IJMMS 30:8 (2002) 459–466
PII. S0161171202109239
http://ijmms.hindawi.com

ADJOINT REGULAR RINGS

HENRY E. HEATHERLY and RALPH P. TUCCI

Received 10 September 2001

Let R be a ring. The *circle operation* is the operation $a \circ b = a + b - ab$, for all $a, b \in R$. This operation gives rise to a semigroup called the *adjoint semigroup* or *circle semigroup* of R. We investigate rings in which the adjoint semigroup is regular. Examples are given which illustrate and delimit the theory developed.

2000 Mathematics Subject Classification: 16N20, 16U99, 20M17, 20M18, 20M19.

1. Introduction. This paper continues the authors' investigation of adjoint semigroups of rings [13, 14]. Here R will always denote a ring (not necessarily commutative and not necessarily with unity). The Jacobson *circle* or *adjoint* operation, $a \circ b = a + b - ab$, for each $a, b \in R$, yields a monoid $(R, \circ)$, the *adjoint semigroup* of R. Here we primarily consider the situation where R is a (von Neumann) regular ring or where $(R, \circ)$ is (von Neumann) regular. Previous work along these lines has been done by Du [7] and Clark [3]. Our viewpoint is that of the interplay between semigroup properties of $(R, \circ)$ or $(R, \cdot)$ and ring properties of R.

For $x \in R$, we use $\mathbf{l}_R(x)$ and $\mathbf{r}_R(x)$ for the left and right annihilator sets of x in R, respectively. When no ambiguity will arise, we use simply $\mathbf{l}(x)$ and $\mathbf{r}(x)$. The Jacobson radical of R is denoted by $\mathcal{J}(R)$.

Let S be a semigroup. We use $E(S)$ for the set of idempotents in S and $Z(S)$ for the center of S. Frequent use will be made of the fact that $E(R, \cdot) = E(R, \circ)$ and $Z(R, \cdot) = Z(R, \circ)$. Consequently we use $E(R)$ and $Z(R)$ for these sets, respectively.

Of particular interest here are the following types of regular semigroups. Let T be a regular semigroup. If the idempotents of T commute among one another, T is said to be *inverse*; this is equivalent to the condition that the von Neumann inverse of each element in T is unique, [5, Theorem 4.11]. If the idempotents of T are central, then T is said to be *Clifford*. It is well known that a Clifford semigroup is a union of groups, [5, Theorem 1.17]. If the idempotents form a subsemigroup, then T is called *orthodox*. If each element commutes with one of its von Neumann pseudoinverses, then T is *completely regular*. This last condition is equivalent to the condition that T is a union of groups, [19, Theorem II.1.4].

Let $\mathcal{P}$ be a semigroup property (or from another vantage point, $\mathcal{P}$ could be thought of as a class of semigroups closed under isomorphism). If $(R, \cdot)$ has property $\mathcal{P}$ we say R is a *$\mathcal{P}$-ring*, and if $(R, \circ)$ has property $\mathcal{P}$ we say R is an *adjoint $\mathcal{P}$-ring*. Exemplary of such properties are regular, completely regular, or Clifford. It is worth noting that what here is called an adjoint completely regular ring is called a generalized radical ring in [3, 8]. (For terminology and basic facts on semigroups, see [5] or [19].)

2. Preliminaries. In this section, we present some preliminary results. Let R^1 be the standard Dorroh extension of a ring R to the ring R^1, which has unity. Recall that this embeds R as an ideal of R^1, so we can identify R as the ideal $\bar{R}$ in R^1. We will at times use R^1 in conjunction with R in order to make use of the simplifying attributes of having an identity for calculation. Recall that the mapping $\phi : x \to 1-x$ is an isomorphism from $(R^1, \circ)$ onto $(R^1, \cdot)$, and ϕ restricted to R yields an injective homomorphism. Observe that if R has identity, then this same mapping, $x \to 1-x$, yields an isomorphism from $(R, \circ)$ onto $(R, \cdot)$. This observation makes the next result immediate. (See also [9, Lemma 20].)

