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THE GALOIS ALGEBRAS AND THE AZUMAYA GALOIS EXTENSIONS

GEORGE SZETO and LIANYONG XUE

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Let B be a Galois algebra over a commutative ring R with Galois group G, C the center of B, $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$, $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in K$, and $B_K = (\oplus \sum_{g \in K} J_g)$. Then B_K is a central weakly Galois algebra with Galois group induced by K. Moreover, an Azumaya Galois extension B with Galois group K is characterized by using B_K.

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1. Introduction. Let B be a Galois algebra over a commutative ring R with Galois group G and C the center of B. The class of Galois algebras has been investigated by DeMeyer [2], Kanzaki [6], Harada [4, 5], and the authors [7]. In [2], it was shown that if R contains no idempotents but 0 and 1, then B is a central Galois algebra with Galois group K and C is a commutative Galois algebra with Galois group G/K where $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$ [2, Theorem 1]. This fact was extended to the Galois algebra B over R containing more than two idempotents [6, Proposition 3], and generalized to any Galois algebra B [7, Theorem 3.8] by using the Boolean algebra B_a generated by $\{0, e_g \mid g \in G$ for a central idempotent $e_g\}$ where $BJ_g = Be_g$ and $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$ [6]. The purpose of this paper is to show that there exists a subalgebra B_K of B such that B_K is a central weakly Galois algebra with Galois group $K|_{B_K}$ induced by K where a weakly Galois algebra was defined in [8] and that $B_K B^K$ is an Azumaya weakly Galois extension with Galois group $K|_{B_K B^K}$ where an Azumaya Galois extension was studied in [1]. Thus some characterizations of an Azumaya Galois extension B of B^K with Galois group K are obtained, and the results as given in [2, 6] are generalized.

2. Definitions and notations. Throughout, let B be a Galois algebra over a commutative ring R with Galois group G, C the center of B, and $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$. We keep the definitions of a Galois extension, a Galois algebra, a central Galois algebra, a separable extension, and an Azumaya algebra as defined in [7]. An Azumaya Galois extension A with Galois group G is a Galois extension A of A^G which is a C^G-Azumaya algebra where C the center of A [1]. A weakly Galois extension A with Galois group G is a finitely generated projective left module A over A^G such that $A_l G \cong \text{Hom}_{A^G}(A, A)$ where $A_l = \{a_l$, a left multiplication map by $a \in A\}$ [8]. We call that A is a weakly Galois algebra with Galois group G if A is a weakly Galois extension with Galois group G such that A^G is contained in the center of A and that

A is a central weakly Galois algebra with Galois group G if A is a weakly Galois extension with Galois group G such that A^G is the center of A. An Azumaya weakly Galois extension A with Galois group G is a weakly Galois extension A of A^G which is a C^G-Azumaya algebra where C the center of A.

3. A weakly Galois algebra. In this section, let B be a Galois algebra over R with Galois group G, C the center of B, $B^G = \{b \in B \mid g(b) = b$ for all $g \in G\}$, and $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$. Then, $B = \oplus\sum_{g\in G} J_g = (\oplus\sum_{g\in K} J_g) \oplus (\oplus\sum_{g\notin K} J_g)$ where $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ [6, Theorem 1] . We denote $\oplus\sum_{g\in K} J_g$ by B_K and the center of B_K by Z. Clearly, K is a normal subgroup of G. We show that B_K is an Azumaya algebra over Z and a central weakly Galois algebra with Galois group $K|_{B_K}$.

Theorem 3.1. *The algebra B_K is an Azumaya algebra over Z.*

Proof. By the definition of B_K, $B_K = \oplus\sum_{g\in K} J_g$, so $C(= J_1) \subset B_K$. Since B is a Galois algebra with Galois group G and $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$, the order of K is a unit in C by [6, Proposition 5]. Moreover, K is an C-automorphism group of B, so B_K is a C-separable algebra by [5, Proposition 5]. Thus B_K is an Azumaya algebra over Z. □

In order to show that B_K is a central weakly Galois algebra with Galois group $K|_{B_K}$, we need two lemmas.

