma 2.12 of [2]).

From Lemma 2.15 (i) of [2] we conclude that $P(A_{n,i}$, i.o.) = 0.

Thus

$$P(\sup_{t-u_{n,i+1} \leq t_1 < t_2 \leq t+v_{n,i-1}} ||W(t_2) - W(t_1)||_1 \leq 2^{1/2} a_{n+1}^{1/2} \phi\, (a_{n+1})\Gamma) = 1$$

for all i and n sufficiently large. Thus $\phi \varepsilon U$.

In case that B is a real separable Hilbert space, then n_1 equals the multiplicity of the maximal eigenvalue of the covariance operator for μ. We have the same result as those of Theorem 3.

THEOREM 4. Let $\{W(t)\colon 0 \leq t < \infty\}$ be μ-Brownian motion in a real separable Hilbert space H with norm $||\cdot||$, and suppose $\phi \in \Phi_\varepsilon$. Then ϕ

is in U with respect to the given norm $||\cdot||$ if and only if

$$\int_{0+} \frac{[\phi(t)]^{n_1+2}}{t} e^{-\phi^2(t)/2} dt < \infty, \tag{3}$$

Where n_1 denotes the multiplicity of the maximal eigenvalue of the covariance operator for μ.

PROOF. Let sequence $\{a_n\}$, $\{u_{n,i}\}$ and $\{v_{n,i}\}$ be the same as those in the proof of Theorem 3. If $u,v \geq 0$ are sufficiently small we choose n sufficiently large such that $a_{n+1} \leq u+v < a_n$ and then fix $i \leq \log n$ such that $u_{n,i} \leq u \leq u_{n,i+1}$ and $v \leq v_{n,i-1}$. If the integral (3) diverges, then we proceed as those in Theorem 3 and conclude that $\phi \in L$. Now if the integral (3) converges define

$$B_{n,i} = \{\omega: \sup_{t-u_{n,i+1} \leq t_1 < t_2 \leq t+v_{n,i}} ||W(t_2) - W(t_1)|| > 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1})\Gamma\}.$$

Then

$$P(B_{n,i}) \leq 4P(||W(1)|| > (\frac{2a_{n+1}}{u_{n,i+1}+v_{n,i-1}})^{1/2} \phi(a_{n+1})\Gamma)$$

$$\leq C[\phi(a_{n+1})]^{n_1-2} \exp\left(-\frac{a_{n+1}\phi^2(a_{n+1})\Gamma^2}{(u_{n,i+1}+v_{n,i-1})\lambda}\right)$$

$$\leq C[\phi(a_{n+1})]^{n_1-2} \exp\{-\phi^2(a_{n+1})\}$$

Where λ is the maximal eigenvalue of the covarience of μ and it is known that $\lambda = \Gamma^2$ [3]. The last inequality comes from [3] and the fact that

$$u_{n,i+1}+v_{n,i-1} \sim \frac{a_{n+1}}{1+3(\log n)^{-1}}.$$

Thus

$$\sum_{n=2}^{\infty} \sum_{i=0}^{\log n} P(B_{n,i}) < C \sum_{n=2}^{\infty} \sum_{i=0}^{\log n} [\phi(a_{n+1})]^{n_1-2} \exp\{-\phi^2(a_{n+1})\}$$

$$= C \sum_{n=2}^{\infty} [\phi(a_{n+1})]^{n_1} \exp\{-\phi^2(a_{n+1})\} < \infty \text{ since}$$

the integral (3) converges. From Lemma 2.15 (i) we have $P(B_{n,i}, \text{ i.o.}) = 0$.

That is $P(B^c_{n,i}) = 1$ for sufficiently large i and n. Thus $\phi \in U$.

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