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METHOD OF REPLACING THE VARIABLES FOR GENERALIZED SYMMETRY OF THE D'ALEMBERT EQUATION

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We show that by the generalized understanding of symmetry, the D'Alembert equation for one component field is invariant with respect to arbitrary reversible coordinate transformations.

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1. Introduction. Symmetries play an important role in particle physics and quantum field theory [1], nuclear physics [11], and mathematical physics [5]. Some receptions are proposed for finding the symmetries, for example, the method of replacing the variables [13], the Lie algorithm [5], and the theoretical-algebraic approach [9]. The purpose of this work is the generalization of the method of replacing the variables. We start from the following definition of symmetry.

2. Main results

Definition 2.1. Let a differential equation $\hat{L}'\phi'(x') = 0$ be given. By symmetry of this equation with respect to the variables replacement $x' = x'(x)$, $\phi' = \phi'(\Phi\phi)$ we will understand the compatibility of the engaging equations system $\hat{A}\phi'(\Phi\phi) = 0$, $\hat{L}\phi(x) = 0$, where $\hat{A}\phi'(\Phi\phi) = 0$ is obtained from the initial equation by replacing the variables, $\hat{L}' = \hat{L}$, $\Phi(x)$ is some weight function. If the equation $\hat{A}\phi'(\Phi\phi) = 0$ can be transformed into the form $\hat{L}(\Psi\phi) = 0$, the symmetry will be named the standard Lie symmetry, otherwise it will be named generalized symmetry.

2.1. Application of Definition 2.1. The elements of Definition 2.1 were used to study the Maxwell equations symmetries [6, 7, 8]. In this paper, we apply this definition for investigation of symmetries of the one-component D'Alembert equation

$$\square'\phi'(x') = \frac{\partial^2\phi'}{\partial x_1'^2} + \frac{\partial^2\phi'}{\partial x_2'^2} + \frac{\partial^2\phi'}{\partial x_3'^2} + \frac{\partial^2\phi'}{\partial x_4'^2} = 0. \tag{2.1}$$

We introduce the arbitrary reversible coordinate transformations $x' = x'(x)$ and the transformation of the field variable $\phi' = \phi(\Phi\phi)$, where $\Phi(x)$ is some unknown function, also we take into account

$$\frac{\partial\phi'}{\partial x_i'} = \sum_j \frac{\partial\phi'}{\partial\xi}\frac{\partial\Phi\phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'},$$

$$\frac{\partial^2\phi'}{\partial x_i'^2} = \sum_j \frac{\partial^2 x_j}{\partial x_i'^2}\frac{\partial\phi'}{\partial\xi}\frac{\partial\Phi\phi}{\partial x_j} + \sum_{jk}\frac{\partial^2\Phi\phi}{\partial x_j\partial x_k}\frac{\partial x_j}{\partial x_i'}\frac{\partial x_k}{\partial x_i'}\frac{\partial\phi'}{\partial\xi} + \sum_{jk}\frac{\partial^2\phi'}{\partial\xi^2}\frac{\partial\Phi\phi}{\partial x_j}\frac{\partial\Phi\phi}{\partial x_k}\frac{\partial x_j}{\partial x_i'}\frac{\partial x_k}{\partial x_i'}, \tag{2.2}$$

where $\xi = \Phi\phi$. After replacing the variables we find that the equation $\Box'\phi' = 0$ transforms into itself, if the system of the engaging equations is fulfilled

$$\begin{aligned}
&\sum_i \sum_j \frac{\partial^2 x_j}{\partial x_i'^2} \frac{\partial \phi'}{\partial \xi} \frac{\partial \Phi\phi}{\partial x_j} + \sum_i \sum_{j=k} \left(\frac{\partial x_j}{\partial x_i'}\right)^2 \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi\phi}{\partial x_j{}^2} \\
&\quad + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi\phi}{\partial x_j \partial x_k} + \sum_i \sum_{j=k} \left(\frac{\partial x_j}{\partial x_i}\right)^2 \frac{\partial^2 \phi'}{\partial \xi^2} \left(\frac{\partial \Phi\phi}{\partial x_j}\right)^2 \\
&\quad + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial^2 \phi'}{\partial \xi^2} \frac{\partial \Phi\phi}{\partial x_j} \frac{\partial \Phi\phi}{\partial x_k} = 0; \quad \Box\phi = 0.
\end{aligned} \tag{2.3}$$