Lemma 3.2. *Let $L = \{g \in K \mid g(a) = a$ for all $a \in B_K\}$. Then, L is a normal subgroup of K such that $\overline{K}(= K/L)$ is an automorphism group of B_K induced by K (i.e., $K|_{B_K} \cong \overline{K}$).*

Proof. Clearly, L is a normal subgroup of K, so for any $h \in K$,

$$h(B_K) = \oplus\sum_{g\in K} h(J_g) = \oplus\sum_{g\in K} J_{hgh^{-1}} = \oplus\sum_{g\in hKh^{-1}} J_g = \oplus\sum_{g\in K} J_g = B_K. \tag{3.1}$$

Thus $K|_{B_K} \cong \overline{K}$. □

Lemma 3.3. *The fixed ring of B_K under K, $(B_K)^K = Z$.*

Proof. Let x be any element in $(B_K)^K$ and b any element in B_K. Then $b = \sum_{g\in K} b_g$ where $b_g \in J_g$ for each $g \in K$. Hence $bx = \sum_{g\in K} b_g x = \sum_{g\in K} g(x) b_g = \sum_{g\in K} x b_g = x\sum_{g\in K} b_g = xb$. Therefore $x \in Z$. Thus $(B_K)^K \subset Z$. Conversely, for any $z \in Z$ and $g \in K$, we have that $zx = xz = g(z)x$ for any $x \in J_g$, so $(g(z)-z)x = 0$ for any $x \in J_g$. Hence $(g(z)-z)J_g = \{0\}$. Noting that $BJ_g = J_gB = B$, we have that $(g(z)-z)B = \{0\}$, so $g(z) = z$ for any $z \in Z$ and $g \in K$. Thus $Z \subset (B_K)^K$. Therefore $(B_K)^K = Z$. □

Theorem 3.4. *The algebra B_K is a central weakly Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$.*

Proof. By Lemma 3.3, it suffices to show that (1) B_K is a finitely generated projective module over Z, and (2) $(B_K)_l\overline{K} \cong \mathrm{Hom}_Z(B_K, B_K)$. Part (1) is a consequence of Theorem 3.1. For part (2), since B_K is an Azumaya algebra over Z by Theorem 3.1 again, $B_K \otimes_Z B_K^o \cong \mathrm{Hom}_Z(B_K, B_K)$ [3, Theorem 3.4, page 52] by extending the map $(a \otimes b)(x) = axb$ linearly for $a \otimes b \in B_K \otimes_Z B_K^o$ and each $x \in B_K$ where B_K^o is the opposite algebra of B_K. By denoting the left multiplication map with $a \in B_K$ by a_l and the right multiplication map with $b \in B_K$ by b_r, $(a \otimes b)(x) = (a_l b_r)(x) = axb$. Since $B_K = \oplus \sum_{g \in K} J_g$, $B_K \otimes_Z B_K^o = \sum_{g \in K} (B_K)_l (J_g)_r$. Observing that $(J_g)_r = (J_g)_l \overline{g}^{-1}$ where $\overline{g} = g|_{B_K} \in K|_{B_K} \cong \overline{K}$, we have that $B_K \otimes_Z B_K^o = \sum_{g \in K} (B_K)_l (J_g)_r = \sum_{g \in K} (B_K)_l (J_g)_l \overline{g}^{-1} = \sum_{g \in K} (B_K J_g)_l \overline{g}^{-1}$. Moreover, since $BJ_g = B$ for each $g \in K$ and $B = \oplus \sum_{h \in G} J_h = B_K \oplus (\oplus \sum_{h \notin K} J_h)$, $B_K \oplus (\oplus \sum_{h \notin K} J_h) = B = BJ_g = B_K J_g \oplus (\oplus \sum_{h \notin K} J_h J_g)$ such that $B_K J_g \subset B_K$ and $\oplus \sum_{h \notin K} J_h J_g \subset \oplus \sum_{h \notin K} J_h$. Hence $B_K J_g = B_K$ for each $g \in K$. Therefore $B_K \otimes_Z B_K^o = \sum_{g \in K} (B_K J_g)_l \overline{g}^{-1} = \sum_{g \in K} (B_K)_l \overline{g}^{-1} = (B_K)_l \overline{K}$. Thus $(B_K)_l \overline{K} \cong \mathrm{Hom}_Z(B_K, B_K)$. This completes the proof of part (2). Thus B_K is a central weakly Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$. □