Proposition 2.1. *Let $a, b \in R$. Then*
(a) *$a \circ b \in E(R)$ if and only if $(1-a)(1-b) = 1-(a \circ b)$;*
(b) *$a \circ b \circ a = a$ if and only if $(1-a)(1-b)(1-a) = (1-a)$;*
(c) *$a \circ R = b \circ R$ if and only if $(1-a)R^1 = (1-b)R^1$;*
(d) *a is adjoint regular in R if and only if $1-a$ is a regular element in R^1.*

The next result is immediate, but it is useful enough to warrant stating.

Proposition 2.2. *An element $a \in R$ is adjoint regular if and only if there exist $b \in R$, $e = e^2 \in R$ such that $a + b - ab = e$ and $ea = e$.*

Note that a quasi-regular element satisfies the conditions of Proposition 2.2 with $e = 0$. Hence, the conditions for adjoint regularity are a natural generalization of the condition for quasi-regularity.

Du [7, Theorem 1] has shown that if R is a regular ring, then $(R, \circ)$ is a regular monoid. We next give a different proof of that result.

Proposition 2.3. *If R is a regular ring, then $(R, \circ)$ is a regular monoid.*

Proof. There is an injective ring homomorphism, $\phi : R \to R^*$, that embeds R as an ideal, $\hat{R}$, in the regular ring R^* which has identity [10, Theorem 1]. Consequently, $\phi : (R, \circ) \to (R^*, \circ)$ is an isomorphism. Since R^* has identity we have that $(R^*, \cdot) \cong (R^*, \circ)$ and that each ideal of $(R^*, \cdot)$ is regular. So each ideal of $(R^*, \circ)$ is regular. Hence $(\hat{R}, \circ)$ is regular and so is its isomorphic image $(R, \circ)$. □

Since every ideal of a regular semigroup is regular, we immediately have that if R is a regular ring and X is an ideal of $(R, \circ)$, then $(X, \circ)$ is regular.

By using the powerful Fuchs-Halperin result, this proof completely bypasses the calculations used in Du's proof [7]. Also, using similar methods, we can obtain analogous results by replacing the term *regular* in Proposition 2.3 by *orthodox* or *inverse.* Finally, note that the converse of Proposition 2.3 does not hold, as any example of a Jacobson radical ring will illustrate.

3. Equivalent conditions under regularity. In this section, we discuss various equivalent conditions to R strongly regular and to R adjoint completely regular. Examples are given which illustrate these results and show limitations to extending them.

It is well known that the following conditions are equivalent for a ring R:
(1) for each $a \in R$ there exists $b \in R$ such that $a = a^2 b$;

(2) for each $a \in R$ there exists $c \in R$ such that $a = c^2 a$;
(3) R is regular and $E(R) \subseteq Z(R)$.

A ring R satisfying any of (and hence all of) these conditions is called a *strongly regular* ring, [6]. Note that these conditions are not equivalent for semigroups, [19, Theorem II.1.4]. The next proposition ties strongly regular with several other conditions on the multiplicative semigroup of a ring. The proposition is a compilation of results from various sources. First some terminology is needed.

Following [17] we say that a semigroup S is *E-solid* if, whenever $e, f, g \in E(S)$ such that $e\mathcal{L}f\mathcal{R}g$, there exists $h \in E(S)$ such that $e\mathcal{R}h\mathcal{L}g$. (Here $\mathcal{L}$ and $\mathcal{R}$ are the standard Green's relations, [5, page 47].) Recall that the *core* of S, denoted by $C(S)$, is the subsemigroup of S generated by $E(S)$, [19, page 89]. So S is orthodox if and only if S is regular and $E(S) = C(S)$. Let $\mathcal{P}$ be a semigroup property. Then S is *locally-*$\mathcal{P}$ if eSe has property $\mathcal{P}$ for every $e \in E(S)$.

Proposition 3.1. *Let R be a regular ring. The following are equivalent:*
(a) *R is strongly regular;*
(b) *R is orthodox;*
(c) *R is completely regular;*
(d) *R is inverse;*
(e) *R is Clifford;*
(f) *R is locally inverse;*
(g) *R is E-solid;*
(h) *R is locally E-solid;*
(i) *$C(R)$ is completely regular.*

Proof. As mentioned above, (a)⇔(e). So (e)⇒(b) is trivial, while (b)⇒(e) follows from [22, Remark 15]. It is known that (a)⇔(c), even for semigroups [19, page 58]. Next, (d)⇔(e) comes from [13], while (d)⇔(f)⇔(g)⇔(h) comes from [17], and (g)⇔(i) comes from [12, Theorem 3]. This completes the logical circuit.