Here $x = (x_1, x_2, x_3, x_4)$, $x_4 = ict$, where c is the speed of light and t is the time. We substitute the solution of the D'Alembert equation ϕ into the first equation of the set (2.3). If the obtained equation has a solution, then the set (2.3) will be compatible. According to Definition 2.1 this compatibility will mean that arbitrary reversible transformations $x' = x'(x)$ are the symmetry transformations of the initial equation $\Box'\phi' = 0$. Owing to the presence of the expressions $(\partial\Phi\phi/\partial x_j)^2$ and $(\partial\Phi\phi/\partial x_j)(\partial\Phi\phi/\partial x_k)$ in the first equation of (2.3), the latter has nonlinear character. Since the analysis of nonlinear systems is difficult we suppose that

$$\frac{\partial^2 \phi'}{\partial \xi^2} = 0. \tag{2.4}$$

In this case, the nonlinear components in the set (2.3) turn to zero and the system will be linear. As a result, we find the field transformation law by integrating (2.4)

$$\phi' = C_1 \Phi\phi + C_2 \longrightarrow \phi' = \Phi\phi. \tag{2.5}$$

Here we suppose for simplicity that the constants of integration are $C_1 = 1$, $C_2 = 0$. It is this law of field transformation that was used within the algorithm [7, 8]. It marks the position of the algorithm in the generalized variables replacement method. Taking into account formulae (2.4) and (2.5), we find the following form for system (2.3):

$$\begin{gathered}
\frac{\partial^2 \phi'}{\partial \xi^2} = 0; \quad \phi' = \Phi\phi; \\
\sum_j \Box' x_j \frac{\partial \Phi\phi}{\partial x_j} + \sum_i \sum_j \left(\frac{\partial x_j}{\partial x_i'}\right)^2 \frac{\partial^2 \Phi\phi}{\partial x_j{}^2} + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial^2 \Phi\phi}{\partial x_j \partial x_k} = 0; \quad \Box\phi = 0.
\end{gathered} \tag{2.6}$$

Since here $\Phi(x) = \phi'(x' \to x)/\phi(x)$, where $\phi'(x')$ and $\phi(x)$ are the solutions of the D'Alembert equation, system (2.6) has a common solution and consequently is compatible. This means that the arbitrary reversible transformations of coordinates $x' = x'(x)$ are symmetry transformations for the one-component D'Alembert equation if the field is transformed with the help of the weight function $\Phi(x)$ according to the law (2.5). The form of this function depends on the D'Alembert equation solutions and the law of the coordinate transformations $x' = x'(x)$.

Next we consider the following examples.

2.1.1. Poincaré group. Let the coordinate transformations belong to the *Poincaré group* P_{10}:

$$x'_j = L_{jk}x_k + a_j, \tag{2.7}$$

where L_{jk} is the matrix of the Lorentz group L_6, a_j are the parameters of the translation group T_4. In this case, we have $\square' x_j = \sum_k L'_{jk}\square' x'_k = 0$, $\sum_i(\partial x_j/\partial x'_i)(\partial x_k/\partial x'_i) = \sum_i L'_{ji}L'_{ki} = \delta_{jk}$. The last term in the second equation of (2.6) turns to zero. The set reduces to the form

$$\square\Phi\phi = 0; \qquad \square\phi = 0. \tag{2.8}$$

According to Definition 2.1 this is a sign of the Lie symmetry. The weight function belongs to the set in [8]:

$$\Phi_{P_{10}}(x) = \frac{\phi'(x)}{\phi(x)} \in \left\{1; \frac{1}{\phi(x)}; \frac{P_j\phi(x)}{\phi(x)}; \frac{M_{jk}\phi(x)}{\phi(x)}; \frac{P_jP_k\phi(x)}{\phi(x)}; \frac{P_jM_{kl}\phi(x)}{\phi(x)}; \dots\right\}, \tag{2.9}$$

where P_j, M_{jk} are the generators of Poincaré group, $j,k,l = 1,2,3,4$. In the space of the D'Alembert equation solutions the set defines a rule of transforming of a solution $\phi(x)$ to another solution $\phi'(x)$. The weight function $\Phi(x) = 1 \in \Phi_{P_{10}}(x)$ determines the transformational properties of the solutions $\phi' = \phi$, which means the well-known relativistic symmetry of the D'Alembert equation [4, 10].