Recall that an algebra A is called an Azumaya weakly Galois extension of A^K with Galois group K if A is a weakly Galois extension of A^K which is a C^K-Azumaya algebra where C is the center of A. Next, we show that $B_K B^K$ is an Azumaya weakly Galois extension with Galois group $K|_{B_K B^K} \cong \overline{K}$. We begin with the following two lemmas about B_K.

Lemma 3.5. *The fixed ring of B under K, $B^K = V_B(B_K)$.*

Proof. For any $b \in B^K$ and $x \in J_g$ for any $g \in K$, we have that $xb = g(b)x = bx$, so $b \in V_B(J_g)$ for any $g \in K$. Thus $b \in V_B(B_K)$. Conversely, for any $b \in V_B(B_K)$ and $g \in K$, we have that $bx = xb = g(b)x$ for any $x \in J_g$, so $(g(b) - b)x = 0$ for any $x \in J_g$. Hence $(g(b) - b)J_g = \{0\}$. But $BJ_g = J_g B = B$ for any $g \in K$, so $(g(b) - b)B = \{0\}$. Thus $g(b) = b$ for any $g \in K$; and so $b \in B^K$. Therefore $B^K = V_B(B_K)$. □

Lemma 3.6. *The algebra B^K is an Azumaya algebra over Z where Z is the center of B_K.*

Proof. Since B is a Galois algebra over R with Galois group G, B is an Azumaya algebra over its center C. By the proof of Theorem 3.1, B_K is a C-separable subalgebra of B, so $V_B(B_K)$ is a C-separable subalgebra of B and $V_B(V_B(B_K)) = B_K$ by the commutator theorem for Azumaya algebras [3, Theorem 4.3, page 57]. This implies that B_K and $V_B(B_K)$ have the same center Z. Thus $V_B(B_K)$ is an Azumaya algebra over Z. But, by Lemma 3.5, $B^K = V_B(B_K)$, so B^K is an Azumaya algebra over Z. □

Theorem 3.7. *Let $A = B_K B^K$. Then A is an Azumaya weakly Galois extension with Galois group $K|_A \cong \overline{K}$.*

Proof. Since B_K is a central weakly Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$ by Theorem 3.4, B_K is a finitely generated projective module over Z and $(B_K)_l \overline{K} \cong \mathrm{Hom}_Z(B_K, B_K)$. By Lemma 3.6, B^K is an Azumaya algebra over Z, so $A (\cong B_K \otimes_Z B^K)$ is a finitely generated projective module over $B^K (= A^{\overline{K}})$. Moreover, since $B^K = V_B(B_K)$ by Lemma 3.5 and $(B_K)_l \overline{K} \cong \mathrm{Hom}_Z(B_K, B_K)$,

$$\begin{aligned} A_l \overline{K} &= (B_K B^K)_l \overline{K} = (B_K)_l \overline{K} (B^K)_r \cong B_K \overline{K} \otimes_Z B^K \cong \mathrm{Hom}_Z (B_K, B_K) \otimes_Z B^K \\ &\cong \mathrm{Hom}_{B^K} (B_K \otimes_Z B^K, B_K \otimes_Z B^K) \cong \mathrm{Hom}_{B^K} (B_K B^K, B_K B^K) \\ &= \mathrm{Hom}_{A^{\overline{K}}}(A, A). \end{aligned} \tag{3.2}$$

