IJMMS 31:1 (2002) 23–29
PII. S0161171202110398
http://ijmms.hindawi.com

ASYMPTOTIC HÖLDER ABSOLUTE VALUES

E. MUÑOZ GARCIA

Received 12 October 2001

We prove that asymptotic Hölder absolute values are Hölder equivalent to classical absolute values. As a corollary we obtain a generalization of Ostrowski's theorem and a classical theorem by E. Artin. The theorem presented implies a new, more flexible, definition of classical absolute value.

2000 Mathematics Subject Classification: 12J20, 12J10, 16W80, 13J99.

1. Introduction. Asymptotic Hölder absolute values generalize the notions of classical absolute value and of Hölder absolute value. A Hölder absolute value (HAV) satisfies an approximate triangle inequality and multiplicative property. More precisely, let $C_1 \geq 1$ and $C_2 \geq 1$. A (C_1, C_2)-Hölder absolute value on a ring R is a mapping $\|\cdot\| : R \to \mathbb{R}_+$ satisfying:

(HAV1) for $x \in R$, $\|x\| = 0 \Leftrightarrow x = 0$;

(HAV2) for $x, y \in R$, $\|x+y\| \leq C_2(\|x\| + \|y\|)$;

(HAV3) for $x, y \in R$, $C_1^{-1}\|x\|\|y\| \leq \|xy\| \leq C_1\|x\|\|y\|$.

It is known that HAV on a ring are Hölder equivalent to a classical ones. More precisely, we have the following theorem (see [2]).

Theorem 1.1 (Hölder rigidity). *Let $\|\cdot\| : R \to \mathbb{R}_+$ be a (C_1, C_2)-Hölder absolute value on a commutative ring R with unit element. There exists an absolute value on R, $|\cdot| : R \to \mathbb{R}_+$, which is (C_1^α, α)-Hölder equivalent to $\|\cdot\|$ with $\alpha = \log_2(2C_2)$, that is, for $x \in R$,*

$$C_1^{-\alpha}|x|^\alpha \leq \|x\| \leq C_1^\alpha |x|^\alpha. \tag{1.1}$$

Moreover, $|\cdot|$ can be defined by

$$|x| = \lim_{n\to+\infty} \left\|x^n\right\|^{1/na}. \tag{1.2}$$

For a ring R with unity, a real constant $C_2 \geq 1$, and a function $C_1(\cdot,\cdot)$ defined on $]1,+\infty[\times\mathbb{N}$ taking values in $[1,+\infty[$, we define a (C_1, C_2)-asymptotic Hölder absolute value (AHAV) on R,

$$|\cdot| : R \longrightarrow \mathbb{R}_+, \tag{1.3}$$

satisfying the three following axioms:

(AHAV1) $|x| = 0$ if and only if $x = 0$;

(AHAV2) for $x, y \in R$, $|x+y| \leq C_2(|x| + |y|)$;

(AHAV3) for $y > 1$ and $n \geq 2$ there is a constant $C_1(y,n) > 1$ such that for $x_1,\ldots,x_n \in R$,

$$C_1(y,n)^{-1}|x_1|^{y^{-1}}\cdots|x_n|^{y^{-1}} \leq |x_1\cdots x_n| \leq C_1(y,n)|x_1|^{y}\cdots|x_n|^{y}, \tag{1.4}$$

and $L = \overline{\lim}_{n\to\infty}(1/n)\log C_1(y,n) < +\infty$.

We prove the following theorem.

Theorem 1.2. *Let R be a commutative ring with unity. Let $C_2 \geq 1$ be a real constant, $\alpha = 1/\log_2(2C_2)$, and $\|\cdot\|$ a (C_1,C_2)-AHAV on R. We have the following dichotomy:*

(i) *if*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log C_1(y,n) = 0, \tag{1.5}$$

then $\|\cdot\|^{\alpha}$ is a classical absolute value on R;

(ii) *if*

$$0 < L = \overline{\lim_{n\to\infty}}\frac{1}{n}\log C_1(y,n) < +\infty, \tag{1.6}$$

then $\|\cdot\|^{\alpha}$ is a Hölder absolute value on R, more precisely, it is $(e^{L\alpha},\alpha)$-Hölder equivalent to an absolute value on R.

As a result of Theorem 1.2(i), we can define classical absolute values as AHAV with $C_2 = 1$ having a sequence of constants $(C_1(y,n))_n$ growing sub-exponentially, that is,

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log C_1(y,n) = 0. \tag{1.7}$$

This is far more flexible than the classical definition.

