A CLASS OF RINGS WHICH ARE ALGEBRAIC OVER THE INTEGERS

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ABSTRACT.   A well-known theorem of N. Jacobson states that any periodic

associative ring is commutative.   Several authors (most notably Kaplansky and

Herstein) generalized the "periodic polynomial" condition and were still able to

conclude that the rings under consideration were commutative.   (See [3])   In

this paper we develop a structure theory for a class of rings which properly

contains the periodic rings.   In particular, an associative ring R is said to be

a quasi-anti-integral (QAI) ring if for every $a \neq 0$ in R there exist a positive

integer k and integers $n_1, n_2, \ldots, n_k$ (all depending on a), so that $0 \neq n_1 a = n_2 a^2 +$.

$+ n_k a^k$.   In the main theorems of this paper, we show that any QAI-ring is a sub-

direct sum of prime QAI-rings, which in turn are shown to be left and right

orders in division algebras which are algebraic over their prime fields.

1. <u>INTRODUCTION</u>.

In this paper we give a characterization of an associative ring which satisfies an algebraic condition over the integers. This condition can be viewed as a natural generalization of one studied by Osborn for a power-associative ring (i.e. a ring in which the subring generated by each element is associative). As in [5] R will be called periodic if for every element a of R, there exists an integer n=n(a) greater than 1, such that $a^n = a$. In [5, p. 321] Osborn proves the following theorem which relates the periodic property to a more general kind of algebraic condition.

THEOREM 1. The following are equivalent for a power-associative ring R:

1. R is a periodic ring.

2. Every element a of R is anti-integral over the ring of integers. That is, there are integers $k, n_2, n_3, \ldots, n_k$ depending on a so that

$$n_k a^k + n_{k-1} a^{k-1} + \ldots + n_2 a^2 + a = 0.$$

3. R is a discrete direct sum of ideals each of which is an algebraic algebra over a periodic prime field and contains no nonzero nilpotent elements.

It follows that if R is a periodic ring, then R = T(R), where T(R) = $\{x \in R | nx = 0$ for some integer $n \neq 0\}$ is the torsion ideal of R. In this paper we consider a class of associative rings, which are not necessarily equal to their torsion ideals, but which satisfy a generalized anti-integral relation. In particular, if F is a commutative, associative ring with 1, and R is a power-associative F-algebra, then a nonzero element a of R is said to be <u>quasi-anti integral (QAI) over F</u> if there are 'scalars' $n_1, n_2, \ldots, n_k$ in F (depending on a) such that $0 \neq n_1 a = n_2 a^2 + \ldots + n_k a^k$. R is a <u>QAI-algebra over F</u> if every nonzero element of R is QAI over F.

After developing some of the basic properties of QAI-algebras over Z, the ring of integers, we show that a prime associative ring which is QAI over Z is

a left and right order in a division algebra which is algebraic over its prime field, and conversely. We then show that an arbitrary associative QAI-algebra over the integers is a subdirect sum of prime QAI-rings. We present an example to show that not every such subdirect sum yields a QAI-ring.

2. PROPERTIES OF QAI-ALGEBRAS.

For the remainder of this paper, unless specifically stated otherwise, the term ring will mean associative ring without any assumption being made about the existence of a unity element. If R is a ring which is a QAI-algebra over Z we will simply call R a QAI-ring. If R is a ring, then a ring Q(R) is said to be a left quotient ring for R and R is called a left order in Q(R) provided (i) there is a monomorphism h of R into Q(R) such that (ii) if x is a regular element (i.e. one which is neither a left nor a right zero divisor) in R, then h(x) is invertible in Q(R), and (iii) Q(R) consists of all elements of the form $h(a)^{-1}h(b)$ for a, b in R with a regular in R. Right quotient ring and right order are defined in a similar manner. If R has a left quotient ring Q(R), we will often assume that R is a subring of Q(R) and so dispense with the monomorphism h.

We state the following well-known results for future reference. See [5], [3] for their proofs.

LEMMA 2. If R is a left order in Q(R) and x_1, x_2, . . . , x_n are arbitrary elements of Q(R), then there are elements a, b_1, b_2, . . . , b_n in R, with a regular in R, such that $x_i = a^{-1}b_i$ (i = 1, 2, . . . , n).

LEMMA 3. Suppose that R is an F-algebra, where F is a commutative and associative ring with 1 and with a left quotient ring Q(F). Then every element x in Q(F) θ_F R can be written as $x = c^{-1} \theta$ r for c regular in F and some r in R.

The following lemma is well-known in its more general form, but we state it here and use it only for associative rings considered as algebras over Z.

The proof in the general case may be found in [5, p. 189] .

LEMMA 4.   Suppose R is an associative algebra over the ring of integers

Z, and that $T(R) = (0)$.   Then the map f from R into $Q \otimes_Z R$ given by

$f(r) = 1 \otimes r$ is a ring monomorphism.   (Q is the field of rationals)

We observe that if A is a prime periodic ring of characteristic $p \neq 0$,

then A is a periodic field and so A is a left order and in fact $A = Q(A)$.

In fact, the class of periodic fields is precisely the class of prime periodic

rings which are left and right orders in fields of nonzero characteristic which

are algebraic over their prime fields.

It follows from Theorem 1 that any periodic ring is a QAI-ring.   In the

characteristic zero case we will show that QAI rings play a role similar to

that of periodic rings in characteristic p.

LEMMA 5.   Let D be any division algebra of characteristic zero which is

algebraic over Q and suppose R is a subring of D.   R is a QAI-ring.

PROOF.   Let a be a nonzero element of R.   Since D is Q-algebraic, there

is a polynomial $p(x) = r_n x^n + \ldots + r_2 x^2 + r_1 x + r_0$ in $Q[x]$ such that

$p(a) = 0$.   Assume such a polynomial of minimal degree has been chosen.   Since

a is not a zero divisor we may assume $r_0 \neq 0$.   But then

$0 \neq r_0 a = -(r_n a^{n+1} + \ldots + r_2 a^3 + r_1 a^2)$.   Suppose $r_i = c_i d_i^{-1}$ and let d be

the least common multiple of $d_0, d_1, \ldots, d_n$.   Then dr_i is an integer for

$i = 0, 1, \ldots, n$ and $0 \neq dr_0 a = -dr_n a^{n+1} - \ldots - dr_2 a^3 - dr_1 a^2$ so a is QAI,

and R is a QAI-ring.