Thus A is a weakly Galois extension of A^K with Galois group $K|_A \cong \overline{K}$. Next, we claim that A has center Z and $A^{\overline{K}}$ is an Azumaya algebra over $Z^{\overline{K}}$. In fact, B_K and B^K are Azumaya algebras over Z by Theorem 3.1 and Lemma 3.6, respectively, so $A(=B_KB^K)$ has center Z and $A^{\overline{K}} = (B_KB^K)^{\overline{K}} = B^K$. Noting that B^K is an Azumaya algebra over Z, we conclude that $A^{\overline{K}}$ is an Azumaya algebra over $Z^{\overline{K}}$. Thus A is an Azumaya weakly Galois extension with Galois group $K|_A \cong \overline{K}$. □

4. An Azumaya Galois extension. In this section, we give several characterizations of an Azumaya Galois extension B by using B_K. This generalizes the results in [2, 6]. The Z-module $\{b \in B_K \mid bx = g(x)b \text{ for all } x \in B_K\}$ is denoted by $J_{\overline{g}}^{(B_K)}$ for $\overline{g} \in \overline{K}$ where $\overline{K}(=K/L)$ is defined in Lemma 3.2.

Lemma 4.1. *The algebra B_K is a central Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$ if and only if $J_{\overline{g}}^{(B_K)} = \oplus\sum_{l\in L} J_{gl}$ for each $\overline{g} \in \overline{K}$.*

Proof. Let B_K be a central Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$. Then $B_K = \oplus\sum_{\overline{g}\in\overline{K}} J_{\overline{g}}^{(B_K)}$ [6, Theorem 1]. Next it is easy to check that $\oplus\sum_{l\in L} J_{gl} \subset J_{\overline{g}}^{(B_K)}$. But $B_K = \oplus\sum_{g\in K} J_g$, so $\oplus\sum_{g\in K} J_g = \oplus\sum_{\overline{g}\in\overline{K}} J_{\overline{g}}^{(B_K)}$ where $\oplus\sum_{l\in L} J_{gl} \subset J_{\overline{g}}^{(B_K)}$. Thus $J_{\overline{g}}^{(B_K)} = \oplus\sum_{l\in L} J_{gl}$ for each $\overline{g} \in \overline{K}$. Conversely, since $J_{\overline{g}}^{(B_K)} = \oplus\sum_{l\in L} J_{gl}$ for each $\overline{g} \in \overline{K}$, $B_K = \oplus\sum_{g\in K} J_g = \oplus\sum_{\overline{g}\in\overline{K}} J_{\overline{g}}^{(B_K)}$. Moreover, by Lemma 3.3, $(B_K)^K = Z$, so $\overline{K}$ is a Z-automorphism group of B_K. Hence $J_{\overline{g}}^{(B_K)} J_{\overline{g}^{-1}}^{(B_K)} = Z$ for each $\overline{g} \in \overline{K}$. Thus B_K is a central Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$ because B_K is an Azumaya Z-algebra by Theorem 3.1 (see [4, Theorem 1]). □

Next, we characterize an Azumaya Galois extension B with Galois group K.

Theorem 4.2. *The following statements are equivalent:*

(1) *B is an Azumaya Galois extension with Galois group K;*

(2) *$Z = C$;*

(3) *$B = B_KB^K$;*

(4) *B_K is a central Galois algebra over C with Galois group $K|_{B_K} \cong K$.*

Proof. (1)⇒(2). Since B is an Azumaya Galois extension with Galois group K, B^K is a C^K-Azumaya algebra. But, by Lemma 3.6, B^K is an Azumaya algebra over Z, so $Z = C^K$. Hence $C \subset Z = C^K \subset C$. Thus $Z = C$.

(2)⇒(3). Suppose that $Z = C$. Then, by Theorem 3.1, B_K is an Azumaya algebra over C. Hence by the commutator theorem for Azumaya algebras, $B = B_KV_B(B_K)$ [3, Theorem 4.3, page 57]. But, by Lemma 3.6, $B^K = V_B(B_K)$, so $B = B_KB^K$.