Note that, in general, Hölder equivalence is a metric property which is stronger than the usual topological equivalence, for example, $\{0\}\cup\{1/n;\ n \geq 1\}$ and $\{0\}\cup\{1/2^n;\ n \geq 1\}$ are homeomorphic, but not Hölder equivalent.

Corollary 1.3. *Consider $|\cdot| : R \to \mathbb{R}^+$ satisfying*

(AV1) *$|x| = 0$ if and only if $x = 0$,*

(AV2) *for $x,y \in R$, $|x+y| \leq |x|+|y|$ then,*

(AV3) *for $x,y \in R$, $|xy| = |x||y|$ is equivalent to:*

(AV3′) *for $y > 1$ and $n \geq 2$ there is a constant $C_1(y,n) > 1$ such that for $x_1,\ldots,x_n \in R$,*

$$C_1(y,n)^{-1}|x_1|^{y^{-1}}\cdots|x_n|^{y^{-1}} \leq |x_1\cdots x_n| \leq C_1(y,n)|x_1|^{y}\cdots|x_n|^{y} \tag{1.8}$$

with $\overline{\lim}_{n\to\infty}(1/n)\log C_1(y,n) = 0$.

Our theorem gives a generalization for discrete rings of Artin's theorem [1].

Corollary 1.4. *If $\|\cdot\|$ is a $(1,C_2)$-AHAV over a discrete field F, there exists an absolute value $|\cdot|$ and an exponent α, such that for all x in F, $\|x\|^{\alpha} = |x|$.*

Also, our theorem implies a generalization of Ostrowski's theorem [3] for classical absolute values $(C_1 = C_2 = y = 1)$ over $\mathbb{Z}$.

Corollary 1.5. *If $\|\cdot\|$ is a (C_1, C_2)-AHAV over $\mathbb{Z}$ normalized, so that $\|1\| = 1$, then $\|\cdot\|$ is $(e^{L\alpha}, \alpha)$-Hölder equivalent to a p-adic absolute value $|\cdot|_p$ or to $|\cdot|_\infty$ or to the trivial absolute value, with $\alpha = 1/\log_2(2C_2)$.*

Remarks. (1) The constant $C_1(\gamma, n)$ in the definition of AHAV can be chosen to satisfy the inequality

$$C_1(\gamma, n) \le C_1\left(\gamma^{1/([\log_2 n]+1)}, 2\right)^n, \tag{1.9}$$

where $[a]$ denotes the integer part of a.

(2) Let $C_2 \ge 1$ and let $|\cdot| : R \to \mathbb{R}_+$ be a (C_1, C_2)-AHAV on R. If $\overline{\lim}_{\gamma \to 1} C_1(\gamma, 2) = C_1 < +\infty$, then $|\cdot|$ is a (C_1, C_2)-Hölder absolute value.

(3) If R is a ring on which a (C_1, C_2)-AHAV $|\cdot|$ is defined, then R is a discrete ring for the topology defined by $|\cdot|$.

1.1. Weak subadditive lemma. We prove a generalization of a classical lemma on subadditive sequences (which might be of independent interest).

Definition 1.6. The real sequence $(b_m)_{m\in\mathbb{N}}$ is weakly subadditive if

(i) for $\gamma > 1$ and $k \ge 1$, there is a constant $K(\gamma, k) > 0$ such that for $m_1, \dots, m_k \in \mathbb{N}$,

$$b_{m_1+\cdots+m_k} \le \gamma \sum_{i=1}^{k} b_{m_i} + K(\gamma, k); \tag{1.10}$$

(ii) for $\gamma > 1$, we have $K^*(\gamma) = \overline{\lim}_{k\to\infty} (1/k) K(\gamma, k) < +\infty$.

Lemma 1.7. *If $(b_m)_{m\in\mathbb{N}}$ is weakly subadditive, then*

$$\underline{\lim}_{m\to\infty} \frac{b_m}{m} = \overline{\lim}_{m\to\infty} \frac{b_m}{m}. \tag{1.11}$$

Proof. Fix $n \ge 1$. For any $m \in \mathbb{Z}$, we consider the Euclidean division

$$m = nq + r, \quad 0 \le r < n. \tag{1.12}$$

Now,

$$b_m = b_{nq+r} \le \gamma(q b_n + b_r) + K(\gamma, q+1). \tag{1.13}$$

Dividing by m,

$$\frac{b_m}{m} = \frac{b_{nq+r}}{nq+r} \le \gamma\left(\frac{q}{nq+r} b_n + \frac{b_r}{nq+r}\right) + \left(\frac{q+1}{nq+r}\right)\frac{K(\gamma, q+1)}{q+1}. \tag{1.14}$$

