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PROJECTIVE REPRESENTATIONS OF QUIVERS

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We prove that $P_1 \xrightarrow{f} P_2$ is a projective representation of a quiver $Q = \bullet \to \bullet$ if and only if P_1 and P_2 are projective left R-modules, f is an injection, and $f(P_1) \subset P_2$ is a summand. Then, we generalize the result so that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} M_{n-1} \xrightarrow{f_{n-1}} M_n$ of a quiver $Q = \bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \to \bullet$ is projective representation if and only if each M_i is a projective left R-module and the representation is a direct sum of projective representations.

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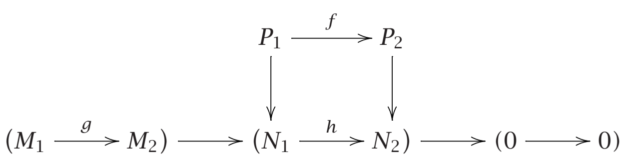
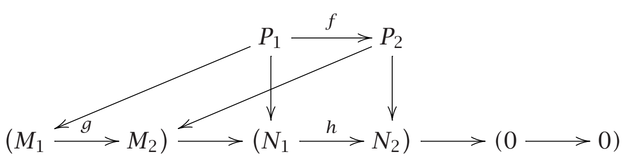
1. Introduction. A quiver is just a directed graph. We allow multiply edges and edges going from a vertex to the same vector. Originally a representation of a quiver assigned a vertex space to each vertex and a linear map to each edge (or arrow) with the linear map going from the vector space assigned to the initial vertex to the one assigned to the terminal vertex. For example, a representation of a quiver $\bullet \to \bullet$ is $V_1 \xrightarrow{f} V_2$. Then we can define a morphism of two representations of the same vector. Now, instead of vector spaces we can use left R-modules and also instead of linear maps we can use R-linear maps. In this paper, we study the properties of projective representations of quivers. Representations of quivers is a new topic in module theory and is recently developed in [1, 2].

Definition 1.1. A representation $P_1 \xrightarrow{f} P_2$ of a quiver $Q = \bullet \to \bullet$ is said to be projective if every diagram of representations



(1.1)

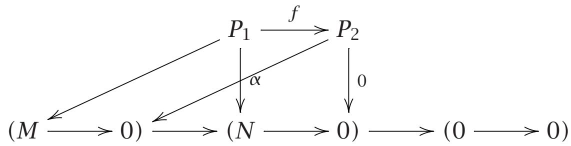
can be completed to a commutative diagram as follows:



(1.2)

Lemma 1.2. *If $P_1 \xrightarrow{f} P_2$ is a projective representation of a quiver $Q = \bullet \to \bullet$, then P_1 and P_2 are projective left R-modules.*

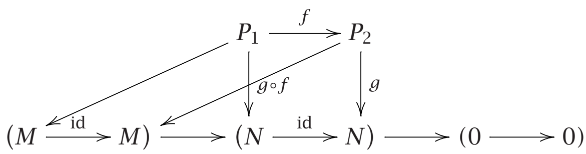
Proof. Let M and N be left R-modules, $\alpha : P_1 \to N$ an R-linear map, and $\beta : M \to N$ an onto R-linear map. Then, since $P_1 \xrightarrow{f} P_2$ is a projective representation, we can complete the diagram



(1.3)

as a commutative diagram. Thus P_1 is a projective left R-module.

Let $g : P_2 \to N$ be an R-linear map and let $h : M \to N$ be an onto R-linear map. Then, since $P_1 \xrightarrow{f} P_2$ is a projective representation, we can complete the diagram

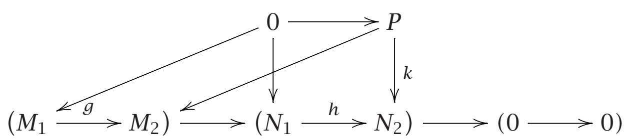


(1.4)

as a commutative diagram. Thus P_2 is a projective left R-module. □

Lemma 1.3. *If P is a projective left R-module, then a representation $0 \to P$ of a quiver $Q = \bullet \to \bullet$ is a projective representation.*

