IJMMS 31:11 (2002) 651–657
PII. S0161171202109033
http://ijmms.hindawi.com

LOCAL COMPLETENESS OF $\ell_p(E)$, $1 \leq p < \infty$

C. BOSCH, T. GILSDORF, C. GÓMEZ, and R. VERA

Received 10 September 2001 and in revised form 26 February 2002

We study the heredity of local completeness and the strict Mackey convergence property from the locally convex space E to the space of absolutely p-summable sequences on E, $\ell_p(E)$ for $1 \leq p < \infty$.

2000 Mathematics Subject Classification: 46A03, 46A17.

1. Introduction. In 1956, Grothendieck [5], introduced the Banach-valued sequence space $\ell_p(E)$, the space of absolutely p-summable sequences on a Banach space E, where he discussed tensor products of ℓ_p and E, with $1 \leq p \leq \infty$. Later, in 1969 Pietsch [8] used Banach-valued sequence spaces $\ell_p(E)$, to study p-summing operators between Banach spaces, also see Diestel et al. [2]. In this paper, we discuss how local completeness and the strict Mackey convergence condition of E imply local completeness and the strict Mackey convergence condition in $\ell_p(E)$ in the case $1 \leq p < \infty$. The case $p = \infty$ was studied in [1].

2. Definitions and notation. Throughout this paper, (E,t) denotes a Hausdorff locally convex space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{\rho_j\}_{j\in J}$ denotes the family of continuous seminorms associated with the topology t on E.

Let $D \subset E$ be a bounded, closed, and absolutely convex set. Denote by $E_D = \cup_{k=1}^{\infty} kD$, and for each $x \in E_D$, $\rho_D(x) = \inf\{r > 0 : x \in rD\}$, the Minkowski seminorm associated with D. Now $E_D \subset E$ and the boundedness of D implies that $i : (E_D, \rho_D) \to (E,t)$ is continuous, and ρ_D is a norm so that, for every $j \in J$ there exists $r_j \in \mathbb{R}^+$ such that $\rho_j|_{E_D} \leq r_j \rho_D$.

Remark 2.1. For each $D \subset E$ bounded, closed, and absolutely convex, the family of seminorms $\{\rho_j\}_{j\in J}$ can be replaced by an equivalent family $\{\rho'_j\}_{j\in J}$ such that $\rho'_j \leq \rho_D$. To construct the family $\{\rho'_j\}_{j\in J}$ we know that there exists $r_j > 0$ such that $\rho_j(x) \leq r_j\rho_D(x)$ for every $x \in E_D$ so it suffices to take $\rho'_j = (1/r_j)\rho_j$ if $r_j > 1$, and we will have $\rho'_j \leq \rho_D$, for every $j \in J$. For simplicity we will always work with an equivalent family of seminorms, also denoted by $\{\rho_j\}_{j\in J}$ such that $\rho_j(x) \leq \rho_D(x)$ holds for every $j \in J$ and $x \in E_D$.

A bounded, closed, and absolutely convex set $D \subset E$, called a disk, is a Banach disk if (E_D, ρ_D) is a Banach space. If every bounded set $A \subset E$ is contained in a Banach disk we say that E is locally complete. Let (E,t) satisfies the strict Mackey convergence condition if for every bounded set $A \subset E$, there exists a disk D that contains A such that the topologies of (E,t) and (E_D, ρ_D) agree on A.

Every metrizable space satisfies the strict Mackey convergence condition, [7]. In addition, the strict Mackey convergence condition is preserved under the formation of closed subspaces, countable products, and countable direct sums, [6]. The strict Mackey convergence condition for webbed spaces is studied in [3, 4].

Remark 2.2. Using the family of seminorms $\{\rho_j\}_{j\in J}$ it is easy to see that the strict Mackey convergence condition is equivalent to: for each D there exists $j_0 \in J$ such that $\rho_{j_0|D} = \rho_D$.

Let p be a real number such that $1 \le p < \infty$. The space $\ell_p(E)$ of absolutely p-summable sequences on E is

$$\ell_p(E) = \left\{(x_n) \subset E : \sum_{n=1}^{\infty} \rho_j^p(x_n) < \infty,\ \forall j \in J\right\}. \tag{2.1}$$

The family of seminorms $\rho_{\rho_j}((x_n)) = (\sum_{n=1}^{\infty} \rho_j^p(x_n))^{1/p}$, $j \in J$, induce a topology of locally convex space in $\ell_p(E)$; we will denote by τ this topology.

