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INTEGRABILITY AND L^1-CONVERGENCE OF REES-STANOJEVIĆ SUMS WITH GENERALIZED SEMICONVEX COEFFICIENTS

KULWINDER KAUR and S. S. BHATIA

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Integrability and L^1-convergence of modified cosine sums introduced by Rees and Stanojević (1973) under a class of generalized semiconvex null coefficients are studied, using Cesaro means of integral order.

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1. Introduction. Let

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{1.1}$$

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_j) \cos kx. \tag{1.2}$$

The problem of L^1-convergence of the Fourier cosine series (1.1) has been settled for various special classes of coefficients. Young [6] found that $a_n \log n = o(1)$, $n \to \infty$ is a necessary and sufficient condition for cosine series with convex $(\Delta^2 a_n \geq 0)$ coefficients, and Kolmogorov [5] extended this result to the cosine series with quasi-convex $(\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty)$ coefficients. Later, Garrett and Stanojević [3] using modified cosine sums (1.2), proved the following theorem.

Theorem 1.1. *Let $\{a_n\}$ be a null sequence of bounded variation. Then the sequence of modified cosine sums*

$$g_n(x) = S_n(x) - a_{n+1} D_n(x), \tag{1.3}$$

where $S_n(x)$ are the partial sums of the cosine series (1.1) and $D_n(x)$ is the Dirichlet kernel, converges in L^1-norm to $g(x)$, the pointwise sum of the cosine series, if and only if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, independent of n, such that

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \epsilon, \quad \text{for every } n. \tag{1.4}$$

This result contains as a special case a number of classical and neo-classical results. In particular, in [3] the following corollary to Theorem 1.1 is proved.

Theorem 1.2. *Let $\{a_n\}$ be a null sequence of bounded variation satisfying condition (1.4). Then the cosine series is the Fourier series of its sum $g(x)$ and $\|S_n(g)-g\| = o(1)$, $n \to \infty$ is equivalent to $a_n \log n = o(1)$, $n \to \infty$.*

In [2] Garrett and Stanojević proved the following theorem.

Theorem 1.3. *If $\{a_n\}$ is a null quasi-convex sequence, then $g_n(x)$ converges to $g(x)$ in the L^1-norm.*

Definition 1.4 (see [4]). A sequence $\{a_n\}$ is said to be semiconvex if $\{a_n\} \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n\left|\Delta^2 a_{n-1} + \Delta^2 a_n\right| < \infty, \quad (a_0 = 0), \tag{1.5}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$, $\Delta a_n = a_n - a_{n+1}$.

It may be remarked here that every quasi-convex null sequence is semi-convex. We generalize semiconvexity of null sequences in the following way: a null sequence $\{a_n\}$ is said to be generalized semiconvex, if

$$\sum_{n=1}^{\infty} n^{\alpha}\left|\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n\right| < \infty, \quad \text{for } \alpha > 0 \ (a_0 = 0). \tag{1.6}$$

For $\alpha = 1$, this class reduces to the class defined in [4]. The object of this paper is to show that Theorem 1.3 of Garrett and Stanojević [2] holds good for cosine sums (1.2) with generalized semi-convex null coefficients.

2. Notation and formulae.

In what follows, we use the following notions [7]:

$$\begin{aligned} &S_n^0 = S_n = a_0 + a_1 + \cdots + a_n; \\ &S_n^k = S_0^{k-1} + S_1^{k-1} + \cdots + S_n^{k-1}, \quad k = 1,2,\ldots,\ n = 0,1,2,\ldots; \\ &A_n^0 = 1, \quad A_n^k = A_0^{k-1} + A_1^{k-1} + \cdots + A_n^{k-1} \quad k = 1,2,\ldots,\ n = 0,1,2,\ldots. \end{aligned} \tag{2.1}$$

The A_n's are called the binomial coefficients and are given by the following relation:

$$\sum_{k=0}^{\infty} A_k^{\alpha} x^k = (1-x)^{(-\alpha-1)}, \tag{2.2}$$

whereas S_n's are given by

$$\sum_{k=0}^{\infty} S_k^{\alpha} x^k = (1-x)^{-\alpha} \sum_{k=0}^{\infty} S_k x^k, \tag{2.3}$$

and

$$\begin{aligned} &A_n^{\alpha} = \sum_{v=0}^{n} A_v^{\alpha-1}, \quad A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}, \\ &A_n^{\alpha} = \binom{n+\alpha}{n} \cong \frac{n^{\alpha}}{\Gamma(\alpha+1)} \qquad (\alpha \neq -1,-2,\ldots). \end{aligned} \tag{2.4}$$

The Cesaro means T_k^{α} of order α is denoted by $T_k^{\alpha} = S_k^{\alpha}/A_k^{\alpha}$.