This by no means exhausts the vast number of known equivalent conditions to R strongly regular, but it suffices for our purposes and gives a good sample of what is known. For more equivalent conditions to R strongly regular see [5, 6, 11, 16, 18, 19, 21, 22].

We next consider various equivalent conditions on the adjoint semigroup of a regular ring. □

Proposition 3.2. *Let R be regular. The following are equivalent:*
(a) *R is adjoint completely regular;*
(b) *R is adjoint Clifford;*
(c) *R is strongly regular.*

Proof. Assume (a). Du [8, Lemma 11] has shown that $(R, \circ)$ completely regular implies that $E(R)$ is closed under the ring multiplication. Since R is regular, this yields that R is orthodox. Thus R is Clifford and by Proposition 3.1 R is strongly regular. Consequently, R is adjoint regular. Conversely, R regular and adjoint Clifford implies that R is Clifford, and hence R is strongly regular. We have established (a)⇒(c)⇔(b). Since a Clifford semigroup is a union of groups, $(R,\circ)$ Clifford implies $(R,\circ)$ completely regular, yielding (b)⇒(a) and finishing the logical chain. □

It is known [13] that R is adjoint inverse if and only if R is adjoint Clifford. Recently Du [9] has shown that adjoint orthodox implies adjoint completely regular. However, an adjoint completely regular ring need not be adjoint Clifford, as the next example shows.

Example 3.3. Let S be a right zero semigroup with more than one element and let R be the semigroup ring $\mathbb{Z}_2[S]$. So every nonzero element of R has the form $x = e_1+\cdots+e_n$, for some distinct $e_1,\dots,e_n \in S$. Let $y = f_1+\cdots+f_m$, where $f_1,\dots,f_m$ are distinct terms from S. Then $xy = ny$. So $xy = 0$, if n is even, and $xy = y$, if n is odd. Then x is an idempotent if and only if n is odd, and if y is also an idempotent, then $xy = y$. Consequently $E(R)$ is closed under ring multiplication. Observe that $R = N(R)\cup E(R)$; hence R is adjoint orthodox. Since elements in $E(R)$ are completely regular in $(R,\circ)$, and since elements in $N(R)$ are quasi-regular in R, and hence are completely regular in R, we have that R is adjoint completely regular. However, since $(e_1+e_2)y(e_1+e_2) = 0$, for each $y \in R$, we see that R is not regular. Also, R is not a Jacobson radical ring. Since $e_1e_2 \neq e_2e_1$, for distinct $e_1,e_2 \in S$, we see that $e_1\circ e_2 \neq e_2\circ e_1$, and hence R is not adjoint inverse.

Having R regular implies that R is adjoint regular, as we have seen. However, R being regular does not imply that R is adjoint inverse, adjoint completely regular, nor adjoint orthodox, as the next example illustrates.

Example 3.4. Let A be a regular ring with identity and let $R = M_2(A)$, the ring of 2×2 matrices over A. So R is regular and hence R is adjoint regular. But there are noncommuting idempotents in R; so R is not adjoint inverse, and hence not adjoint completely regular by Proposition 3.2. By [9, Theorem 14], R is not adjoint orthodox.

4. Decomposition. Let R be a ring, and let $S_1,\dots,S_n$ be subrings of R. If $R = S_1+\cdots+S_n$ and whenever $s_i \in S_i$, $i = 1,\dots,n$ such that $s_1+\cdots+s_n = 0$, then $s_i = 0$, $i = 1,\dots,n$, then we say that R is a *supplementary sum* of the S_i, $i = 1,\dots,n$, [1]. This is equivalent to $R^+ = \Sigma_{i=1}^n \oplus S_i^+$, as a direct sum of abelian groups. We write $R = S_1 \dotplus \cdots \dotplus S_n$ for such a supplementary sum. We next state as a lemma the well-known two-sided Peirce decomposition, given here without using an identity in the ring. (See [1, 15].)

Lemma 4.1. *Let $e \in E(R)$. Then $R = eRe \dotplus e\cdot \mathbf{l}_R(e) \dotplus \mathbf{r}_R(e)\cdot e \dotplus \mathbf{r}_R(e)\cap \mathbf{l}_R(e)$.*

Recall [2] that $e \in E(R)$ is said to be a *left semicentral idempotent* in R if $eRe = Re$, (equivalently, $ere = re$, for each $r \in R$). It is well known that in this case $\mathbf{l}_R(e)$ is an ideal of R and $R/\mathbf{l}_R(e) \cong eRe$.

