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A NEW TRIPLE SUM COMBINATORIAL IDENTITY

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We prove a new triple sum combinatorial identity derived from $r_p(x,y,z) = (x+y-z)^p - (x^p + y^p - z^p)$, extending a previous result by Sinyor et al.

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1. Introduction. The following combinatorial identity

$$\sum_{i=0}^{n}\sum_{j=0}^{2l}\sum_{k=0}^{j}\frac{(-1)^{i+j-k}}{m-l}\binom{m+l-k}{2l-k+1}\binom{m-l+k-1}{k}\binom{2l-j}{n-i}\binom{j}{i} = \begin{cases}\dfrac{1}{2m+1}\dbinom{2m+1}{2m-2l+n}\dbinom{m-l+n/2}{n/2}, & \text{for } n \text{ even},\\ 0, & \text{for } n \text{ odd},\end{cases} \tag{1.1}$$

for $0 \le l \le m-1$, $0 \le n \le 2l$, and m, l, n nonnegative integers, arose from considering

$$r_p(x,y,z) = (x+y-z)^p - (x^p + y^p - z^p), \tag{1.2}$$

where x, y, z are any nonzero integers, and p is odd and $p > 2$. We will use an identity proved in [2] which also arose from considering (1.2).

2. Proof of the identity. Define $f_p(x,y,z) = r_p(x,y,z)/(p(x+y)(z-x)(z-y))$. It is easily seen that $f_p(x,y,z)$ is a nonzero rational function and can be expressed as a polynomial in $(z-x)(z-y)$. Define coefficients $a_{i,(m-j)}$ such that

$$f_p(x,y,z) = \sum_{j=0}^{m-1}\sum_{i=0}^{2j} a_{i,(m-j)} x^{2j-i} y^i (z-x)^{m-j-1}(z-y)^{m-j-1}, \tag{2.1}$$

where $p = 2m+1$. In [2] we proved the following combinatorial identity:

$$\sum_{l'\le l}\sum_{j'\le j}\frac{1}{m-l'}\binom{m+l'-j'}{2l'-j'+1}\binom{m-l'+j'-1}{j'}\binom{m-l'}{2(l-l')-(j-j')}\binom{m-l'}{j-j'} = \frac{1}{2(m-l)}\binom{2m}{2l+1}\binom{2l+1}{j} = \frac{1}{2l+1}\binom{2m}{2l}\binom{2l+1}{j}, \tag{2.2}$$

and we showed that $a_{i,(m-j)}$ defined above satisfy a recurrence and are of the form

$$a_{i,(m-j)}=\sum_{k=0}^{i}\frac{(-1)^{i-k}}{m-j}\binom{m+j-k}{2j-k+1}\binom{m-j+k-1}{k}. \tag{2.3}$$

Now substitute (2.3) in (2.1) and change the variables x and y to $r=(x+y)/2$, $s=(x-y)/2$, which gives

$$\begin{aligned} f_p(x,y,z) &= \sum_{j=0}^{m-1}\sum_{i=0}^{2j} a_{i,(m-j)}(r+s)^{2j-i}(r-s)^i(z-x)^{m-j-1}(z-y)^{m-j-1} \\ &= \sum_{j=0}^{m-1}\sum_{i=0}^{2j}\sum_{u=0}^{2j}\sum_{v=0}^{u}\sum_{k=0}^{i}\frac{(-1)^{v+i-k}}{m-j}\binom{m+j-k}{2j-k+1}\binom{m-j+k-1}{k}\binom{2j-i}{u-v}\binom{i}{v} \\ &\qquad \times r^{2j-u}s^u(z-x)^{m-j-1}(z-y)^{m-j-1}. \end{aligned} \tag{2.4}$$

We now derive another form of $f_p(x,y,z)$ which will establish the identity.

Define $a=z-x$ and $b=z-y$. Then

$$\begin{aligned} r_p(x,y,z) &= (x+y-z)^p+z^p-x^p-y^p \\ &= \left(\frac{2r-a-b}{2}\right)^p+\left(\frac{2r+a+b}{2}\right)^p-\left(\frac{2r-a+b}{2}\right)^p-\left(\frac{2r+a-b}{2}\right)^p \\ &= r^p\left[\left(1-\frac{a+b}{2r}\right)^p+\left(1+\frac{a+b}{2r}\right)^p-\left(1+\frac{b-a}{2r}\right)^p-\left(1-\frac{b-a}{2r}\right)^p\right]. \end{aligned} \tag{2.5}$$

Using an identity from Gould [1], $\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x^k=((1+\sqrt{x})^n+(1-\sqrt{x})^n)/2$, (2.5) becomes

$$\begin{aligned} r_p(x,y,z) &= 2r^p\sum_{t=0}^{\lfloor p/2\rfloor}\binom{p}{2t}\left[\left(\frac{a+b}{2r}\right)^{2t}-\left(\frac{b-a}{2r}\right)^{2t}\right], \\ f_p(x,y,z) &= \frac{r^{p-1}}{pab}\sum_{t=0}^{\lfloor p/2\rfloor}\binom{p}{2t}\left[\left(\frac{a+b}{2r}\right)^{2t}-\left(\frac{b-a}{2r}\right)^{2t}\right] \\ &= \sum_{t=0}^{\lfloor p/2\rfloor}\frac{r^{2m-2t}}{pab}\binom{p}{2t}\left[\left(\frac{a+b}{2}\right)^{2t}-\left(\frac{b-a}{2}\right)^{2t}\right] \\ &= \sum_{t=0}^{\lfloor p/2\rfloor}\frac{r^{2m-2t}}{pab}\binom{p}{2t}\left[(ab+s^2)^t-s^{2t}\right] \\ &= \sum_{t}\frac{r^{2m-2t}}{p}\binom{p}{2t}\sum_{i}\binom{t}{i}s^{2i}a^{t-i-1}b^{t-i-1}. \end{aligned} \tag{2.6}$$

Substituting $j = m - t + i$, we obtain

$$f_p(x,y,z) = \sum_j \sum_i \frac{1}{2m+1} \binom{2m+1}{2m-2j+2i} \binom{m-j+i}{i} r^{2j-2i} s^{2i} a^{m-j-1} b^{m-j-1}, \quad (2.7)$$

where $0 \le j \le m-1$ and $0 \le i \le j$.

Note that (2.7) can be considered as an expression in $r^{2j-i}s^i$ where all terms with odd i are zero and $0 \le i \le 2j$, that is,

$$f_p(x,y,z) = \begin{cases} \displaystyle\sum_{j=0}^{m-1} \sum_{i=0}^{2j} \frac{1}{2m+1} \binom{2m+1}{2m-2j+i} \binom{m-j+i/2}{i/2} r^{2j-i} s^i (ab)^{m-j-1}, & i \text{ even}, \\ 0, & i \text{ odd}. \end{cases} \quad (2.8)$$

We now equate (2.4) and (2.8), removing the common summation terms, which gives the identity.

Remark 2.1. By using Wegschaider's MultiSum, a computer algebra package which is available from http://www.risc.uni-linz.ac.at/research/combinat/risc/software/, we found recurrence equations satisfied by the summand of our identity but we could not use them to give a short proof for our identity.

References

[1] H. W. Gould, *Combinatorial Identities. A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Henry W. Gould, West Virginia, 1972.

[2] J. Sinyor, T. Speevak, and A. Tefera, *A new combinatorial identity*, Int. J. Math. Math. Sci. **25** (2001), no. 6, 361–363.

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