(3)⇒(4). By hypothesis, $B = B_KB^K$, so $L = \{1\}$ where L is given in Lemma 3.2. By the proofs of Theorem 3.1 and Lemma 3.6, B_K and B^K are C-separable subalgebras of the Azumaya C-algebra B such that $B = B_KB^K$, so B_K and B^K are Azumaya algebras over C [3, Theorem 4.4, page 58]. Thus C is the center of B_K. Next, we claim that $J_g = J_g^{(B_K)}$ for each $g \in K$. In fact, it is clear that $J_g \subset J_g^{(B_K)}$. Conversely, for each $a \in J_g^{(B_K)}$ and $x \in B$ such that $x = yz$ for some $y \in B_K$ and $z \in B^K$, noting that $B^K = V_B(B_K)$, we have that $ax = ayz = g(y)az = g(y)za = g(yz)a = g(x)a$. Thus $J_g^{(B_K)} \subset J_g$. This proves that $J_g = J_g^{(B_K)}(= J_{\overline{g}}^{(B_K)}$ since $L = \{1\})$ for each $g \in K$. Hence, B_K is a central Galois algebra over C with Galois group $K|_{B_K} \cong K$ by Lemma 4.1.

(4)⇒(1). Since B is a Galois algebra with Galois group G, B is a Galois extension with Galois group K. By hypothesis, B_K is a central Galois algebra over C with Galois group $K|_{B_K} \cong K$, so the center of B_K is C, that is, $Z = C$. Hence B^K is an Azumaya algebra over $C(= C^K)$ by Lemma 3.6. Thus B is an Azumaya Galois extension with Galois group K. □

Theorem 4.2 generalizes the following result of Kanzaki [6, Proposition 3].

Corollary 4.3. *If $J_g = \{0\}$ for each $g \notin K$, then B is a central Galois algebra with Galois group K and C is a Galois algebra with Galois group G/K.*

Proof. This is the case in Theorem 4.2 that $B = B_K B^K = B_K$ where $B^K = C$. □

We conclude the present paper with two examples, one to illustrate the result in Theorem 4.2, and another to show that $Z \neq C$.

Example 4.4. Let $A = \mathbb{R}[i,j,k]$, the real quaternion algebra over the field of real numbers $\mathbb{R}$, $B = (A \otimes_{\mathbb{R}} A) \oplus A \oplus A \oplus A \oplus A$, and G the group generated by the elements in $\{g_1, k_i, k_j, k_k, h_i, h_j, h_k\}$ where g_1 is the identity of G and for all $(a \otimes b, a_1, a_2, a_3, a_4) \in B$,

$$\begin{aligned}
k_i(a\otimes b, a_1, a_2, a_3, a_4) &= (iai^{-1}\otimes b, ia_1i^{-1}, ia_2i^{-1}, ia_3i^{-1}, ia_4i^{-1}),\\
k_j(a\otimes b, a_1, a_2, a_3, a_4) &= (jaj^{-1}\otimes b, ja_1j^{-1}, ja_2j^{-1}, ja_3j^{-1}, ja_4j^{-1}),\\
k_k(a\otimes b, a_1, a_2, a_3, a_4) &= (kak^{-1}\otimes b, ka_1k^{-1}, ka_2k^{-1}, ka_3k^{-1}, ka_4k^{-1}),\\
h_i(a\otimes b, a_1, a_2, a_3, a_4) &= (a\otimes ibi^{-1}, a_2, a_1, a_4, a_3),\\
h_j(a\otimes b, a_1, a_2, a_3, a_4) &= (a\otimes jbj^{-1}, a_3, a_4, a_1, a_2),\\
h_k(a\otimes b, a_1, a_2, a_3, a_4) &= (a\otimes kbk^{-1}, a_4, a_3, a_2, a_1).
\end{aligned} \tag{4.1}$$