Proposition 4.2. *Let e be a left semicentral idempotent in R.*

(a) *$eRe = Re$ is a left ideal of R; $l_R(e) = l_R(Re)$ is an ideal of R; and $e\cdot l_R(e)$ is a right ideal of R;*

(b) $R = Re \oplus_l l_R(Re)$ *as a direct sum of left ideals of* R *and* $R/l_R(e) \cong eRe$*, a ring with unity* e*;*

(c) $eR = Re \dotplus e \cdot l_R(Re)$*.*

Proof. By definition, $Re = eRe$. The rest is routine. □

In a strictly analogous fashion, we define right semicentral idempotent and obtain a dual result. For a central idempotent we naturally obtain a stronger result.

Corollary 4.3. *If* e *is a central idempotent in* R*, then* $R = eRe \oplus \mathrm{Ann}_R(eRe)$*, as a direct sum of ideals of* R*.*

Following Clark and Lewin [4], $e \in E(R)$ is said to be a *principal idempotent* in R if the homomorphic image of e in $\bar{R} = R/J(R)$ is the identity in $\bar{R}$. This implies that if $u \in E(R)$ such that $eu = 0 = ue$, then $u = 0$. (The latter condition, together with $e \neq 0$, is what Albert used in defining the *principal idempotent* [1, page 25].)

Principal idempotents play a key role in our next decomposition, whose proof makes use of the following result due to Du [7, Corollary 2].

Lemma 4.4. *If* R *is adjoint regular and* e *is a principal idempotent in* R*, then* $R = eRe \dotplus J(R)$*, as a supplementary sum of subrings.*

We are now ready to give a much shorter proof of the main result in [3, Theorem B].

Proposition 4.5. *Let* e *be an idempotent in a ring* R*. The following are equivalent:*

(a) R *is adjoint completely regular and* e *is a principal idempotent in* R*;*

(b) $R = eRe \dotplus J(R)$ *and* eRe *is a strongly regular ring.*

Proof. Assume (a). By Lemma 4.4, $R = eRe \dotplus J(R)$; so $R/J(R) \cong eRe$. Since the ring eRe inherits the adjoint completely regular condition from R and eRe is regular, by Proposition 3.2 we have that ring eRe is strongly regular.

Assume (b). Then eRe is adjoint regular, so by [7] the ring R is adjoint regular. Since all the idempotents of R are central, R is adjoint completely regular. Clearly e is a principal idempotent of R. □

In view of Proposition 4.5 and the results of Section 3, we immediately have a plethora of conditions equivalent to part (a) of Proposition 4.5.

Proposition 4.6. *Let* $R = A \oplus B$ *as a direct sum of left (right) ideals. If* R *is adjoint regular, then* A *and* B *are adjoint regular rings.*

Proof. Let $a \in A$. Then there exist $a_1 \in A$, $b_1 \in B$ such that $a = a \circ (a_1 + b_1) \circ a = a \circ (a_1 + b_1) + a - [a \circ (a_1 + b_1)]a$. So $a \circ (a_1 + b_1) = [a \circ (a_1 + b_1)]a \in A$. Then $a \circ (a_1 + b_1) = a + a_1 + b_1 - a(a_1 + b_1)$, or $b_1 - ab_1 = a \circ (a_1 + b_1) - a - a_1 \in A$. But $b_1 - ab_1 \in B$; so $b_1 - ab_1 = 0$. Then $a = a \circ (a_1 + b_1) \circ a = [a + a(a_1 + b_1) - a(a_1 + b_1)] \circ a$, or $a = (a + a_1 - aa_1) \circ a = a \circ a_1 \circ a$. Proceed similarly for right ideals. □

Lemma 4.7. *Let* R *be adjoint regular.*

(a) *Either* $J(R) = R$ *or* R *contains a nonzero idempotent.*

(b) *If the module* ${}_RR$ *is indecomposable and* $R \neq J(R)$*, then* $R = eRe \oplus \mathbf{r}(e)$*, as a direct sum of right ideals, with* eRe *a regular ring with identity* e*, and* $\mathbf{r}(e)$ *is a square zero ideal of* R*.*