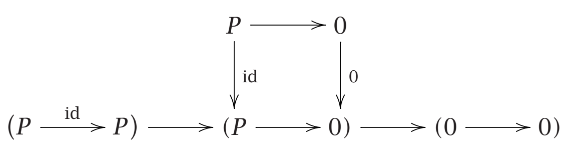
Proof. The lemma follows by completing the diagram



(1.5)

as a commutative diagram. □

Remark 1.4. A representation $P \to 0$ of a quiver $Q = \bullet \to \bullet$ is not a projective representation if $P \neq 0$, because we cannot complete the diagram

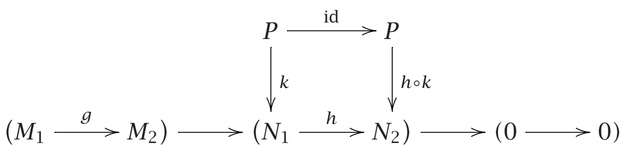


(1.6)

as a commutative diagram.

Lemma 1.5. *If P is a projective left R-module, then a representation $P \xrightarrow{\text{id}} P$ of a quiver $Q = \bullet \to \bullet$ is a projective representation.*

Proof. Let M_1, M_2, N_1, and N_2 be left R-modules and let $g : M_1 \to M_2$ and $h : N_1 \to N_2$ be R-linear maps. Let $k : P \to N_1$ be an R-linear map and choose $h \circ k : P \to N_2$ as an R-linear map. And consider the following diagram:



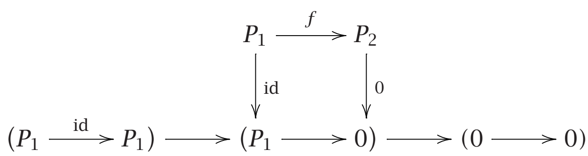
(1.7)

Then, since P is a projective left R-module, there exists a map $\alpha : P \to M_1$. Now choose $g \circ \alpha : P \to M_2$ as an R-linear map. Then α and $g \circ \alpha$ complete the above diagram as a commutative diagram. Therefore $P \xrightarrow{\text{id}} P$ is a projective representation. □

2. Direct sum of projective representations

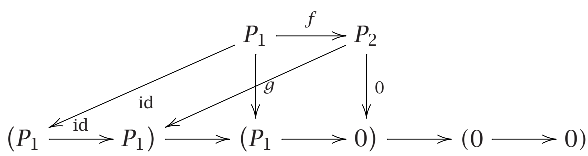
Theorem 2.1. *A representation $P_1 \xrightarrow{f} P_2$ of a quiver $Q = \bullet \to \bullet$ is projective if and only if P_1 and P_2 are projective left R-modules, f is an injection, and $f(P_1) \subset P_2$ is a summand.*

Proof. Consider the following diagram:



(2.1)

Since $P_1 \xrightarrow{f} P_2$ is a projective representation, we can complete the above diagram as a commutative diagram as follow:



(2.2)

Thus $g \circ f = \text{id}_{P_1}$. Therefore, $P_2 \cong f(P_2) \oplus \ker(g)$ and

$$(P_1 \longrightarrow P_2) \cong (P_1 \longrightarrow (P_1 \oplus \ker(g))) \cong (P_1 \xrightarrow{\text{id}} P_1) \oplus (0 \longrightarrow \ker(g)). \tag{2.3}$$

This completes the proof. □

Now let $Q = \bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \to \bullet$ be a quiver with n vertices and $n-1$ arrows. Then, we can easily generalize the results of Lemmas 1.3 and 1.5 as follows: the representations

$$\begin{array}{c} 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_n, \\ 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow P_{n-1} \xrightarrow{\text{id}} P_{n-1}, \\ \vdots \\ 0 \longrightarrow P_2 \xrightarrow{\text{id}} \cdots \longrightarrow P_2 \xrightarrow{\text{id}} P_2 \xrightarrow{\text{id}} P_2, \\ P_1 \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} P_1 \end{array} \tag{2.4}$$

are all projective representations of a quiver $Q = \bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \to \bullet$, if each P_i is a projective left R-module. We can also generalize Lemma 1.2 so that if $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n$ of a quiver $Q = \bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \to \bullet$ is projective representation, then each M_i is a projective left R-module.