Then,

(1) we can check that B is a Galois algebra over B^G with Galois group G where $B^G = \{(r_1 \otimes r_2, r, r, r, r) \mid r_1, r_2,\ r \in \mathbb{R}\} \subset C$, and $C = (\mathbb{R}\otimes\mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, the center of B;

(2) $K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} = \{g_1, k_i, k_j, k_k\}$;

(3) $J_1 = C$, $J_{k_i} = (\mathbb{R}i \otimes 1) \oplus \mathbb{R}i \oplus \mathbb{R}i \oplus \mathbb{R}i \oplus \mathbb{R}i$, $J_{k_j} = (\mathbb{R}j \otimes 1) \oplus \mathbb{R}j \oplus \mathbb{R}j \oplus \mathbb{R}i \oplus \mathbb{R}j$, $J_{k_k} = (\mathbb{R}k \otimes 1) \oplus \mathbb{R}k \oplus \mathbb{R}k \oplus \mathbb{R}i \oplus \mathbb{R}k$, so $B_K = (A \otimes_{\mathbb{R}} \mathbb{R}) \oplus A \oplus A \oplus A \oplus A$. Hence B_K has center C, that is $Z = C$, and B_K is a central Galois algebra over C with Galois group $K|_{B_K} \cong K$;

(4) $B^K = (\mathbb{R} \otimes A) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and $B = B_K B^K$, that is, B is an Azumaya Galois extension with Galois group K.

Example 4.5. Let $A = \mathbb{R}[i,j,k]$, the real quaternion algebra over the field of real numbers $\mathbb{R}$, $B = A \oplus A \oplus A$, $G = \{1, g_i, g_j, g_k\}$, and for all $(a_1, a_2, a_3) \in B$,

$$\begin{aligned}
g_i(a_1, a_2, a_3) &= (ia_1i^{-1}, ia_2i^{-1}, ia_3i^{-1}),\\
g_j(a_1, a_2, a_3) &= (ja_1j^{-1}, ja_3j^{-1}, ja_2j^{-1}),\\
g_k(a_1, a_2, a_3) &= (ka_1k^{-1}, ka_3k^{-1}, ka_2k^{-1}).
\end{aligned} \tag{4.2}$$

Then,

(1) B is a Galois algebra over B^G where $B^G = \{(r_1, r, r) \mid r_1,\ r \in \mathbb{R}\} \subset C$, and $C = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, the center of B. The G-Galois system is $\{a_i; b_i \mid i = 1, 2, \ldots, 8\}$ where

$$\begin{array}{llll} a_1 = (1,0,0), & a_2 = (i,0,0), & a_3 = (j,0,0), & a_4 = (k,0,0), \\ a_5 = (0,1,0), & a_6 = (0,j,0), & a_7 = (0,0,1), & a_8 = (0,0,k); \\ b_1 = \frac{1}{4}a_1, & b_2 = -\frac{1}{4}a_2, & b_3 = -\frac{1}{4}a_3, & b_4 = -\frac{1}{4}a_4, \\ b_5 = \frac{1}{2}a_5, & b_6 = -\frac{1}{2}a_6, & b_7 = \frac{1}{2}a_7, & b_8 = -\frac{1}{2}a_8, \end{array} \tag{4.3}$$

(2) $K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} = \{1, g_i\}$ where $J_{g_i} = \mathbb{R}i \oplus \mathbb{R}i \oplus \mathbb{R}i$, so $B_K = \mathbb{R}[i] \oplus \mathbb{R}[i] \oplus \mathbb{R}[i]$ which is a commutative ring not equal to C, that is, $Z \neq C$.

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George Szeto: Department of Mathematics, Bradley University, Peoria, IL 61625, USA
E-mail address: szeto@hilltop.bradley.edu

Lianyong Xue: Department of Mathematics, Bradley University, Peoria, IL 61625, USA
E-mail address: lxue@hilltop.bradley.edu