Taking the upper limit when $m \to \infty$,

$$\overline{\lim}_{m\to\infty} \frac{b_m}{m} \le \gamma\left(\frac{b_n}{n} + 0\right) + \frac{1}{n} K^*(\gamma). \tag{1.15}$$

That is, for all $q \ge 1$,

$$\overline{\lim}_{m\to\infty} \frac{b_m}{m} \le \gamma \frac{b_n}{n} + \frac{1}{n} K^*(\gamma). \tag{1.16}$$

Now, taking the lower limit on the right side when $n \to \infty$,

$$\overline{\lim_{m\to\infty}} \frac{b_m}{m} \le \gamma \underline{\lim}_{n\to\infty} \frac{b_n}{n}. \tag{1.17}$$

This holds for all $\gamma > 1$, thus making $\gamma \to 1$,

$$\overline{\lim_{m\to\infty}} \frac{b_m}{m} \le \underline{\lim}_{m\to\infty} \frac{b_m}{m}, \qquad \overline{\lim_{m\to\infty}} \frac{b_m}{m} = \underline{\lim}_{m\to\infty} \frac{b_m}{m}. \tag{1.18}$$

□

1.2. Proof of Theorem 1.1

Lemma 1.8. *Define $|||\cdot||| : R \to \mathbb{R}_+$ by $|||x||| = \|x\|^\alpha$. Then, $|||\cdot|||$ is a $(C_1^\alpha, 2)$-AHAV on R.*

Proof. (AHAV1) $|||x||| = 0$ if and only if $\|x\| = 0$ if and only if $x = 0$.

(AHAV2) $|||x+y||| = \|x+y\|^\alpha \le (2C_2)^\alpha(\max(\|x\|,\|y\|))^\alpha \le 2(\|x\|^\alpha + \|y\|^\alpha) = 2(|||x||| + |||y|||)$.

(AHAV3) For all $\gamma > 1$ and for all $n \ge 2$ there is a constant $C_1(\gamma,n)^\alpha > 1$ such that for all $x_1,\dots,x_n$ in R,

$$\begin{aligned}(C_1(\gamma,n))^{-\alpha} |||x_1|||^{\gamma^{-1}} \cdots |||x_n|||^{\gamma^{-1}} \\ \le |||x_1\cdots x_n||| \le C_1(\gamma,n)^\alpha |||x_1|||^\gamma \cdots |||x_n|||^\gamma.\end{aligned} \tag{1.19}$$

□

Lemma 1.9. *Let $x \in R$ and define the real sequence $(a_n)_{n\in\mathbb{N}}$ by $a_n = |||x^n|||$. The sequence $(a_n^{1/n})$ is converging and*

$$e^{-L}|||x||| \le \lim_{n\to\infty} a_n^{1/n} \le e^L |||x|||, \tag{1.20}$$

where $L = \overline{\lim}_{n\to\infty}(1/n)\log C_1(\gamma,n) < +\infty$.

Proof. Let $b_m = \log a_m$. The sequence $\{b_m\}$ is weakly subadditive, since for all $\gamma > 1$ and for all $k \ge 1$ there is a constant $K(\gamma,k) = (C_1(\gamma,k))^\alpha$, such that

$$b_{m_1+\cdots+m_k} \le \gamma \sum_{i=1}^k b_{m_i} + \log K(\gamma,k), \tag{1.21}$$

and for all $\gamma > 1$,

$$\lim_{k\to+\infty} \frac{1}{k}\log K(\gamma,k) < +\infty. \tag{1.22}$$

Therefore, by Lemma 1.7,

$$\underline{\lim}_{m\to\infty} \frac{b_m}{m} = \overline{\lim_{m\to\infty}} \frac{b_m}{m}. \tag{1.23}$$

Thus, to prove the convergence of $\{a_n^{1/n}\}$, we only have to prove that $\{a_n^{1/n}\}$ is bounded.

Let $\gamma > 1$, for $n \in \mathbb{N}$ there is $C_1(\gamma,n)^\alpha$ satisfying

$$C_1(\gamma,n)^{-\alpha}|||x|||^{n/\gamma} \le |||x^n||| \le C_1(\gamma,n)^\alpha |||x|||^{n\gamma}. \tag{1.24}$$

Taking nth roots,

$$C_1(\gamma,n)^{-\alpha/n}|\|x\||^{1/\gamma} \leq a_n^{1/n} \leq C_1(\gamma,n)^{\alpha/n}|\|x\||^{\gamma}. \tag{1.25}$$

Since $L = \overline{\lim}_{n\to\infty}(1/n)\log C_1(\gamma,n) < +\infty$, we obtain

$$e^{-\alpha L}|\|x\||^{1/\gamma} \leq \lim_{n\to\infty} a_n^{1/n} \leq e^{\alpha L}|\|x\||^{\gamma}. \tag{1.26}$$

This inequality holds for any $\gamma > 1$. Taking the limit when $\gamma \to 1$,

$$e^{-\alpha L}|\|x\|| \leq a_n^{1/n} \leq e^{\alpha L}|\|x\||. \tag{1.27}$$

□

Now we define that $|\cdot| : R \to \mathbb{R}_+$ by $|0| = 0$ and that $|x| = \lim_{n\to\infty}|\|x^n\||^{1/n}$ for $x \neq 0$.