The class of QAI-rings, although containing the periodic rings, can now

be seen to be quite different from the class of periodic rings.   The ring of

integral quaternions is a noncommutative QAI-ring (compare with [2, Thm. 11]).

A periodic ring is Jacobson semisimple since every element generates an

idempotent, while the subring of Q consisting of all quotients of the form

$2r/_{2s+1}$, $(2r, 2s + 1) = 1$, is a Jacobson radical ring which is also QAI by Lemma 5.

THEOREM 6. Let R be an associative QAI-ring with torsion ideal $T(R)$. Then

   (a) R has no nonzero nilpotent elements, and thus every
       idempotent of R is contained in the center of R, $Z(R)$.

   (b) $T(R)$ is a QAI-ring in its own right with the following
       decompositions:

      (i) $T(R)$ is a periodic ring which is a direct sum of
           algebraic Z_p-algebras each of which has no nilpotent
           elements.

      (ii) $T(R)$ is a periodic ring which is a subdirect sum of
           primitive periodic rings (i.e. a subdirect sum of
           periodic fields).

   (c) If S is any subring of R, then S is a QAI-ring.

   (d) Any homomorphic image of R which is torsion-free is a QAI-ring.

   PROOF. (a) It suffices to prove that R has no nonzero elements which square to 0. Let $r \neq 0$ be an element of R. Then there are integers n, k, a_2, a_3, ... , a_k such that $0 \neq nr = a_2 r^2 + a_3 r^3 + ... + a_k r^k$. Then clearly, $r^2 = 0$ implies $r^{2+1} = 0$ for all positive integers i and this yields, $a_2 r^2 + a_3 r^3 + ... + a_k r^k = 0$ a contradiction. Thus $r^2 \neq 0$ and hence R has no nonzero nilpotent elements. If $e^2 = e$ and x is an arbitrary element of R, then $(ex - exe)^2 = 0 = (xe - exe)^2$. Thus $ex = exe = xe$ and so e is in $Z(R)$.

   (b) $T(R)$ is an ideal of R and it is clear that the property of being
        QAI is inherited by subrings, so $T(R)$ is itself a QAI-ring.

(i)   Let $R_p = \{x \text{ in } R | p^m x = 0 \text{ for some positive integer } m\}$,

for each prime number p. R_p is a subset of $T(R)$ and is in

fact an ideal of R and hence of $T(R)$. $T(R)$ is actually a

discrete direct sum of the collection of ideals $\{R_p\}$, p a

prime. Let x be an arbitrary element of R_p. There exists

a positive integer m such that $p^m x = 0$. This implies

$(px)^m = p^m x^m = (p^m x) x^{m-1} = 0$ so px is a nilpotent element

of R. By (a), $px = 0$ and hence R_p is an algebra over Z_p.

Also $0 \neq nx = a_2 x^2 + a_3 x^3 + \ldots + a_t x^t$ where $n, a_2, \ldots, a_t$

are integers which may be considered as coefficients from Z_p

since $nx \neq 0$. Therefore, each R_p is an algebraic Z_p-algebra

with no nonzero nilpotent elements and so a periodic ring by

Theorem 1.

(ii) By (i) $T(R)$ is a periodic ring and by $\begin{bmatrix} 2, \text{ Thm. } 11 \end{bmatrix}$ $T(R)$ is

commutative. If a is an element of $T(R)$, then for some

positive integer m greater than 1, $a^m = a$ which implies

$e = a^{m-1}$ is an idempotent. But the Jacobson radical $J(T(R))$

contains no idempotents which yields a is not an element of

$J(T(R))$. Thus $J(T(R)) = (0)$ and $T(R)$ is a subdirect sum of

primitive rings which must also be commutative and periodic,

since they are homomorphic images of $T(R)$. Therefore $T(R)$ is

a subdirect sum of periodic fields.

(c) This is clear since the QAI-property is an element-wise condition.

(d) If I is an ideal of R for which R/I is torsion-free, then R/I is

a QAI-ring follows from the fact that

$0 \neq na = n_2 a^2 + n_3 a^3 + \ldots + n_k a^k$ implies

$\overline{0} \neq \overline{na} = n_2 \overline{a}^2 + n_3 \overline{a}^3 + \ldots + n_k \overline{a}^k$ if $\overline{a} = a + I \neq \overline{0}$ in R/I.

LEMMA 7. If $T(R) = 0$ then R is a QAI-ring if and only if every element of R satisfies some polynomial with integer coefficients and has no nonzero nilpotent elements.

PROOF. The necessity follows directly from Theorem 6. Conversely, let w be a nonzero element of R. Then there is some polynomial $p(x) = a_t x^t + \ldots + a_2 x^2 + a_1 x$ with $a_1, a_2, \ldots, a_t$ integers so that $p(w) = 0$. If $a_1 \neq 0$ then $a_1 w \neq 0$ and we are finished. Otherwise, let k be minimal so that $a_k w \neq 0$ and set $z = a_t w^{t-(k-1)} + \ldots + a_{k+1} w^2 + a_k w$. If $z = 0$, w is QAI. Otherwise, $z \neq 0$ but $z^{k+1} = 0$, a contradiction. Therefore R is a QAI-ring.

It will sometimes be convenient to use the obviously equivalent form of the definition: a in R is QAI if and only if there exists a polynomial $p(x) = a_1 x + a_2 x^2 + \ldots + a_k x^k$ with integer coefficients so that $p(a) = 0$ and $a_1 a \neq 0$. (p(x) will be called an integral polynomial.)

THEOREM 8. Suppose R and S are both power-associative QAI-rings. Then so is their direct sum $R \oplus S$.

PROOF. Let (r, s) be an element of $R \oplus S$. We may assume that both r and s are nonzero for otherwise (r, s) is QAI trivially. The argument will be divided into four cases determined by the additive orders of r and s, which will be denoted by $o(r)$ and $o(s)$, respectively.

Case 1. $o(r) = n$, $o(s) = m$; n, m both finite. Then $r \in T(R)$ and $s \in T(S)$. By Theorem 6 $T(R)$ and $T(S)$ are both periodic rings, and by the proof of Theorem 13.2 [5] so is $T(R) \oplus T(S)$. But $T(R) + T(S) = T(R \oplus S)$. Thus $(r, s) \in T(R \oplus S)$, a periodic ring and hence (r, s) is QAI.