The space $\ell_p(E_D)$ is defined by $\ell_p(E_D) = \{(x_n) \subset E_D : \sum_{n=1}^{\infty} \rho_D^p(x_n) < \infty\}$ and endowed with the topology generated by the norm

$$\rho_{\rho_D}((x_n)) = \left[\sum_{n=1}^{\infty} \rho_D^p(x_n)\right]^{1/p}. \tag{2.2}$$

We denote $A_D = \{(x_n) \in \ell_p(E) : (x_n)_{n\in\mathbb{N}} \subset D\}$.

Note that $\rho_{\rho_j}|_{\ell_p(E_D)} \le \rho_{\rho_D}$ for every $j \in J$ since $\rho_j|_{E_D} \le \rho_D$.

3. Bounded sets.

In this section, we characterize the bounded sets of $\ell_p(E)$ in terms of the bounded sets of E.

Lemma 3.1. *Let D be a disk in (E,t); then*

(i) $\ell_p(E_D) \subseteq \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ *for some* $k \in \mathbb{N}\}$;

(ii) *if there exists $j_0 \in J$, depending on D, such that $\rho_{j_0|D} = \rho_D$ (i.e., the strict Mackey convergence condition holds), then* $\{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ *for some* $k \in \mathbb{N}\} \subset \ell_p(E_D)$.

Proof. (i) Let $(x_n) \in \ell_p(E_D)$. Then $\sum_{n=1}^{\infty}[\rho_D(x_n)]^p < \infty$ so that given $\varepsilon = 1$ there exists $n_0 \in \mathbb{N}$, such that for each $n > n_0$, we have $\rho_D(x_n) \le (\sum_{n_0}^{\infty} \rho_D^p(x_n))^{1/p} \le 1$ which means that $x_n \in D$ for every $n > n_0$.

Now for $i = 1,2,\ldots,n_0$ there exists $k_i \ge 0$ such that $x_i \in k_i D$. We take $k = \max\{1,k_1,\ldots,k_{n_0}\}$. Then $\{x_n\} \subset kD$ and we have $\ell_p(E_D) \subset \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ for some $k \in \mathbb{N}\}$.

(ii) Let $(x_n) \in \{(y_n) \in \ell_p(E) : \{y_n\} \subset kD$ for some $k \in \mathbb{N}\}$. Thus $x_n \in E_D$ for every $n \in \mathbb{N}$ since $\{x_n\} \subset kD$.

Now observe that $\sum_{n=1}^{\infty} \rho_D^p(x_n) = \sum_{n=1}^{\infty} \rho_{j_0}^p(x_n) < \infty$ since $(x_n) \in \ell_p(E)$. Hence in this case we have the equality $\ell_p(E_D) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD$ for some $k \in \mathbb{N}\}$. $\square$

Remark 3.2. Note that $kA_D = A_{kD}$ for every $k \in \mathbb{N}$.

Corollary 3.3. *If E satisfies the strict Mackey convergence condition, then $\ell_p(E)_{A_D} = \ell_p(E_D)$.*

Proof. It follows from the equality in the proof of Lemma 3.1(ii) that $\ell_p(E)_{A_D} \subset \ell_p(E_D)$. Now let $(x_n) \in \ell_p(E_D)$. Then by Lemma 3.1(i), $(x_n) \subset kD$ for some $k \in \mathbb{N}$ so $\{x_n\} \subset A_{kD} = kA_D$ and $(x_n) \in \ell_p(E)_{A_D}$. □

Remark 3.4. If (E,t) satisfies the strict Mackey convergence condition, then

$$\ell_p(E)_{A_D} = \ell_p(E_D) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset A_{kD} \text{ for some } k \in \mathbb{N}\}. \tag{3.1}$$

Lemma 3.5. (i) $\rho_{A_D}((x_n)) = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$;

(ii) $\rho_{A_D}((x_n)) \le \rho_{\rho_D}((x_n))$ *for every* $(x_n) \in \ell_p(E_D)$.

Proof. (i) Let $s = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$. Then $\{x_n\} \subset sD$ so $\{x_n\} \subset A_{sD} = sA_D$ and then $\rho_{A_D}((x_n)) \le s$. Now take $r = \rho_{A_D}((x_n))$. Then $\{x_n\} \subset rA_D = A_{rD}$ and then $\{x_n\} \subset rD$ which means that $r \ge s$.