2.1.2. Weyl group. Let the transformations of coordinates belong to the *Weyl Group* W_{11}:

$$x'_j = \rho L_{jk}x_k + a_j, \tag{2.10}$$

where $\rho = \text{const}$ is the parameter of the scale transformations of the group Δ_1. In this case we have $\square' x_j = \rho' \sum_k L'_{jk}\square' x'_k = 0$, $\sum_i(\partial x_j/\partial x'_i)(\partial x_k/\partial x'_i) = \sum_i \rho'^2 L'_{ji}L'_{ki} = \rho'^2\delta_{jk} = \rho^{-2}\delta_{jk}$. The set (2.6) reduces to the set (2.8) and has the solution $\Phi_{W_{11}} = C\Phi_{P_{10}}$, where $C = \text{const}$. The weight function $\Phi(x) = C$ and the law $\phi' = C\phi$ means the well-known Weyl symmetry of the D'Alembert equation [4, 10]. Here $C = \rho^l$, where l is the conformal dimension of the field $\phi(x)$ [2]. Consequently, the D'Alembert equation is W_{11}-invariant for the field ϕ with arbitrary conformal dimension l. This property is essential for the Voigt [13] and Umov [12].

2.1.3. Inversion group. Let the coordinate transformations belong to the *inversion group* I:

$$x'_j = -\frac{x_j}{x^2}; \quad x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2; \quad x^2x'^2 = 1. \tag{2.11}$$

In this case, we have $\square' x_j = 4x'_j/x'^4 = -4x_jx^2$, $\sum_i(\partial x_j/\partial x'_i)(\partial x_k/\partial x'_i) = \rho'^2(x')\delta_{jk} = 1/x'^4\delta_{jk} = x^4\delta_{jk}$. The set (2.6) reduces to the set

$$-4x_j\frac{\partial\Phi\phi}{\partial x_j} + x^2\square\Phi\phi = 0; \qquad \square\phi = 0. \tag{2.12}$$

The substitution of $\Phi(x) = x^2\Psi(x)$ transforms equation (2.12) for $\Phi(x)$ into the equation $\square\Psi\phi = 0$ for $\Psi(x)$. It is a sign of the Lie symmetry. The equation has the solution $\Psi = 1$. The result is $\Phi(x) = x^2$. Consequently, the field transforms according to the law $\phi' = x^2\phi(x) = \rho^{-1}(x)\phi(x)$. This means the conformal dimension $l = -1$ of the field $\phi(x)$ in the case of the D'Alembert equation symmetry with respect to the inversion group I in agreement with [4, 9]. In a general case the weight function belongs to the set

$$\Phi_I(x) = x^2\Psi(x) \in \left\{x^2; \frac{x^2}{\phi(x)}; x^2\frac{P_j\phi(x)}{\phi(x)}; x^2\frac{M_{jk}\phi(x)}{\phi(x)}; x^2\frac{P_jP_k\phi(x)}{\phi(x)}; \dots\right\}. \tag{2.13}$$

2.1.4. Special conformal group. Let the coordinate transformations belong to the *special conformal group* C_4:

$$x'_j = \frac{x_j - a_jx^2}{\sigma(x)}; \quad \sigma(x) = 1 - 2a\cdot x + a^2x^2; \quad \sigma\sigma' = 1. \tag{2.14}$$

In this case, we have $\square' x_j = 4(a_j - a^2x_j)\sigma(x)$, $\sum_i(\partial x_j/\partial x'_i)(\partial x_k/\partial x'_i) = \rho'^2(x')\delta_{jk} = \sigma^2(x)\delta_{jk}$. The set (2.6) reduces to the set

$$4\sigma(x)(a_j - a^2x_j)\frac{\partial\Phi\phi}{\partial x_j} + \sigma^2(x)\square\Phi\phi = 0; \qquad \square\phi = 0. \tag{2.15}$$

The substitution of $\Phi(x) = \sigma(x)\Psi(x)$ transforms (2.15) into the equation $\square\Psi\phi = 0$ which corresponds to the Lie symmetry. From this equation, we have $\Psi = 1$, $\Phi(x) = \sigma(x)$. Therefore, $\phi' = \sigma(x)\phi(x)$ and the conformal dimension of the field is $l = -1$ as above. Analogously to (2.13), the weight function belongs to the set

$$\Phi_{C_4}(x) = \sigma(x)\Psi(x) \in \left\{\sigma(x); \frac{\sigma(x)}{\phi(x)}; \sigma(x)\frac{P_j\phi(x)}{\phi(x)}; \sigma(x)\frac{M_{jk}\phi(x)}{\phi(x)}; \dots\right\}. \tag{2.16}$$

Thus, we can see that $\phi(x) = 1/\sigma(x)$ is the solution of the D'Alembert equation. Combination of W_{11}, I, and C_4 symmetries means the well-known D'Alembert equation conformal C_{15}-symmetry [4, 9, 10].