Theorem 2.2. *A representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is projective if and only if M_1, M_2, and M_3 are projective left R-modules, $f_1(M_1)$ is a summand of M_2, and $f_2(M_2)$ is a summand of M_3. That is,*

$$(M_1 \longrightarrow M_2 \longrightarrow M_3) \cong (P_1 \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} P_1) \oplus (0 \longrightarrow P_2 \xrightarrow{\text{id}} P_2) \oplus (0 \longrightarrow 0 \longrightarrow P_3). \tag{2.5}$$

Proof. The diagram

$$\begin{array}{ccccccccccc} & & & & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & & \\ & & & & \big\downarrow{\scriptstyle \text{id}} & & \big\downarrow & & \big\downarrow & & \\ (M_1 \xrightarrow{\text{id}} M_1 \longrightarrow 0) & \longrightarrow & & & (M_1 & \longrightarrow & 0 & \longrightarrow & 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array} \tag{2.6}$$

can be completed to a commutative diagram by $\text{id} : M_1 \to M_1$, $g_{21} : M_2 \to M_1$, and $0 : M_3 \to 0$. Then we can get $g_{21} \circ f_1 = \text{id}_{M_1}$ so that $M_2 \cong M_1 \oplus \ker(g_{21})$. Now the diagram

$$\begin{array}{ccccccccccc} & & & & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & & \\ & & & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow & & \big\downarrow & & \\ (M_2 \xrightarrow{\text{id}} M_2 \xrightarrow{\text{id}} M_2) & \longrightarrow & & & (M_2 & \xrightarrow{\text{id}} & M_2 & \longrightarrow & 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array} \tag{2.7}$$

can be completed to a commutative diagram by $f_1 : M_1 \to M_2$, $\text{id} : M_2 \to M_2$, and $g_{32} : M_3 \to M_2$. Then, we can get $g_{32} \circ f_2 = \text{id}_{M_2}$ so that $M_3 \cong M_2 \oplus \text{Ker}(g_{32})$. Therefore, $M_3 \cong M_2 \oplus \text{Ker}(g_{32}) \cong M_1 \oplus \text{Ker}(g_{21}) \oplus \text{Ker}(g_{32})$. Hence,

$$(M_1 \longrightarrow M_2 \longrightarrow M_3) \cong (P_1 \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} P_1) \oplus (0 \longrightarrow P_2 \xrightarrow{\text{id}} P_2) \oplus (0 \longrightarrow 0 \longrightarrow P_3). \tag{2.8}$$

This completes the proof. $\square$

Now, we can easily generalize Theorem 2.2 so that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} M_{n-1} \xrightarrow{f_{n-1}} M_n$ of a quiver $Q = \bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \to \bullet$ is projective representation if and only if each M_i is a projective left R-module and the representation is the direct sum of the following projective representations:

$$\begin{gathered} 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_n, \\ 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow P_{n-1} \xrightarrow{\text{id}} P_{n-1}, \\ \vdots \\ 0 \longrightarrow P_2 \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} P_2 \xrightarrow{\text{id}} P_2 \xrightarrow{\text{id}} P_2, \\ P_1 \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} P_1 \xrightarrow{\text{id}} P_1. \end{gathered} \tag{2.9}$$

Remark 2.3. The representations of a quiver $Q = \bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \to \bullet$:

$$\begin{gathered} P \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0, \\ P \xrightarrow{\text{id}} P \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0, \\ \vdots \\ P \xrightarrow{\text{id}} P \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} P \xrightarrow{\text{id}} P \longrightarrow 0 \end{gathered} \tag{2.10}$$

are not projective representations if $P \neq 0$.

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