(ii) $\rho_{\rho_D}((x_n)) = (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \ge \rho_D(x_n)$ for every $n \in \mathbb{N}$. Using (i) we have $\rho_{\rho_D}((x_n)) \ge \rho_{A_D}((x_n))$. □

Note that A_D is not bounded in $\ell_p(E)$; we need to construct a "smaller" set, in the sense of boundedness.

Define for each $j \in J$ and $m \in \mathbb{N}$ the set $A_D(j,m) = \{(x_n)_n \in A_D : \rho_{\rho_j}((x_n)) \le m\}$ and for each $B \subset \ell_p(E)$, let $B^* = \{x \in E : x \in \{x_n\} \text{ and } (x_n) \in B\}$.

The next proposition gives a way to look at the bounded sets in $\ell_p(E)$.

Proposition 3.6. *If $\beta = \{D_\lambda\}_{\lambda \in \wedge}$ is a fundamental system of bounded disks in E, then $\{C = \cap_{j \in J}\{A_{D_\lambda}(j,m_j)\} : \lambda \in \Lambda,\ (m_j) \in \mathbb{N}^J\}$ is a fundamental system of τ-bounded sets in $\ell_p(E)$.*

Proof. Let $B \subset \ell_p(E)$ be a bounded set. Then B^* is bounded in E so $B^* \subset D_\lambda$ for some λ. For each $x \in B^*$, if $x \in (x_n)$ then given $j \in J$ there is some s_j such that $\rho_j(x) \le \rho_{\rho_j}((x_n)) \le s_j$ so that $\rho_{\rho_j}(B) \le s_j$. Now let $m_j \in \mathbb{N}$ be such that $s_j \le m_j$. We have $B \subset C = \cap_{j \in J} A_{D_\lambda}(j,m_j)$. □

Remark 3.7. (i) If D is bounded in E, then for each $j \in J$, by Remark 2.1 $\rho_{j|E_D} \le \rho_D$.

(ii) If C is bounded in $\ell_p(E)$, then for each $j \in J$, by Remark 2.1 $\rho_{\rho_j} \mid \ell_p(E)_C \le \rho_C$.

4. Main results

Proposition 4.1. *If for some D there exists $j_0 \in J$, such that $\rho_{j_0}|_D = \rho_D$ in E, then $\rho_{\rho_{j_0}|C} = \rho_C$ where $C = \cap_{j \in J} A_D(j,m_j)$ in $\ell_p(E)$. Equivalently, if E satisfies the strict Mackey convergence condition, then $\ell_p(E)$ also satisfies the strict Mackey convergence condition.*

Proof. Let $(x_n) \in C$. Then $s = \rho_{\rho_{j_0}}(x_n) = (\sum_{n=1}^{\infty} \rho_{j_0}^p(x_n))^{1/p} = (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \ge \rho_D(x_n) \ge \rho_{\rho_j}(x_n)$ for every $j \in J$ and $n \in \mathbb{N}$. So we have $(x_n) \in \cap_{j \in J} A_D(j,s) = s[\cap_{j \in J} A_D(j,1)] \subset sC$. Thus $\rho_C((x_n)) \le s = \rho_{\rho_{j_0}}(x_n)$ and since C is bounded in $\ell_p(E)$ we have $\rho_{\rho_j} \le \rho_C$ for each $j \in J$; now $\rho_{\rho_j}|_C \le \rho_C$ for every $j \in J$, so for j_0 we have $\rho_{\rho_{j_0}}|_C = \rho_C$.

Notice that if B is a bounded set in $\ell_p(E)$, then $\rho_{\rho_j}(B) \le m_j$ for all $j \in J$ with $m_j \in N$ and then $B \subset \cap_{j\in J} A_{B^*}(j, m_j)$.

This gives the property we need to characterize the bounded sets in $\ell_p(E)$. □

Theorem 4.2. *If E is locally complete and satisfies the strict Mackey convergence condition, then $(\ell_p(E)_C, \rho_C)$ where $C = \cap_{j\in J} A_D(j, m_j)$ in $\ell_p(E)$, is a Banach space so $\ell_p(E)$ is locally complete.*