2.1.5. Galilei group. Let the coordinate transformations belong to the *Galilei group* G_1:

$$x'_1 = x_1 + i\beta x_4; \qquad x'_2 = x_2; \qquad x'_3 = x_3; \qquad x'_4 = \gamma x_4; \qquad c' = \gamma c, \tag{2.17}$$

where $\beta' = -\beta/\gamma$, $\gamma' = 1/\gamma$, $\beta = V/c$, $\gamma = (1 - 2\beta n_x + \beta^2)^{1/2}$. In this case, we have

$$\begin{gathered}
\square' x_j = 0; \qquad \sum_i\left(\frac{\partial x_1}{\partial x'_i}\right)^2 = 1 - \beta'^2; \\
\sum_i\left(\frac{\partial x_2}{\partial x'_i}\right)^2 = \sum_i\left(\frac{\partial x_3}{\partial x'_i}\right)^2 = 1; \qquad \sum_i\left(\frac{\partial x_4}{\partial x'_i}\right)^2 = \gamma'^2; \\
\sum_i\frac{\partial x_1}{\partial x'_i}\frac{\partial x_2}{\partial x'_i} = \sum_i\frac{\partial x_1}{\partial x'_i}\frac{\partial x_3}{\partial x'_i} = \sum_i\frac{\partial x_2}{\partial x'_i}\frac{\partial x_3}{\partial x'_i} = \sum_i\frac{\partial x_2}{\partial x'_i}\frac{\partial x_4}{\partial x'_i} = 0; \\
\sum_i\frac{\partial x_1}{\partial x'_i}\frac{\partial x_4}{\partial x'_i} = i\beta'\gamma' = -\frac{i\beta}{\gamma^2}.
\end{gathered} \tag{2.18}$$

After putting these expressions into the set (2.6) we find (see [8])

$$\square\Phi\phi - \frac{\partial^2\Phi\phi}{\partial x_4{}^2} - \left(i\frac{\partial}{\partial x_4} + \beta\frac{\partial}{\partial x_1}\right)^2\frac{\Phi\phi}{\gamma^2} = \left[\frac{(i\partial_4 + \beta\partial_1)^2}{\gamma^2} - \triangle\right]\Phi\phi = 0. \tag{2.19}$$

In accordance with Definition 2.1 the Galilei symmetry of the D'Alembert equation is the generalized symmetry (being the conditional one [8]). The weight function belongs to the set (see [7])

$$\Phi_{G_1}(x) = \frac{\phi'(x' \to x)}{\phi(x)} \in \left\{ \frac{\phi(x')}{\phi(x)}; \frac{1}{\phi(x)}; \frac{P'_j \phi(x')}{\phi(x)}; \frac{[\Box', H'_1]\phi(x')}{\phi(x)}; \dots \right\}, \tag{2.20}$$

where $H'_1 = it'\partial_{x'}$ is the generator of the pure Galilei transformations. For the plane waves the weight function $\Phi(x)$ is (see [6, 7, 8])

$$\Phi_{G_1}(x) = \frac{\phi(x' \to x)}{\phi(x)} = \exp\left\{ -\frac{i}{\gamma}\left[(1-\gamma)k \cdot x - \beta\omega\left(n_x t - \frac{x}{c}\right)\right]\right\}, \tag{2.21}$$

where $k = (\mathbf{k}, k_4)$, $\mathbf{k} = \omega\mathbf{n}/c$ is the wave vector, $\mathbf{n}$ is the wave front guiding vector, ω is the wave frequency, $k_4 = i\omega/c$, $k'_1 = (k_1 + i\beta k_4)/\gamma$, $k'_2 = k_2/\gamma$, $k'_3 = k_3/\gamma$, $k'_4 = k_4$, $\mathbf{k}'^2 = \mathbf{k}^2$-inv, where inv means invariant. (For comparison, in the relativistic case we have $k'_1 = (k_1 + i\beta k_4)/(1-\beta^2)^{1/2}$, $k'_2 = k_2$, $k'_3 = k_3$, $k'_4 = (k_4 - i\beta k_1)/(1-\beta^2)^{1/2}$, $\mathbf{k}'^2 + k_4'^2 = \mathbf{k}^2 + k_4{}^2$-inv as is well known.)