Lemma 1.10. *The function $|\cdot| : R \to \mathbb{R}_+$ defined as above is an absolute value on R. Moreover, if $\overline{\lim}_{n\to+\infty}(1/n)\log C_1(\gamma,n) = 0$, then $|x| = \|x\|^{\alpha}$ for all $x \in R$.*

Proof. From Lemma 1.9, if $\overline{\lim}_{n\to\infty}(1/n)\log C_1(\gamma,n) = 0$, we obtain

$$|\|x\|| \leq |x| \leq |\|x\||. \tag{1.28}$$

That is, $|x| = \|x\|^{\alpha}$. □

It is clear that, $|x| = 0$ if and only if $x = 0$. Next we check the multiplicative property. For $\gamma > 1$ and for $n \geq 2$ there exists $C_1(\gamma,2)^{\alpha} > 1$, such that for $n \in \mathbb{N}$ and x, y in R,

$$\begin{aligned} &C_1(\gamma,2)^{-\alpha}|\|x^n\||^{\gamma^{-1}}|\|y^n\||^{\gamma^{-1}} \\ &\quad \leq |\|x^n\||\,|\|y^n\|| \leq C_1(\gamma,n)^{\alpha}|\|x^n\||^{\gamma}|\|y^n\||^{\gamma}. \end{aligned} \tag{1.29}$$

Taking nth roots and passing to the limit when $n \to +\infty$, we obtain

$$|x|^{\gamma^{-1}}|y|^{\gamma^{-1}} \leq |xy| \leq |x|^{\gamma}|y|^{\gamma}. \tag{1.30}$$

Taking the limit when $\gamma \to 1$, we have the desired multiplicative property.

Finally, we have to check the triangle inequality. This is a corollary of the following general proposition that gives an equivalent, apparently weaker, definition of absolute value.

Proposition 1.11. *Let R be a commutative ring with unity. Let $|\cdot| : R \to \mathbb{R}_+$ be a function satisfying the following three properties:*

(A1) *$|x| = 0$ if and only if $x = 0$;*

(A2) *(approximate triangle inequality) there exists a real constant $B > 0$, such that for all x, y in R, $|x+y| \leq B(|x|+|y|)$;*

(A3) *for x, y in R, $|xy| = |x||y|$.*

Then, $|\cdot|$ is an absolute value on R, that is, $|\cdot|$ satisfies the triangle inequality.

Lemma 1.12. *For $x, y \in R$,*

$$|x+y| \leq B(|x|+|y|) \leq 2B\max(|x|,|y|). \tag{1.31}$$

Lemma 1.13. *Let $|\cdot|' : R \to \mathbb{R}_+$, such that for $x, y \in R$,*

$$|x+y|' \le M \max\left(|x|', |y|'\right), \tag{1.32}$$

for some positive constant M. Then for $x_1, x_2, \ldots, x_n \in R$,

$$\left|\sum_{i=1}^{n} x_i\right|' \le M^{[\log_2 n]+1} \max_{1\le i\le n}\left(|x_i|'\right), \tag{1.33}$$

where $[a]$ denotes the integer part of a.

Proof. Let $m = [\log_2 n]+1$ and complete the sequence $(x_i)_{1\le i\le n}$ into $(x_i)_{1\le i\le 2^m}$ adjoining 0 elements.

$$\begin{aligned}
\left|\sum_{i=1}^{2^m} x_i\right|' &\le M \max\left(\left|\sum_{i=1}^{2^{m-1}} x_i\right|', \left|\sum_{i=2^{m-1}+1}^{2^m} x_i\right|'\right) \\
&\le M^2 \max\left(\left|\sum_{i=1}^{2^{m-2}} x_i\right|', \left|\sum_{i=2^{m-2}+1}^{2^{m-1}} x_i\right|', \left|\sum_{i=2^{m-1}+1}^{3\cdot 2^{m-2}} x_i\right|', \left|\sum_{i=3\cdot 2^{m-2}+1}^{2^m} x_i\right|'\right) \\
&\le \cdots \le M^m \max_{1\le i\le 2^m} |x_i|'.
\end{aligned} \tag{1.34}$$