Proof. Let D be a bounded closed disk such that $(E_{D,}\rho_D)$ is a Banach space and let $C = \cap_{j\in J} A_D(j, m_j)$. By Remark 2.1 there is a $j_0 \in J$ such that $\rho_{j_0}|_D = \rho_D$. We will show that $(\ell_p(E)_C, \rho_C)$ is a Banach space. By Corollary 3.3 we have $\ell_p(E)_{A_D} = \ell_p(E_D)$ and since $C \subset A_D$, $\ell_p(E)_C \subset \ell_p(E)_{A_D}$. Let $(x_n^k)_{k\in\mathbb{N}} \subset \ell_p(E)_C$ be a ρ_C-Cauchy sequence. Thus for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \ge N$ we have $\rho_C((x_n^k) - (x_m^k)) < \varepsilon$. Using Remark 3.7(ii) we have that $\rho_{\rho_j} \mid \ell(E)_C \le \rho_C$. Hence (x_n^k) is also a ρ_{ρ_j}-Cauchy sequence and then a $\rho_{\rho_{j_0}}$-Cauchy sequence. Thus $\rho_D(x_n^k - x_m^k) = \rho_{j_0}(x_n^k - x_m^k) \le \rho_{\rho_{j_0}}((x_n^k) - (x_m^k))$, then the sequence $(x_n^k)_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$ is also a ρ_D-Cauchy sequence in (E_D, ρ_D) which is a Banach space, so there exists z^k in E_D such that (x_n^k) converges to z^k with respect to the norm ρ_D. Using Remark 3.7(i) we have $\rho_{j|E_D} \le \rho_D$. Hence, we have the following claims.

Claim 1. We have that (x_n^k) converges to z^k with respect to the seminorm ρ_j for every $j \in J$.

Claim 2. Consider the sequence formed by the $(z^k)_{k\in\mathbb{N}} \in \ell_p(E_D)$. We compute

$$
\begin{aligned}
\sum_{k=1}^{\infty} (\rho_D(z^k))^p &= \lim_{m\to\infty} \sum_{k=1}^{m} (\rho_D(z^k))^p \\
&= \lim_{m\to\infty} \sum_{k=1}^{m} (\rho_{j_0}(z^k))^p \\
&= \lim_{m\to\infty} \sum_{k=1}^{m} \rho_{j_0}\Big(\lim_{n\to\infty} x_n^k\Big)^p \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{m} \rho_{j_0}(x_n^k)^p \\
&\le \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p \\
&\le \lim_{n\to\infty} \rho_{\rho_{j_0}}((x_n)) \\
&\le \varepsilon + \rho_{\rho_{j_0}}((x_N)) < \infty, \quad \text{for some } N \in \mathbb{N}.
\end{aligned}
\tag{4.1}
$$

In this last inequality we used $x_n = (x_n^k)_{k\in\mathbb{N}}$ and since it is a $\rho_{\rho_{j_0}}$-Cauchy sequence, given $\varepsilon > 0$, $\rho_{\rho_{j_0}}(x_n^k) - \rho_{\rho_{j_0}}(x_m^k) \le \rho_{\rho_{j_0}}((x_n^k) - (x_m^k)) < \varepsilon$ for every $n, m > N$, so $\rho_{\rho_{j_0}}((x_n)) \le \varepsilon + \rho_{\rho_{j_0}}((x_N))$. Notice that (x_n) is a ρ_{ρ_j}-Cauchy sequence for every $j \in J$.

Therefore for j_0 and consequently for ρ_{ρ_D}, then for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\rho_D(x_n^k - z^k) = \rho_D(x_n^k - \lim_{m\to\infty} x_m^k) = \lim_{m\to\infty} \rho_D(x_n^k - x_m^k) < \varepsilon$.

Claim 3. The sequence (x_n^k) converges to $(z^k)_{k\in\mathbb{N}}$ in $\ell_p(E_D)$. Since

$$\begin{aligned} \rho_{\rho_D}\left(x_n^k - (z^k)_k\right) &= \left[\sum_{k=1}^{\infty} \rho_D^p(x_n^k - z^k)\right]^{1/p} \\ &\le \left[\sum_{k=1}^{N} \rho_D^p(x_n^k - z^k) + \frac{\varepsilon^p}{2}\right]^{1/p} \\ &\le \left(\underbrace{\frac{\varepsilon^p}{2N} + \cdots + \frac{\varepsilon^p}{2N}}_{N \text{ factors}} + \frac{\varepsilon^p}{2}\right)^{1/p} \\ &= \varepsilon, \quad \text{for } n > N. \end{aligned} \tag{4.2}$$

In the first inequality we used Claim 2. This completes the proof of the convergence.