Case 2. $o(r) = n$, finite and $o(x) = \infty$. Since s is QAI there exists an integral polynomial $p(x) = a_1 x + a_2 x^2 + \ldots + a_k x^k$ with $p(s) = 0$ and $a_1 s \neq 0$. Let $f(x) = np(x) = (na_1)x + (na_2)x^2 + \ldots + (na_k)x^k$. f(x) is an integral polynomial and $na_1 s \neq 0$ since s has infinite additive order. Thus

$na_1(r,s) = (na_1r, na_1s) = (0, na_1s) \neq (0, 0)$ in $R \oplus S$, yet

$f((r,s)) = na_1(r,s) + na_2(r,s)^2 + \ldots + na_k(r,s)^k$

$$= (na_1r, na_1s) + (na_2r^2, na_2s^2) + \ldots + (na_kr^k, na_ks^k)$$

$$= (f(r), f(s)) = (np(r), np(s)) = (0, 0)$$

Therefore, (r,s) is QAI.

Case 3. $o(r) = \infty$, $o(s) = n$, finite. Same as Case 2 - interchange the roles of r and s.

Case 4. $o(r) = \infty$, $o(s) = \infty$. Since R is a QAI-ring, there is an integral polynomial $p(x) = a_1x + a_2x^2 + \ldots + a_kx^k$ with $a_1r \neq 0$ and $p(r) = 0$. Now, $p((r,s)) = a_1(r,s) + a_2(r,s)^2 + \ldots + a_k(r,s)^k$

$$= (a_1r, a_1s) + (a_2r^2, a_2s^2) + \ldots + (a_kr^k, a_ks^k)$$

$$= (p(r), p(s)) = (0, p(s)).$$

If $p(s) = 0$ in S, then $p((r,s)) = (0, 0)$ and $a_1(r,s) \neq (0, 0)$ in $R \oplus S$ so (r,s) is QAI. Otherwise, $p(s) \neq 0$ in S. But S is a QAI-ring so there is an integral polynomial $h(x) = b_1x + b_2x^2 + \ldots + b_nx^n$ with $b_1p(s) \neq 0$ and $h(p(s)) = 0$ in S. Define $g(x)$ to be the composition of the two polynomials $p(x)$ and $h(x)$. That is, set

$g(x) = h(p(x)) = b_1(a_1x + a_2x^2 + \ldots + a_kx^k)$

$$+ b_2(a_1x + a_2x^2 + \ldots + a_kx^k)^2 + \ldots$$

$$+ b_n(a_1x + a_2x^2 + \ldots + a_kx^k)^n.$$

Upon inspection of this polynomial one sees that the coefficient of the linear term is $b_1a_1 \neq 0$ and that $g(x)$ is clearly an integral polynomial. Since $o(r) = \infty = o(s)$, we get

$b_1a_1(r,s) = (b_1a_1r, b_1a_1s) \neq (0, 0)$ the zero element of $R \oplus S$. However,

$g((r,s)) = h(p((r,s))) = h((0, p(s))) = (0, h(p(s))) = (0, 0)$, and (r,s) is QAI. Since these four cases are exhaustive, the theorem is proved.

COROLLARY 9. Suppose R_1, R_2, ... , R_n are QAI-rings. Then their direct sum $R_1 \oplus R_2 \oplus ... \oplus R_n$ is also a QAI-ring.

PROOF. This can be proved easily by induction.

COROLLARY 10. Let $\{R_t\}_{t \ \epsilon \ A}$ be any collection of QAI-rings, and suppose $R = \oplus \sum_{t \ \epsilon \ A} R_t$ is their weak direct sum. Then R is a QAI-ring.

PROOF. Let a be any nonzero element of R. Then all but a finite number of the components of a are zero; i.e. if $a = (a_t)_{t \ \epsilon \ A}$ then there exist a finite number of elements t_1, t_2, ... , t_k in A such that $a_s = 0$ if s is not a member of $\{t_1, t_2, ... , t_k\}$. Then a can be considered to be an element of $R_{t_1} \oplus R_{t_2} \oplus ... \oplus R_{t_k}$ which is a QAI-ring. Thus a is QAI and $R = \oplus \sum_{t \ \epsilon \ A} R_t$ is a QAI-ring.

As can be seen from Theorem 6 and Corollary 10, the class of QAI-rings is closed under taking subrings, direct sums and torsion-free homomorphic images. However, Z/nZ is a homomorphic image of Z which is not a QAI-ring whenever n is not square-free. To see that the class of QAI-rings is not closed under complete direct products consider the following example.

EXAMPLE. Let $W_i = Z$, the ring of integers for $i = 1, 2, 3, ...$, and set $R = \overset{\infty}{\underset{i=1}{\Pi}} W_i$. Let $x \ \epsilon \ R$ be that element $x = (x_i)$ such that $x_i = i$.

Suppose x is QAI. That is, suppose there exist n, a_2, a_3, ... , a_k integers so that $0 \neq nx = a_2 x^2 + a_3 x^3 + ... + a_k x^k$. Since two elements of R are equal if and only if they are equal in each component, this equation yields the following

infinite system of equations:

$$n = a_2 + a_3 + \ldots + a_k$$

$$2n = 4a_2 + 8a_3 + \ldots + 2^k a_k \text{ - which implies that 2 divides n}$$

$$3n = 9a_2 + 27a_3 + \ldots + 3^k a_k \text{ - which implies that 3 divides n}$$

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$$mn = m^2 a_2 + m^3 a_3 + \ldots + m^k a_k \text{ - which implies that m divides n}$$

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This is clearly a contradiction since n can have only a finite number of

divisors. Therefore R is not a QAI-ring.

3. THE PRIME CASE

Let R be an arbitrary QAI-ring, and P(R) be the prime radical of R. P(R)

is the intersection of all the prime ideals of R and is also a nil ideal of R,

hence equals (0), since R has no nilpotent elements and thus certainly no nil

ideals. This fact, together with examples of QAI-rings which are Jacobson

radical rings, led us to investigate QAI-rings via the prime-semi-prime route

instead of the primitive-semisimple one.

LEMMA 11. If R is a prime associative ring then $Q \otimes_Z R$ is also prime.

PROOF. If R is prime, then it has a characteristic p. If $p \neq 0$, then

$Q \otimes_Z R = (0)$ since $m/n \otimes r = mp/np \otimes r = m/np \otimes pr = m/np \otimes 0 = 0$.