Also for $0 < x \le \pi$, let

$$\begin{aligned}
\bar{D}_0(x) &= -\frac{1}{2}\cot\frac{x}{2},\\
\bar{S}_n(x) &= \bar{D}_0(x) + \sin x + \sin 2x + \cdots + \sin nx,\\
\bar{S}_n^1(x) &= \bar{S}_0(x) + \bar{S}_1(x) + \bar{S}_2(x) + \cdots + \bar{S}_n(x),\\
\bar{S}_n^2(x) &= \bar{S}_0^1(x) + \bar{S}_1^1(x) + \bar{S}_2^1(x) + \cdots + \bar{S}_n^1(x),\\
&\ \ \vdots\\
\bar{S}_n^k(x) &= \bar{S}_0^{k-1}(x) + \bar{S}_1^{k-1}(x) + \bar{S}_2^{k-1}(x) + \cdots + \bar{S}_n^{k-1}(x).
\end{aligned} \tag{2.5}$$

The conjugate Cesaro means $\bar{T}_k^\alpha$ of order α is denoted by $\bar{T}_k^\alpha = \bar{S}_k^\alpha / A_k^\alpha$.
We use the following lemma for the proof of our result.

Lemma 2.1 (see [1]). *If $\alpha \ge 0$, $p \ge 0$,*

$$\begin{gathered}
\epsilon_n = o(n^{-p}),\\
\sum_{n=0}^{\infty} A_n^{\alpha+p} \left|\Delta^{\alpha+1}\epsilon_n\right| < \infty,
\end{gathered} \tag{2.6}$$

then

$$\sum_{n=0}^{\infty} A_n^{\lambda+p} \left|\Delta^{\lambda+1}\epsilon_n\right| < \infty \quad \text{for } -1 \le \lambda \le \alpha, \tag{2.7}$$

$A_n^{\lambda+p}\Delta^\lambda \epsilon_n$ is of bounded variation for $0 \le \lambda \le \alpha$ and tends to zero as $n \longrightarrow \infty$.

3. Main result.

The main result of this paper is the following theorem.

Theorem 3.1. *If $\{a_n\}$ is a generalized semiconvex null sequence, then $g_n(x)$ converges to $g(x)$ in L^1-metric if and only if $\lim_{n\to\infty} \Delta a_n \log n = o(1)$, as $n \to \infty$.*

Proof. We have

$$\begin{aligned}
g_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx\\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x)\\
&= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \quad (a_0 = 0)\\
&= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} - a_{n+1} D_n(x),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned} D_n(x) &= \frac{\sin nx + \sin(n+1)x}{2\sin x}, \\ g_n(x) &= \sum_{k=1}^{n-1}(a_{k-1}-a_{k+1})\frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x} \\ &\quad - a_{n+1}\frac{\sin nx}{2\sin x} - a_{n+1}\frac{\sin(n+1)x}{2\sin x} \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}+\Delta a_k)\sin kx + \Delta a_n\frac{\sin(n+1)x}{2\sin x}. \end{aligned} \tag{3.2}$$

Applying Abel's transformation, we have

$$\begin{aligned} g_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\sum_{v=1}^{k}\sin vx + (\Delta a_{n-1}+\Delta a_n)\sum_{v=1}^{n}\sin vx \\ &\quad + \Delta a_n\frac{\sin(n+1)x}{2\sin x} \\ &= \frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)(\bar{S}_k^0(x)-\bar{S}_0(x)) + (\Delta a_{n-1}+\Delta a_n)(\bar{S}_n^0(x)-\bar{S}_0(x))\right] \\ &\quad + \Delta a_n\frac{\sin(n+1)x}{2\sin x} \\ &= \frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\bar{S}_k^0(x) - \sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\bar{S}_0(x)\right] \\ &\quad + \frac{1}{2\sin x}[(\Delta a_{n-1}+\Delta a_n)\bar{S}_n^0(x) - (\Delta a_{n-1}+\Delta a_n)\bar{S}_0(x)] + \Delta a_n\frac{\sin(n+1)x}{2\sin x} \\ &= \frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)(\bar{S}_k^0(x)) - (\Delta a_{n-1}+\Delta a_n)\bar{S}_n^0(x) + a_2\bar{S}_0(x)\right] \\ &\quad + \Delta a_n\frac{\sin(n+1)x}{2\sin x}. \end{aligned} \tag{3.3}$$