Claim 4. We have $(z^k)_{k\in\mathbb{N}} \in \ell_p(E)_C$. $(x_n^k)_{k\in\mathbb{N}}$ is a ρ_C-Cauchy sequence so it is bounded and there is an $s \in \mathbb{N}$ such that $(x_n^k) \subset sC$. Using Claim 3, (x_n^k) converges to (z^k) in $\ell_p(E)_C$ with respect to ρ_{ρ_D} and since $\rho_{\rho_j}|_{\ell_p(E_D)} \le \rho_{\rho_D}$ for every $j \in J$ the sequence (x_n^k) is τ-convergent to (z^k), it is convergent for each ρ_{ρ_j}. Now for each $\varepsilon > 0$ there exists N_j such that $\rho_{\rho_j}((z^k)) \le \rho_{\rho_j}((z^k) - (x_n^k)) + \rho_{\rho_j}((x_n^k)) < \varepsilon + sm_j$ for every $j \in J$ and $n \ge N_j$, this means that $(z^k) \in sC \subset \ell_p(E)_C$.

Claim 5. The sequence (x_n^k) converges to $(z^k)_{k\in\mathbb{N}}$ in $\ell_p(E)_C$. Let $\varepsilon > 0$, since (x_n^k) is a ρ_C-Cauchy sequence, there is $N \in \mathbb{N}$ such that $(x_n^k) - (x_m^k) \in \varepsilon C$ for every $n, m \ge N$. C is τ-closed so $(x_n^k) - (\tau - \lim(x_m^k)) \in \varepsilon C$; then $(x_n^k) - (z^k) \in \varepsilon C$ for every $n \ge N$ which means $\rho_C((x_n^k) - (z^k)) \le \varepsilon$ for every $n \ge N$.

Notice that this is true for every $1 \le p < \infty$. The case $p = \infty$ also follows from this and we get the characterization given in [1], although under a stronger hypothesis. Here we need E to satisfy the strict Mackey convergence condition. □

Lemma 4.3. *If $D \subset E$ is t-complete and the net $\{x_\lambda\}_\Lambda$ is a τ-Cauchy net bounded with respect to ρ_C, that is if there exists $s \in \mathbb{N}$ such that $\{x_\lambda\}_\Lambda \subset sC$ then there exists $z \in 2sC$ such that x_λ converges to z with respect to the τ topology in $\ell_p(E)$.*

Proof. Let $\{x_\lambda\}_\Lambda$ be a τ-Cauchy net, $x_\lambda = (x_\lambda^1, x_\lambda^2, \ldots)$, then for every $\varepsilon > 0$ there exists $\lambda_j \in \Lambda$ such that for every $j \in J$, $\rho_j(x_\lambda^k - x_{\lambda'}^k) \le \rho_{\rho_j}(x_\lambda - x_{\lambda'}) < \varepsilon$ for every $\lambda, \lambda' \ge \lambda_j$ and $k \in \mathbb{N}$. So $\{x_\lambda^k\}_\Lambda \subset D$ is t-Cauchy for each $k \in \mathbb{N}$, and since D is complete there is a z^k such that x_λ^k converges to z^k with respect to the topology t for each $k \in \mathbb{N}$. Let $z = \{z^1, z^2, \ldots\}$. Then $z \subset D$, and for each $j \in J$ and $k \in \mathbb{N}$ we have $\rho_j(x_\lambda^k - z^k) = \rho_j(x_\lambda^k - (\rho_j - \lim_{\lambda'} x_{\lambda'}^k)) = \lim_{\lambda'} \rho_j(x_\lambda^k - x_{\lambda'}^k)$, so raising to the pth power and adding with respect to k we have

$$\begin{aligned} \sum_{k=1}^{\infty} \rho_j(x_\lambda^k - z^k)^p &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho_j(x_\lambda^k - z^k)^p \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \lim_{\lambda'} \rho_j(x_\lambda^k - x_{\lambda'}^k)^p \end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty}\lim_{\lambda'}\sum_{k=1}^{n}\rho_j(x_\lambda^k - z^k)^p\\
&\le \lim_{\lambda'}\sum_{k=1}^{\infty}\rho_j(x_\lambda^k - z^k)^p\\
&= \lim_{\lambda'}\rho_{\rho_j}(x_\lambda - x_{\lambda'}) < \varepsilon^p,
\end{aligned} \tag{4.3}$$

for every $\lambda \ge \lambda_j$.