If $p = 0$, then by Lemma 4, R is embedded in $Q \otimes_Z R$ via $f: r \longrightarrow 1 \otimes r$.

Suppose there exist elements x, y in $Q \otimes_Z R$ such that $x(Q \otimes_Z R)y = (0)$. By

Lemma 3, $x = s^{-1} \otimes r_1$, $y = t^{-1} \otimes r_2$ for some s, t ϵ Z and r_1, r_2 ϵ R. Then

$0 = (s^{-1} \otimes r_1)(Q \otimes R)(t^{-1} \otimes r_2) = (s^{-1}Qt^{-1}) \otimes (r_1 R r_2)$. In particular

$0 = (s^{-1}(st)t^{-1}) \theta (r_1 R r_2) = 1 \theta (r_1 R r_2)$. But $f: R \longrightarrow Q\theta_Z R$ is one-to-one and so $r_1 R r_2 = 0$. But R is prime and so $r_1 = 0$ or $r_2 = 0$.

Therefore, $x(Q\theta_Z R)y = (0)$ implies $x = 0$ or $y = 0$ and $Q\theta_Z R$ is prime.

LEMMA 12. Suppose R is a prime ring with no nonzero nilpotent elements. Then R has no nonzero, zero-divisors.

PROOF. Suppose $xy = 0$. By hypothesis R has no nonzero nilpotent elements. Now $(yRx)^2 = (yRx)(yRx) = yR(xy)Rx = 0$ and thus $yRx = 0$. The primeness of R now forces $x = 0$ or $y = 0$. Therefore $xy = 0$ implies $x = 0$ or $y = 0$.

LEMMA 13. Suppose R is a prime QAI-ring of characteristic zero. Then $S = Q\theta_Z R$ is also a QAI-ring.

PROOF. Let $x \in Q\theta_Z R$. By Lemma 3, there is an integer c and an element r in R such that $x = c^{-1}\theta r$. S is a Q-algebra, so we may rewrite x as $x = c^{-1}(1 \theta r)$.

R is a QAI-ring and $r \in R$ so there are integers $n, n_2, n_3, \ldots , n_k$ such that $0 \neq nr = n_2 r^2 + n_3 r^3 + \ldots + n_k r^k$. Since S is a Q-algebra, it is torsion-free so

$$0 \neq n(1\theta r) = n\theta r = 1\theta nr = 1\theta(n_2 r^2 + n_3 r^3 + \ldots + n_k r^k)$$
$$= (1\theta n_2 r^2) + (1\theta n_3 r^3) + \ldots + (1\theta n_k r^k)$$
$$= n_2(1\theta r)^2 + n_3(1\theta r)^3 + \ldots + n_k(1\theta r)^k. \quad \text{Now}$$

$$0 \neq nx = n(c^{-1}\theta r) = c^{-1}(n\theta r) = c^{-1}(n_2(1\theta r)^2 + n_3(1\theta r)^3 + \ldots + n_k(1\theta r)^k)$$
$$= c^{-1}n_2(1\theta r)^2 + c^{-1}n_3(1\theta r)^3 + \ldots + c^{-1}n_k(1\theta r)^k$$
$$= n_2 c(c^{-1}\theta r)^2 + n_3 c^2(c^{-1}\theta r)^3 + \ldots + n_k c^{k-1}(c^{-1}\theta r)^k, \text{ since S}$$

is a Q-algebra. But if $a_i = n_i c^{i-1}$ for $i = 2, 3, \ldots , k$ then we have shown that $0 \neq nx = a_2 x^2 + a_3 x^3 + \ldots + a_k x^k$ with $a_i \in Z$. Therefore since $x \in S$ was arbitrary, S is a QAI-ring which is also a Q-algebra.

COROLLARY 14. If R is a prime QAI-ring of characteristic zero, then R can be embedded as a subring in a Q-algebra S which is a prime QAI-ring.

PROOF. Let $S = Q \otimes_Z R$. By Lemma 11 and Lemma 13, S is a prime QAI-ring and is clearly a Q-algebra. By Lemma 4 R is embedded inside S by $r \longrightarrow 1 \otimes r$.

If we require, in addition to the hypothesis of the last corollary, that our ring R have a unity element 1, then we can also conclude that $Q \otimes_Z R$ is a division algebra. In particular, let S be a Q-algebra which is a prime QAI-ring with 1 and let x be a nonzero element of S. We wish to show that x is invertible. There exist integers $m, a_2, a_3, \ldots, a_t$ so that

$$0 \neq mx = a_2 x^2 + a_3 x^3 + \ldots + a_t x^t. \text{ Then}$$

$0 = mx - (a_2 x^2 + \ldots + a_t x^t) = x(m \cdot 1 - (a_2 x + \ldots + a_t x^{t-1}))$. But by Lemma 12, R has no zero-divisors and then we get $m \cdot 1 - (a_2 x + a_3 x^2 + \ldots + a_t x^{t-1}) = 0$. Thus $m \cdot 1 = a_2 x + a_3 x^2 + \ldots + a_t x^{t-1} = (a_2 \cdot 1 + a_3 x + \ldots + a_t x^{t-2}) \cdot x$. So x is invertible, $x^{-1} = m^{-1}(a_2 \cdot 1 + a_3 x + \ldots + a_t x^{t-2})$, and we have proven the following.

THEOREM 15. If R is a prime Q-algebra with 1 which is also a QAI-ring then R is a division algebra.

If one assumes that R is a prime QAI-ring with 1, then the resulting Q-algebra $Q \otimes_Z R$ satisfies the hypothesis of Theorem 15. The following theorem shows that we need not assume the existence of 1.

THEOREM 16. If R is a prime Q-algebra which is also a QAI-ring, then R is a division algebra, algebraic over Q.

PROOF. Let a be a nonzero element of R. There exist positive integers n, k and integers $n_2, n_3, \ldots, n_k$ such that $0 \neq na = n_2 a^2 + \ldots + n_k a^k$. We may assume that $n_k \neq 0$ - for otherwise the polynomial is of smaller degree and at least one of the $n_i \neq 0$. Since R is a Q-algebra,

$a^k = n_k^{-1}(na - (n_2 a^2 + \ldots + n_{k-1} a^{k-1}))$. That is, A, the Q-subalgebra of R generated by a is finite dimensional.