□

Lemma 1.14. *Let $\overline{\mathbb{Z}}$ be the image of $\mathbb{Z}$ in R. For $n \in \mathbb{N}$,*

$$|\bar{n}| \le 2n|1|. \tag{1.35}$$

Proof. We use Lemma 1.13 with $M = 2$ and $|\cdot|' = |\cdot|$. Take $m = [\log_2 n]+1$, $n \le 2^m \le 2n$, and $x_i = 1$ for $1 \le i \le n$. We have

$$|n| = \left|\sum_{i=1}^{n} x_i\right| \le 2^m|1| \le 2n|1|. \tag{1.36}$$

□

Lemma 1.15. *Let $\overline{\mathbb{Z}}$ be the image of $\mathbb{Z}$ in R. For $n \in \mathbb{N}$,*

$$|\bar{n}| \le n. \tag{1.37}$$

Proof. Using Lemma 1.14,

$$\left|\overline{n^k}\right| = \left|\bar{n}^k\right| \le 2n^k|1|, \tag{1.38}$$

and $|\bar{n}^k|^{1/k} \le 2^{1/k} n |1|^{1/k}$. Taking $k \to +\infty$, we have $|\bar{n}| \le n$. □

Proof of Proposition 1.11. Let $x, y \in R$ and $n \ge 1$. Let $m = [\log_2 n]+1$. Using Lemmas 1.12 and 1.14, we have

$$\begin{aligned}
\left|(x+y)^n\right| &= \left|\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\right| \\
&\le (B)^m \max_{0\le i\le n} \left|\binom{n}{i} x^i y^{n-i}\right|.
\end{aligned} \tag{1.39}$$

Now using Lemma 1.14,

$$\begin{aligned} \left|(x+y)^n\right| &\le (2B)^m \max_{0\le i\le n} \left|\binom{n}{i}\right| |x|^i |y|^{n-i} \\ &\le (2B)^m \max_{0\le i\le n} \binom{n}{i} |x|^i |y|^{n-i} \\ &\le (2B)^m \sum_{i=0}^{n} \binom{n}{i} |x|^i |y|^{n-i} \\ &\le (2B)^m (|x|+|y|)^n. \end{aligned} \tag{1.40}$$

Finally,

$$|x+y| = \left|(x+y)^n\right|^{1/n} \le (2B)^{(1/n)([\log_2 n]+1)}(|x|+|y|), \tag{1.41}$$

and passing to the limit $n \to +\infty$ we get the sharp triangle inequality $|x+y| \le |x|+|y|$. □

Proof of Theorem 1.2.

Case 1. Assume $\overline{\lim}_{n\to\infty}(1/n)\log C_1(y,n) = 0$. By Lemma 1.8, for all x, y in R we have

$$|x+y| = |\|x+y\|| \le 2(|\|x\||+|\|y\||) \le 4\max(|\|x\||,|\|y\||) = 4\max(|x|,|y|). \tag{1.42}$$

Therefore, by Proposition 1.11, the function $|\cdot|$ satisfies the triangle inequality.

Case 2. Assume $0 < L = \overline{\lim}_{n\to\infty}(1/n)\log C_1(y,n) < +\infty$. From Lemma 1.9, for any x in R,

$$e^{-\alpha L}|\|x\|| \le |x| \le e^{\alpha L}|\|x\||. \tag{1.43}$$

Therefore,

$$|x+y| \le e^{\alpha L}|\|x+y\|| \le 2e^{\alpha L}(|\|x\||+|\|y\||) \le 2e^{2\alpha L}(|x|+|y|). \tag{1.44}$$

Thus by Proposition 1.11, the function $|\cdot|$ satisfies the triangle inequality, it is an absolute value, and $\|\cdot\|^\alpha$ is $(e^{L\alpha},\alpha)$-equivalent to $|\cdot|$. □

References

[1] E. Artin, *Theory of Algebraic Numbers*, George Striker, Göttingen, 1959.

[2] E. Muñoz Garcia, *Hölder absolute values are equivalent to classical ones*, Proc. Amer. Math. Soc. **127** (1999), no. 7, 1967–1971.

[3] A. Ostrowski, *Über einige Lösungen der Funktionalgleichung $\phi(x)\cdot\phi(y) = \phi(x\cdot y)$*, Acta Math. **41** (1914), 271–284 (German).

E . Muñoz Garcia: Department of Mathematics, University of California, Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90095-1555, USA

E-mail address: munoz@math.ucla.edu