So we have $\rho_{\rho_j}(x_\lambda - z)^p = \sum_{k=1}^{\infty}\rho_j(x_\lambda^k - z^k)^p < \varepsilon^p$, for every $\lambda \ge \lambda_j$. This means that x_λ converges to z with respect to the topology τ. We still need to prove that $z \in \ell_p(E)$

$$\begin{aligned}
\rho_{\rho_j}(z)^p &= \sum_{k=1}^{\infty}\rho_j(z^k)^p\\
&= \sum_{k=1}^{\infty}\rho_j(z^k + x_\lambda^k - x_\lambda^k)^p\\
&\le \sum_{k=1}^{\infty}2^p\left[\rho_j(z^k - x_\lambda^k)^p + \rho_j(x_\lambda^k)^p\right]\\
&= 2^p\sum_{k=1}^{\infty}\rho_j(z^k - x_\lambda^k)^p + 2^p\sum_{k=1}^{\infty}\rho_j(x_\lambda^k)^p\\
&< 2^p\varepsilon^p + 2^p\rho_{\rho_j}(x_\lambda)^p\\
&\le 2^p\varepsilon^p + 2^p m_j
\end{aligned} \tag{4.4}$$

($x_\lambda \in C = \cap_{j\in J}A_D(j,m_j)$), then if we let $\varepsilon \to 0$ we get $\rho_{\rho_j}(z) \le 2m_j$, and finally $z \in 2C \subset \ell_p(E)$. □

Theorem 4.4. *If D is t-complete, then $\ell_p(E)_C$ is ρ_C-complete.*

Proof. Let (x_n^k) be a ρ_C-Cauchy sequence; it is clearly ρ_C-bounded and τ-Cauchy, so $(x_n^k) \subset sC$ for some $s \in \mathbb{N}$. Then by Lemma 4.3, there is a $z = (z^k) \in 2sC \subset \ell_p(E)_C$ such that the sequence (x_n^k) converges to z with respect to the topology τ. Note that A_D is τ-closed so $A_D(j,m)$ is also τ-closed for every $j \in J$ and $m \in \mathbb{N}$; then $C = \cap_{j\in J}A_D(j,m_j)$ is τ-closed. For $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $(x_n^k) - (x_m^k) \in \varepsilon C$ for every $n, m \ge N$, and since C is τ-closed $(x_n^k) - (\tau - \lim(x_m^k)) \in \varepsilon C$ then $(x_n^k) - (z^k) \in \varepsilon C$ for every $n \ge N$. This means that (x_n^k) converges to (z^k) with respect to ρ_C. □

Theorem 4.5. *If E is t-complete, then $\ell_p(E)$ is τ-complete.*

Proof. The proof of Lemma 4.3 can be repeated here to get the τ-completeness of $\ell_p(E)$. □

Acknowledgments. C. Bosch and C. Gómez were partially supported by Asociación Mexicana de Cultura A.C. T. Gilsdorf was partially supported by a Fulbright scholarship. R. Vera was supported by Coordinación de la Investigación Científica de la Universidad Michoacana.

References

[1] C. Bosch, A. García-Martínez, and T. Gilsdorf, *Some hereditary properties of $l_\infty(E)$ from E*, to appear in Internat. J. of Math.

[2] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.

[3] A. García-Martínez, *Webbed spaces, double sequences, and the Mackey convergence condition*, Int. J. Math. Math. Sci. **22** (1999), no. 3, 521–524.

[4] T. E. Gilsdorf, *Boundedly compatible webs and strict Mackey convergence*, Math. Nachr. **159** (1992), 139–147.

[5] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1953), 1–79 (French).

[6] J. Horváth, *Topological Vector Spaces and Distributions. Vol. I*, Addison-Wesley, Massachusetts, 1966.

[7] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Mathematics Studies, vol. 131, North-Holland, Amsterdam, 1987.

[8] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1967), 333–353 (German).

C. Bosch: Departamento Académico de Matemáticas, Itam, Rio Hondo #1, Col. Tizapán San Ángel, C.P. 01000 México D.F., Mexico
E-mail address: bosch@itam.mx

T. Gilsdorf: Department of Mathematics, University of North Dakota, Grand Forks, ND 58202-8376, USA
E-mail address: thomas_gilsdorf@und.nodak.edu

C. Gómez: Departamento Académico de Matemáticas, Itam, Rio Hondo #1, Col. Tizapán San Ángel, C.P. 01000 México D.F., Mexico
E-mail address: claudiag@itam.mx

R. Vera: Esc. de C. Físico-Matemáticas, Universidad Michoacana, Santiago Tapia No. 403 Edif. B, Ciudad Universitaria, 58060 Morelia, Mich., Mexico
E-mail address: rvera@zeus.ccu.umich.mx