Consider $a_r : A \longrightarrow A$, the right multiplication map given by $a_r(x) = xa$. a_r is a Q-linear transformation which is one-to-one since R, being a prime QAI-ring, has no nonzero zero-divisors. Now A is finite dimensional so a_r is also onto, and by a standard argument a_r is invertible and hence a is invertible in A. Similarly, every b in A which is nonzero is invertible, and since A is a commutative subalgebra, it is a field. Let e in A be the identity of A. $e^2 = e$, and for any $y \neq 0$ in R, $e(ey - y) = 0 = (ye - y)e$. Since R has no nonzero zero-divisors, $ey = y = ye$ and e is an identity for R. Thus every nonzero element of R is invertible and R is a division algebra. The fact that R is algebraic over Q follows from the definition of QAI-ring.

Having characterized prime Q-algebras which are QAI-rings as being division algebras which are algebraic over the rationals, we now wish to investigate the nature of the embedding of a prime, characteristic zero QAI-ring R into $Q \otimes_Z R$. It is interesting that R turns out to be a left and right order in $S = Q \otimes_Z R$. To demonstrate this, it is convenient to know the inverse of an arbitrary nonzero element $c^{-1} \otimes r$ in S.

LEMMA 17. Suppose $e = t^{-1} \otimes s$ is the identity in S and $c^{-1} \otimes r$ is any nonzero element of S. Then the multiplicative inverse of $c^{-1} \otimes r$ is given by $d^{-1} \otimes u$ where $tru = cds = tur$.

PROOF. By Theorem 16, $c^{-1} \otimes r$ is invertible since S is a division algebra. Suppose $(c^{-1} \otimes r)^{-1} = d^{-1} \otimes u$. Then $(c^{-1} \otimes r)(d^{-1} \otimes u) = t^{-1} \otimes s = (d^{-1} \otimes u)(c^{-1} \otimes r)$, if and only if $(cd)^{-1} \otimes ru = t^{-1} \otimes s = (dc)^{-1} \otimes ur$. Since S is a Q-algebra, upon multiplying by tcd in Z, we get $t \otimes ru = cd \otimes s = t \otimes ur$. But the tensor product is taken over Z, so $1 \otimes tru = 1 \otimes cds = 1 \otimes tur$. Applying Lemma 14 this is equivalent to $tru = cds = tur$.

We now proceed to characterize prime QAI-rings of characteristic zero.

THEOREM 18. Let R be a prime QAI-ring of characteristic zero. Then R is a left and right order in a rational division algebra which is algebraic over Q.

PROOF. By Corollary 14 and Theorem 16 the function $f : R \longrightarrow Q\theta_Z R$ given by $f(r) = 1\theta r$ is a ring monomorphism of R into the Q-division algebra $S = Q\theta_Z R$ which is algebraic over the rational field. It remains then to show that f actually embeds R as a left and right order in S.

By definition, we must show:

(i) If x is regular in R, then $f(x)$ is invertible in S; and

(ii) Every element z in S can be written as $z = f(x)^{-1}f(y)$ (and
    $z = f(u)f(v)^{-1}$) where x, y, u, v are in R with x and v regular.

To establish (i), it is necessary only to observe that since f is one-to-one and S is a division algebra, $f(x) = 1\theta x$ will be invertible for any $x \neq 0$ in R.

Let $e = t^{-1}\theta s$ be the identity for S. From $(1\theta m)(t^{-1}\theta s) = 1\theta m = (t^{-1}\theta s)(1\theta m)$ it follows that $t^{-1}\theta ms = 1\theta m = t^{-1}\theta sm$ which implies (multiplying by t) $1\theta ms = 1\theta tm = 1\theta sm$ and hence, because f is one-to-one, $ms = tm = sm$.

Let z be an arbitrary element of S, say $z = p^{-1}\theta m$ where $p \neq 0$ is an integer and m is in R. If $m = 0$, then write $z = (1\theta y)^{-1}(1\theta 0)$ for any $y \neq 0$ in R. So assume $m \neq 0$; then $pm \neq 0$ since R is prime of characteristic zero. Consider $1\theta pm \neq 0$. By Lemma 17 $(1\theta pm)^{-1} = q^{-1}\theta w$ where $tpwm = tpmw = qs$. Then $(1\theta pm)^{-1}(1\theta m^2) = (q^{-1}\theta w)(1\theta m^2) = q^{-1}\theta wm^2$. But $tpwm = qs$, and upon multiplying by m in R, we get $tpwm^2 = qsm = sqm = tqm$. Thus $t(pwm^2 - qm) = 0$ and R being torsion-free implies $pwm^2 = qm$. It follows that $1\theta pwm^2 = f(pwm^2) = f(qm) = 1\theta qm$. Since S is a Q-algebra, if we multiply by $(pq)^{-1}$ then we see that $q^{-1}\theta wm^2 = p^{-1}\theta m = z$.

Therefore $z = p^{-1}\theta m = q^{-1}\theta w m^2 = (q^{-1}\theta w)(10m^2) = (10pm)^{-1}(10m^2)$. S is

then seen to be the collection of all elements of the form $(10x)^{-1}(10y) = $

$f(x)^{-1}f(y)$. Thus R is a left order in S. The proof for 'R is a right order

in S' is similar to the one above.

We now turn our attention to the remaining case in the prime situation--

that of prime QAI-rings of nonzero characteristic. Indeed it is here that we

find the overlap with periodic rings.

THEOREM 19. Suppose R is a prime QAI-ring of characteristic $p \neq 0$. Then

R is a periodic field.

PROOF. Because R = T(R), by Theorem 6 R is a periodic ring and by Lemma

12, R has no nonzero zero-divisors. If $x \neq 0$ in R then there exists a positive

integer n greater than 1 such that $x^n = x$. Then if $e = x^{n-1}$, $e^2 = e \neq 0$. We

claim that e must be the identity element for R. If not, then there is y in

R such that $ey \neq y$ (or $ye \neq y$). But then $ey - y \neq 0$ and yet $e(ey - y) = 0$

(or $ye - y \neq 0$ and $(ye - y)e = 0$). This yields a contradiction since R has no

nonzero zero-divisors and hence $e = x^{n-1}$ is an identity for R. For any w in

R , $w^k = w$ for some k greater than 1 and as above w^{k-1} is an idempotent. By

the same argument as before $w^{k-1} = e$. Thus every nonzero element of R is

invertible and by [2, Thm. 11] R is commutative. Therefore a prime QAI-ring

of characteristic $p \neq 0$ is a periodic field.