(c) *If the modules* ${}_RR$ *and* R_R *are indecomposable, then either* $R = J(R)$ *or* R *is a division ring.*

Proof. (a) Let $r \in R$, $r \neq 0$. Then there exists $\bar{r} \in R$ such that $r \circ \bar{r} \circ r = r$, and $r \circ \bar{r}$ and $\bar{r} \circ r$ are nonzero idempotents in $(R, \circ)$. So $r \circ \bar{r}$ and $\bar{r} \circ r$ are also idempotents in R. If R has no nonzero idempotent, then $r \circ \bar{r} = \bar{r} \circ r$. So in this case each element of R is quasi-regular, that is, $R = J(R)$.

(b) If $R \neq J(R)$, then there exists a nonzero $r \in R$ such that $e = r \circ \bar{r}$ is a nonzero idempotent in R. Then $R = Re \oplus \mathbf{l}(e)$, as a direct sum of left ideals. Since Re and $\mathbf{l}(e)$ are submodules of ${}_RR$, and since $Re \neq 0$, we have $\mathbf{l}(e) = 0$, and hence e is a right identity of R. So $R = eR \oplus \mathbf{r}(e)$, as a direct sum of right ideals of R. However, $R \cdot \mathbf{r}(e) = (Re)\mathbf{r}(e) = 0$, so $\mathbf{r}(e)$ is an ideal of R and $\mathbf{r}(e) \subseteq \mathbf{r}(R)$. Since R is adjoint regular, we have that eRe is a regular ring [7, Proposition 2].

(c) Continuing from the proof of (b), since R_R is indecomposable and since $eRe \neq 0$, we have that $\mathbf{r}(e) = 0$, and hence e is also a left identity. Thus $R = eRe$, a regular ring with identity. Since the ring is indecomposable, either $R = J(R)$ or R is a division ring. □

Note that from Lemma 4.7, we have that if R is indecomposable in terms of both left and right ideal decompositions, then either R is a Jacobson radical ring and the regular radical (see [20, Chapter VI]) is zero, or R is equal to its regular radical and the Jacobson radical is zero.

Proposition 4.8. *Let R be adjoint regular. If R has a right (left) nonzero semicentral idempotent, then there exist submonoids A and B of $(R, \circ)$ such that*

(a) $R = A \circ B$ *and* $A \cap B = 0$*;*
(b) $R = A \oplus B$ *as a direct sum of adjoint regular right (left) ideals of* R*;*
(c) A *is a regular ring with identity and* B *is a two-sided ideal of* R*.*

Proof. Let e be a nonzero right semicentral idempotent in R. Then $R = eR \oplus r(e)$ as a direct sum of right ideals, where $eR = eRe$ is a ring with unity and $r(e)$ is a two-sided ideal of R. Let $A = eR$ and $B = r(e)$. Observe that $R = A \circ B$ because $AB = 0$. By Proposition 4.6, $(A, \circ)$ is regular. Since A is a ring with unity, A is regular. Proceed similarly for e left semicentral. □

5. Radicals for adjoint Clifford rings. We show the equivalence of several standard radicals for rings which are adjoint Clifford and obtain a characterization for the Jacobson radical of such rings.

Proposition 5.1. *If R is adjoint Clifford, then the Brown-McCoy radical, $\mathcal{G}(R)$, and $\mathcal{J}(R)$ are equal.*

Proof. For purposes of contradiction suppose that there is an adjoint Clifford ring R with $\mathcal{J}(R) \neq \mathcal{G}(R)$. Then $R/\mathcal{J}(R)$ is also adjoint Clifford and $\mathcal{J}(R/\mathcal{J}(R)) = 0$. So, without loss of generality, take $\mathcal{J}(R) = 0$. Since R is adjoint Clifford it is a subdirect product of division rings by [13, Proposition 3.7]. Because $\mathcal{G}(R) \neq 0$, at least one of these homomorphic image division rings must have nonzero Brown-McCoy radical, a contradiction. □

Example 5.2. Proposition 5.1 cannot be extended to all adjoint regular rings, even if the ring is also adjoint simple, as the following example illustrates. Let V be a vector space over a field F with $\dim_F V = \aleph_\omega$, and let ω be the first infinite limit ordinal. Let $R = \{\phi \in \mathrm{End}_F V \mid \operatorname{rank}\phi < \aleph_\omega\}$. It is well known that R is a regular ring with no maximal ideal. So $J(R) = 0$ and $G(R) = R$. It is also known that $(R, \circ)$ is simple, [4, Example 3B].