□

If we use Abel's transformation α times, we have similarly,

$$\begin{aligned} g_n(x) &= \frac{1}{2\sin x}\left[\sum_{k=1}^{n-\alpha}(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\bar{S}_k^{\alpha-1}(x) + \sum_{k=1}^{\alpha}\Delta^k a_{n-k}\bar{S}_{n-k+1}^{k-1}(x)\right] \\ &\quad + \frac{1}{2\sin x}\left[\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\bar{S}_{n-k+1}^{k-1}(x) + a_2\bar{S}_0(x)\right] + \Delta a_n\frac{\sin(n+1)x}{2\sin x}. \end{aligned} \tag{3.4}$$

Since $\bar{S}_n(x)$ and $\bar{T}_n(x)$ are uniformly bounded on every segment $[\epsilon, \pi-\epsilon]$, $\epsilon > 0$.

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g_n(x) \\ &= \frac{1}{2\sin x}\left[\sum_{k=1}^{\infty}(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\bar{S}_k^{\alpha-1}(x) + a_2\bar{S}_0(x)\right]. \end{aligned} \tag{3.5}$$

Thus

$$g(x)-g_n(x)=\frac{1}{2\sin x}\left[\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\bar{S}_k^{\alpha-1}(x)-\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\bar{S}_{n-k+1}^{k-1}(x)\right]$$
$$-\frac{1}{2\sin x}\left[\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\bar{S}_{n-k+1}^{k-1}(x)\right]-\Delta a_n\frac{\sin(n+1)x}{2\sin x},$$
$$\|g(x)-g_n(x)\|\le C\left[\int_0^{\pi}\left|\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\bar{S}_k^{\alpha-1}(x)\right|dx\right]$$
$$+C\left[\int_0^{\pi}\left|\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\bar{S}_{n-k+1}^{k-1}(x)\right|dx+\int_0^{\pi}\left|\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\bar{S}_{n-k+1}^{k-1}(x)\right|dx\right]$$
$$+\int_0^{\pi}\left|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\right|dx,$$
$$\|g(x)-g_n(x)\|\le C\left[\sum_{k=n-\alpha+1}^{\infty}\left|(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\right|\int_0^{\pi}\left|\bar{S}_k^{\alpha-1}(x)\right|dx\right]$$
$$+C\left[\sum_{k=1}^{\alpha}|\Delta^k a_{n-k}|\int_0^{\pi}|\bar{S}_{n-k+1}^{k-1}(x)|\,dx+\sum_{k=1}^{\alpha}|\Delta^k a_{n-k+1}|\int_0^{\pi}|\bar{S}_{n-k+1}^{k-1}(x)|\,dx\right]$$
$$+\int_0^{\pi}\left|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\right|dx$$
$$\le C\left[\sum_{k=n-\alpha+1}^{\infty}A_k^{\alpha}\left|(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\right|\int_0^{\pi}\left|\bar{T}_k^{\alpha}(x)\right|dx\right]$$
$$+C\left[\sum_{k=1}^{\alpha}A_{n-k+1}^{k}\left|\Delta^k a_{n-k}\right|\int_0^{\pi}\left|\bar{T}_{n-k+1}^{k}(x)\right|dx\right]$$
$$+C\left[\sum_{k=1}^{\alpha}A_{n-k+1}^{k}\left|\Delta^k a_{n-k+1}\right|\int_0^{\pi}\left|\bar{T}_{n-k+1}^{k}(x)\right|dx\right]$$
$$+\int_0^{\pi}\left|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\right|dx. \tag{3.6}$$

The first three terms of the above inequality are of $o(1)$ by Lemma 2.1 and the hypothesis of Theorem 3.1.

Moreover, since

$$\int_0^{\pi}\left|\frac{\sin(n+1)x}{2\sin x}\right|dx\le C\log n,\quad n\ge 2, \tag{3.7}$$

therefore

$$\int_0^{\pi}\left|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\right|dx\sim\Delta a_n\log n. \tag{3.8}$$

It follows that $\int_0^\pi |g(x)-g_n(x)|dx \to 0$, if and only if $\Delta a_n \log n \to o(1)$ as $n \to \infty$. This completes the proof of the theorem. □

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Kulwinder Kaur: School of Basic and Applied Sciences, Thapar Institute of Engineering and Technology, P.O. Box 32, Patiala 147004, India
E-mail address: mathkk@hotmail.com

S. S. Bhatia: School of Basic and Applied Sciences, Thapar Institute of Engineering and Technology, P.O. Box 32, Patiala 147004, India
E-mail address: ssbhatia63@yahoo.com