THEOREM 20. For a ring R to be a prime QAI-ring it is necessary and

sufficient that it be a left and right order in a division algebra which is

algebraic over its prime field.

PROOF. The necessity follows directly from Theorem 18 (characteristic

zero) and Theorem 19 (characteristic p). For sufficiency, let R be a left

and right order in a division algebra D, where D is algebraic over its prime

field F. Suppose $x \neq 0$ in R; then there exist a positive integer n and
$a_0 \neq 0$, a_1, ... , a_n in F such that $a_n x^n + ... + a_1 x + a_0 = 0$. Multiplying by
x gives $a_n x^{n+1} + ... + a_1 x^2 + a_0 x = 0$.

Case 1. If char $F = p \neq 0$ then the a_i can be considered as integers
(between 0 and p).

Case 2. If char $F = 0$, then $F = Q$ and $a_i = b_i c_i^{-1}$, $i = 0, 1, ... , n$.
Set d equal to the least common multiple of c_0, c_1, ... , c_n. Then da_i is an
integer for $i = 0, 1, ... , n$ and $(da_n)x^{n+1} + ... + (da_1)x^2 + (da_0)x = 0$.

In either case x is QAI so R is a QAI-ring. Since a subring of a division
ring is always prime R is a prime QAI-ring.

In fact, we can drop the assumption that R be an order in Theorem 20.

COROLLARY 21. A ring R is a prime QAI-ring if and only if R is a subring
of a division algebra which is algebraic over its prime field.

There are several instances where a QAI-ring R, being a special type of
prime ring, must not only be a left order in a division ring but must itself
be a division ring.

THEOREM 22. Suppose R is a primitive QAI-ring. Then R is a division
ring which is algebraic over its prime field.

PROOF. A primitive ring is prime; so if R has nonzero characteristic, it
will be a periodic field by Theorem 19. We thus assume that R is a primitive
QAI-ring of characteristic zero. By the Density Theorem R is isomorphic to a
dense ring of linear transformations of a vector space V over a division ring D.
We claim that the D-dimension of V is one. Suppose not; that is, suppose that
$\{v_1, v_2\}$ is a D-linearly independent subset of V. Since R is dense, there exist
elements x and y in R so that $yv_1 = v_2$, $yv_2 = 0$, $xv_1 = 0$, $xv_2 = v_2$. But R is a
QAI-ring so there exist integers m, m_2, m_3, ... , m_t and n, n_2, n_3, ... , n_s so
that $0 \neq mx = m_2 x^2 + m_3 x^3 + ... + m_t x^t$ and $0 \neq ny = n_2 y^2 + n_3 y^3 + ... + n_s y^s$.

Then $mnxy = mx(n_2 y^2 + \ldots + n_s y^s) = ay^2$, where $a = mx(n_2 + n_3 y + \ldots + n_s y^{s-2})$.
But $0 = 0 \cdot v_1 = (mnxy - ay^2)v_1 = mnx(yv_1) - ay(yv_1) = mnv_2$, which is a contra-
diction since D has characteristic zero if R does. Thus the dimension of V over
D is one and so R is isomorphic to $D = Hom_D(V,V)$, and R is a division ring. R
is algebraic over its prime field by the definition of QAI.

COROLLARY 23. Suppose R is a QAI-ring which is an algebra over a field F.
Then R is a subdirect sum of division algebras over F.

PROOF. Since R is a QAI-ring, R is algebraic over F. Thus the Jacobson
radical is a nil ideal of R and hence (0). R is then a subdirect sum of
primitive F-algebras, $\{D_t\}_{t \epsilon A}$. Each D_t is a homomorphic image of R and has
the same characteristic as R so is a QAI-ring. The corollary follows from
Theorem 22.

THEOREM 24. A simple QAI-ring must be a division ring.

PROOF. For each positive integer m, let $mR = \{mr | r \epsilon R\}$. mR is an ideal
of R, and since R is simple, either $mR = (0)$ or $mR = R$. If there is an m for
which $mR = (0)$, then R is a prime QAI-ring of nonzero characteristic. By
Theorem 19 R is a periodic field. Otherwise, $mR = R$ for every positive integer
m. In this case R is a Q-algebra and so a division ring by Theorem 16.

4. THE GENERAL CASE.

In our study of prime QAI-rings we considered two separate cases – one of
characteristic zero and another of prime characteristic. If R is any QAI-ring,
then R is semiprime; however, a semiprime ring may have characteristic zero and
still contain elements of finite order. The dichotomy we will use in the general
case will be the torsion T(R) and the torsion-free R/T(R) cases, then
attempting to put these two extremes together to say something about R. As is
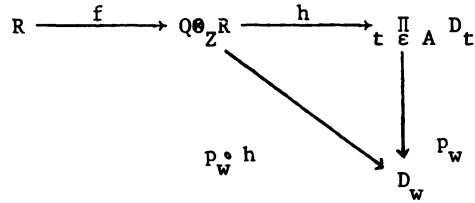often the case, it is the latter which is the most difficult.

THEOREM 25. Let R be a QAI-ring. The torsion ideal, T(R), is a subdirect sum of periodic fields and R/T(R) is a subring of an algebra S over the rationals Q, so that S is a subdirect sum of Q-division algebras which are algebraic over Q.

PROOF. The conclusions about T(R) are just Theorem 6. Let $M = R/T(R)$, a torsion-free QAI-ring by Theorem 6. Set $S = Q \otimes_Z M$. Since M is a QAI-ring, so is S by Lemma 13. If f: $M \longrightarrow S$ is given by $f(m) = 1 \otimes m$ then f is a ring monomorphism since M is torsion-free. Thus we may consider M as a subring of a Q-algebra S which is itself a QAI-ring. We claim that S is a subdirect sum of prime Q-algebras which are QAI-rings. S has no nonzero nilpotent elements and hence no nonzero nil ideals. Suppose a is a nonzero element of S, and set $J_a = \{I \mid I$ is a Q-algebra ideal of S and $I \cap \{a^n \mid n$ is a positive integer$\} = \phi\}$. $(0) \in J_a$ so $J_a \neq \phi$. By Zorn's Lemma pick I_a maximal in J_a. Then I_a is a prime algebra ideal, and the intersection of the collection $\{I_a \mid a \in S\}$ of ideals is (0). Each S/I_a is a Q-algebra which forces it to be torsion-free and hence a QAI-ring by Theorem 6. Thus S is a subdirect sum of the collection $\{S/I_a\}_{a \in S}$ of prime Q-algebras which are QAI-rings, and we are done by applying Theorem 16.