In the next proposition, when R does not have unity we use the unity element in the Dorroh extension for convenience of expression.

Proposition 5.3. *If R is adjoint Clifford, then $\mathcal{J}(R) = \cap_{e\in E}(1-e)R$.*

Proof. Let $R = eR \oplus (1-e)R$ as ideals. Then eR is adjoint Clifford because the map $\phi : R \to eR$ is a ring homomorphism and so $\phi : (R, \circ) \to (eR, \circ)$ is a semigroup homomorphism. Therefore $\mathcal{J}(R) \subseteq (1-e)R$. Since e is arbitrary, we have that $\mathcal{J}(R) \subseteq \cap_{e\in E}(1-e)R$.

Conversely, let $I = \cap_{e\in E}(1-e)R$. Then I is an ideal, hence adjoint Clifford. But I contains no nonzero idempotent. Therefore $(I, \circ)$ is a group, so $I \subseteq \mathcal{J}(R)$. □

References

[1] A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, American Mathematical Society, New York, 1939.

[2] G. F. Birkenmeier, *Idempotents and completely semiprime ideals*, Comm. Algebra **11** (1983), no. 6, 567–580.

[3] W. E. Clark, *Generalized radical rings*, Canad. J. Math. **20** (1968), 88–94.

[4] W. E. Clark and J. Lewin, *On minimal ideals in the circle composition semigroup of a ring*, Publ. Math. Debrecen **14** (1967), 99–104.

[5] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Vol. I*, Mathematical Surveys, no. 7, American Mathematical Society, Rhode Island, 1961.

[6] M. P. Drazin, *Rings with central idempotent or nilpotent elements*, Proc. Edinburgh Math. Soc. (2) **9** (1958), 157–165.

[7] X. K. Du, *The rings with regular adjoint semigroups*, Northeast. Math. J. **4** (1988), no. 4, 463–468.

[8] ———, *The structure of generalized radical rings*, Northeast. Math. J. **4** (1988), no. 1, 101–114.

[9] ———, *The adjoint semigroup of a ring*, preprint, 2001.

[10] L. Fuchs and I. Halperin, *On the imbedding of a regular ring in a regular ring with identity*, Fund. Math. **54** (1964), 285–290.

[11] K. R. Goodearl, *Von Neumann Regular Rings*, Monographs and Studies in Mathematics, vol. 4, Pitman, Massachusetts, 1979.

[12] T. E. Hall, *On regular semigroups*, J. Algebra **24** (1973), 1–24.

[13] H. E. Heatherly and R. P. Tucci, *Adjoint Clifford rings*, to appear in Acta Math. Hungar.

[14] ———, *The circle semigroup of a ring*, Acta Math. Hungar. **90** (2001), 231–242.

[15] N. Jacobson, *Structure of Rings*, American Mathematical Society Colloquium Publications, vol. 37, American Mathematical Society, Rhode Island, 1968.

[16] L. Kovács, *A note on regular rings*, Publ. Math. Debrecen **4** (1956), 465–468.

[17] J. O. Loyola, *e-free objects in e-varieties of inverse rings*, Semigroup Forum **54** (1997), no. 3, 375–380.

[18] J. Luh, *A note on strongly regular rings*, Proc. Japan Acad. **40** (1964), 74–75.

[19] M. Petrich and N. R. Reilly, *Completely Regular Semigroups*, Canadian Mathematical Society Series of Monographs and Advanced Texts, vol. 23, John Wiley & Sons, New York, 1999.

[20] F. A. Szász, *Radicals of Rings*, John Wiley & Sons, Chichester, 1981.
[21] P. G. Trotter, *Congruence extensions in regular semigroups*, J. Algebra **137** (1991), no. 1, 166–179.
[22] J. Zeleznikov, *Orthodox rings and semigroups*, J. Austral. Math. Soc. Ser. A **30** (1980), 50–54.

Henry E. Heatherly: Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA

Ralph P. Tucci: Department of Mathematics and Computer Science, Loyola University, New Orleans, LA 70118, USA
E-mail address: tucci@loyno.edu