Suppose R is a torsion-free QAI-ring. It follows from Theorem 25 that we may embed R in a rational algebra which has a nice representation. Can we then say that R also has a similar representation?

THEOREM 26. Suppose R is a torsion-free QAI-ring. Then R can be written as a subdirect sum of orders in Q-division algebras each of which is algebraic over Q.

PROOF. Since Q is the quotient field of Z, the homomorphism f: $R \longrightarrow Q \otimes_Z R$ given by $f(r) = 1 \otimes r$ is a ring embedding. By the proof of Theorem 25, $Q \otimes_Z R$ is a subdirect sum of a collection of Q-division algebras $\{D_t\}_{t \in A}$. For each t in A, D_t is algebraic over Q. Consider the following diagram:

$$R \xrightarrow{\quad f \quad} Q \otimes_Z R \xrightarrow{\quad h \quad} t \underset{\epsilon\ A}{\Pi} D_t$$

$$p_w \cdot h \qquad\qquad\qquad \downarrow\ p_w$$

$$D_w$$

h is the embedding of $Q \otimes_Z R$ into the complete direct product, $t \underset{\epsilon\ A}{\Pi} D_t$ and p_w

is just the projection homomorphism onto the wth coordinate. Set $E_w = p_w(h(f(R)))$.

E_w is a subring of D_w and a ring homomorphic image of R. In fact, E_w is an order

in D_w. For if d is an element of D_w and 1 is the unity of D_w, then there are

elements $n^{-1}\theta e$ and $m^{-1}\theta r$ such that $p_w(h(n^{-1}\theta e)) = 1$ and $p_w(h(m^{-1}\theta r)) = d$. Then

$p_w(h(f(me))) = mn \cdot 1$ and $p_w(h(f(nr))) = nmd$. Thus $d = (mn \cdot 1)^{-1}(mnd) =$

$(p_w(h(f(me))))^{-1}(p_w(h(f(nr))))$. By the way E_t is defined for each t in A, it is

clear that R is a subdirect sum of $\{E_t\}_{t\ \epsilon\ A}$. (The proof that E_w is an order in

D_w is due to Professor F. Kosier.)

COROLLARY 27. If R is a torsion-free QAI-ring, then R is a subdirect sum

of prime QAI-rings of characteristic zero.

PROOF. Using the notation of the proof of Theorem 26, R is a subdirect sum

of $\{E_t\}_{t\ \epsilon\ A}$. E_t is a subring (order) of a Q-division algebra and is a QAI-ring.

Thus E_t is prime of characteristic zero.

Reviewing what we have done up to this point in the general case, we see

that by Theorem 25 and Theorem 26 if R is either a torsion QAI-ring or a torsion-

free QAI-ring, then R is a subdirect sum of orders in division algebras, each

being algebraic over its prime field. Our present goal is to get a similar

decomposition for an arbitrary QAI-ring.

DEFINITION. If R is a ring then a subset M of R is called a **multiplicative**

system if

1) 0 is not in M, and

2) whenever m and n are elements of M, so is their product mn.

M is a <u>maximal</u> multiplicative system if it is not a proper subset of another

multiplicative system.

Observe that if R is a QAI-ring, then R has no nonzero nilpotent elements;

so if r is a nonzero element of R, $N_r = \{r^n | n = 1, 2, 3, \ldots\}$ is a multiplicative

system. We state the following three lemmas without proof; see [1].

LEMMA 28. If R is a QAI-ring and a $\neq$ 0 in R, then there is a maximal

multiplicative system containing a.

LEMMA 29. If M is a maximal multiplicative system in a ring R without

nilpotent elements, then the set-theoretic complement c(M) is an ideal.

LEMMA 30. Let R be a ring without nilpotent elements and for each x $\neq$ 0

in R let M_x be a maximal multiplicative system of R containing x with complement

$c(M_x)$. Then R is a subdirect sum of the integral domains $\{R/c(M_x)\}_{x \neq 0 \in R}$.

As we observed earlier, all homomorphic images of a QAI-ring need not

be QAI-rings. In Theorem 6 we showed that if I is an ideal of R such that R/I

is torsion-free then R/I is a QAI-ring. However, in the decomposition of R, we

can make use of the following:

LEMMA 31. Let R be a QAI-ring and M a maximal multiplicative system in R.

Then R/c(M) is a QAI-ring.

PROOF. For notational convenience we denote R/c(M) by A and let $\bar{x}$ represent

the image of x under the natural homomorphism f: R $\longrightarrow$ A. From Lemma 30 we

see that A is an integral domain. If A has characteristic zero, then we are

finished since it will then be torsion-free. Thus we may assume A has charac-

teristic a prime p $\neq$ 0. We make two observations:

(i) Since A has no zero divisors, it suffices to show that for a in

R with $\bar{a} \neq \bar{0}$, $\bar{a}$ satisfies some relation of the form

$n_k x^k + n_{k-1} x^{k-1} + \ldots + n_2 x^2 + n_1 x = 0$ with at least one--any

one--of the integers n_i not divisible by p.

(ii) If a is a nonzero element of R of finite additive order m,

then $m\overline{a} = mf(a) = f(ma) = f(0) = \overline{0}$. Thus $f(a) = \overline{a}$ has finite

additive order k and k divides m. Therefore, since the

characteristic of A is p, if a in R is of order q where q is

a prime, $q \neq p$, then $\overline{a} = f(a) = \overline{0}$ in A.

$T(R) = \oplus\Sigma R_p$ where $R_p = \{x \in R | p^k x = 0 \text{ for some } k\}$

$= \{x \in R | px = 0\}$ (R has no nilpotent elements)

Let $a \neq 0$ be an element of R.

Case 1. $a \in T(R)$ $a = a_1 + a_2 + \ldots + a_t$, $a_i \in R_{p_i}$

If $\overline{a} = \overline{0}$, we are done. Otherwise $\overline{a} \neq \overline{0}$ and by (ii) there exists an integer

s such that $1 \leq s \leq t$ with $a_s \in R_{p_s} = R_p$ and $\overline{a} = \overline{a_s}$ in A. But for some

collection of integers $m_1, m_2, \ldots, m_k$, $m_k(a_s)^k + \ldots + m_2(a_s)^2 + m_1 a_s = 0$

and $m_1 a_s \neq 0$. This implies that p does not divide m_1 and

$\overline{0} = m_k(\overline{a_s})^k + m_{k-1}(\overline{a_s})^{k-1} + \ldots + m_2(\overline{a_s})^2 + m_1\overline{a_s}$

$= m_k(\overline{a})^k + m_{k-1}(\overline{a})^{k-1} + \ldots + m_2(\overline{a})^2 + m_1\overline{a}$.

Since $\overline{a} = \overline{a_s} \neq \overline{0}$ we also have $m_1\overline{a} = m_1\overline{a_s} \neq \overline{0}$ (p does not divide m_1).

Hence $\overline{a}$ is QAI.

Case 2. $a \notin T(R)$

If $\overline{a} = \overline{0}$ in A, we are done. Suppose that $\overline{a} \neq \overline{0}$. Since R is a QAI-ring,

there exist integers n, $r_1, r_2, \ldots, r_n$ such that

$r_n a^n + r_{n-1} a^{n-1} + \ldots + r_2 a^2 + r_1 a = 0$ with $r_1 a \neq 0$.

If p does not divide r_1, then $r_1\overline{a} \neq \overline{0}$ in A and we are finished.

If p divides r_1, then $\overline{0} = r_n\overline{a}^n + r_{n-1}\overline{a}^{n-1} + \ldots + r_2\overline{a}^2 + r_1\overline{a}$

$$= r_n\overline{a}^n + r_{n-1}\overline{a}^{n-1} + \ldots + r_2\overline{a}^2$$

$$= (r_n\overline{a}^{n-1} + r_{n-1}\overline{a}^{n-2} + \ldots + r_2\overline{a})\overline{a}$$

But A has no zero divisors and $\overline{a} \neq \overline{0}$ so $r_n\overline{a}^{n-1} + \ldots + r_3\overline{a}^2 + r_2\overline{a} = \overline{0}$.

If p does not divide r_2, then $r_2\overline{a} \neq \overline{0}$ in A and we are finished. If p divides

r_2, then as above we have $r_n(\overline{a})^{n-2} + \ldots + r_3\overline{a} = \overline{0}$. Continuing in this manner,

it follows that either $\overline{a}$ is QAI by (i) or p divides r_i for i = 1, 2, $\ldots$, n.

Therefore, assume p divides r_i for i = 1, 2, $\ldots$, n, and suppose p^t is

the highest power of p which divides all of the r_i. Write $r_j = p^t b_j$ for some

integers b_1, b_2, $\ldots$, b_n; at least one of these integers, say b_y, is not

divisible by p.

Now, $0 = r_n a^n + \ldots + r_2 a^2 + r_1 a = p^t(b_n a^n + \ldots + b_2 a^2 + b_1 a)$.

Set $z = b_n a^n + \ldots + b_2 a^2 + b_1 a$. If z = 0 in R, then $\overline{a}$ is QAI by i, since

$(b_y, p) = 1$. Otherwise $z \neq 0$ and since $p^t z = 0$, $z \in T(R)$; in fact, $z \in R_p$.

By Case 1, either $\overline{z} = \overline{0}$ in A which implies $b_n\overline{a}^n + \ldots + b_2\overline{a}^2 + b_1\overline{a} = \overline{0}$

and we are finished again by i; or $\overline{z} \neq \overline{0}$ in A. If this is the case, (R is QAI)

then for some integers

u, c_1, c_2, $\ldots$, c_u; $c_1 z \neq 0$ and $c_u z^u + \ldots + c_2 z^2 + c_1 z = 0$.

$c_1 z \neq 0$ implies p does not divide c_1. Then $\overline{0} = c_u(\overline{z})^u + \ldots + c_2(\overline{z})^2 + c_1\overline{z}$

and $c_1\overline{z} \neq \overline{0}$ in A. This gives

$\overline{0} = c_u(\overline{z})^u + \ldots + c_2(\overline{z})^2 + c_1(\overline{z}) = c_u(b_n\overline{a}^n + \ldots + b_2\overline{a}^2 + b_1\overline{a})^u + \ldots$

$$+ c_2(b_n\overline{a}^n + \ldots + b_2\overline{a}^2 + b_1\overline{a})^2 + c_1(b_n\overline{a}^n + \ldots + b_2\overline{a}^2 + b_1\overline{a}).$$

Note that the coefficient of $\overline{a}$ in the last equation is $c_1 b_1$. If p does not divide b_1, then p does not divide $c_1 b_1$ which implies $\overline{a}$ is QAI by (1).

If p divides b_1 but does not divide b_2, let
$w = z - b_1 a = b_n a^n + \ldots + b_3 a^3 + b_2 a^2$. Then $\overline{w} = \overline{z - b_1 a} = \overline{z} - b_1 \overline{a} = \overline{z}$,
because A has characteristic p, which divides b_1.

$$\overline{0} = c_u (\overline{w})^u + \ldots + c_2 (\overline{w})^2 + c_1 \overline{w} = c_u (b_n \overline{a}^n + \ldots + b_2 \overline{a}^2)^u + \ldots$$

$$+ c_2 (b_n \overline{a}^n + \ldots + b_2 \overline{a}^2)^2 + c_1 (b_n \overline{a}^n + \ldots + b_2 \overline{a}^2).$$

Note that the coefficient of $\overline{a}^2$ here is $c_1 b_2$, and since p does not divide $c_1 b_2$, $\overline{a}$ is QAI by (1). If p divides b_1 and b_2 but not b_3, let $v = z - b_1 a - b_2 a^2$ and do as before, since again $\overline{v} = \overline{z}$. Since there is b_y which is not divisible by p we eventually reach a stage where $\overline{a}$ is QAI by (1). In any case, a in R implies $\overline{a}$ is QAI. Hence $A = R/c(M)$ is a QAI-ring.

THEOREM 32. Suppose R is a QAI-ring. Then R is a subdirect sum of prime QAI-rings and hence a subdirect sum of periodic fields and orders in algebraic rational division algebras.

PROOF. Let F be the family of all maximal multiplicative systems in R. By Lemma 30, R is a subdirect sum of the integral domains $\{R/c(M)\}_{M \in F}$, and Lemma 31 implies that each $R/c(M)$ is a prime QAI-ring. Theorem 18 and Theorem 19 yield the desired form for the subdirect summands.

The example at the end of Section 2 shows that the converse of Theorem 32 fails to hold. The reader is encouraged to compare Theorem 32 to [5, Theorem 14.7].

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