3. Comparison of the results. Table 3.1 illustrates the results obtained above.

TABLE 3.1

Group	P_{10}	W_{11}	I	C_4	G_1
$WF\Phi(x)$	1	ρ^l	x^2	$\sigma(x)$	$\exp\{-i[(1-\gamma)k \cdot x - \beta\omega(n_x t - x/c)]/\gamma\}$

For the different transformations $x' = x'(x)$, the weight functions $\Phi(x)$ may be found in a similar way.

Note that in the symmetry theory of the D'Alembert equation, conditions (2.6) for transforming this equation into itself combine the requirements formulated by various authors, as can be seen in Table 3.2, where m_α, m_0 are some numbers, $D_{\alpha\beta}$ and $M_{\alpha\beta}$ are the 6×6 numerical matrices.

According to Table 3.2 for the field $\phi' = \phi$ with conformal dimension $l = 0$ and the linear homogeneous coordinate transformations from the group $L_6 X \triangle_1 \in W_{11}$ with $\rho = (1-\beta^2)^{1/2}$, the formulae were proposed by Voigt [13] and cited by Pauli [10]. In the plain waves case, they correspond to the transformations of the 4-vector $k = (\mathbf{k}, k_4)$ and proper frequency ω_0 according to the law $k'_1 = (k_1 + i\beta k_4)/\rho(1-\beta^2)^{1/2}$, $k'_2 = k_2/\rho$, $k'_3 = k_3/\rho$, $k'_4 = (k_4 - i\beta k_1)/\rho(1-\beta^2)^{1/2}$, $\omega'_0 = \omega_0/\rho$, $k'x' = kx$-inv. In the case of the W_{11}-coordinate transformations belonging to the set of arbitrary transformations $x' = x'(x)$, the requirements for the one component field with $l = 0$ were found by Umov [12]. The requirement that the second derivative $\partial^2\phi'/\partial\phi_\alpha\partial\phi_\beta = 0$ with $\Phi = 1$ is turned into zero was introduced by Di Jorio [3]. The weight function $\Phi \neq 1$ and the set (2.6) were proposed by the author of the present work [6, 7, 8].

By now only the D'Alembert equation symmetries corresponding to the linear systems of the type (2.8), (2.12), and (2.15) have been well studied. These are the well-known relativistic and conformal symmetry of the equation. The investigations corresponding to the linear conditions (2.6) are much more scanty and presented only

TABLE 3.2

Author	Coordinates transform.	Group	Conditions of invariance	Fields transform.
Voigt [13]	$x'_j = A_{jk}x_k$	$L_6 X \Delta_1$	$A'_{ji}A'_{ki} = \rho'^2\delta_{jk}$	$\phi' = \phi$
Umov [12]	$x'_j = x_j{}'(x)$	W_{11}	$\frac{\partial x_j}{\partial x'_i}\frac{\partial x_k}{\partial x'_i} = \rho'^2\delta_{jk}$ $\square' x_j = 0$	$\phi' = \phi$
Di Jorio [3]	$x'_j = L_{jk}x_k + a_j$	P_{10}	$L'_{ji}L'_{ki} = \delta_{jk}$ $\frac{\partial^2\phi'}{\partial\phi_\alpha\partial\phi_\beta} = 0$	$\phi' = m_\alpha\phi_\alpha + m_0$; $\alpha = 1,\dots,n$
Kotel'nikov [6, 7, 8]	$x'_j = x'_j(x)$	C_4	$\frac{\partial x_j}{\partial x'_i}\frac{\partial x_k}{\partial x'_i} = \rho'^2(x')\delta_{jk}$ $\frac{\partial^2\phi'_\alpha}{\partial\xi_\beta\partial\xi_\gamma} = 0$ $\square'\phi'_\alpha = 0 \to$ $\hat{A}\phi'_\alpha(\psi\phi_1,\dots,\psi\phi_6) = 0,\ \square\phi_\beta = 0$	$\phi'_\alpha = \psi D_{\alpha\beta}\phi_\beta$ $\xi_\alpha = \psi\phi_\alpha$ $\alpha,\beta = 1,\dots,6$
	$x'_j = x'_j(x)$	G_1	$\frac{\partial^2\phi'_\alpha}{\partial\xi_\beta\partial\xi_\gamma} = 0$ $\square'\phi'_\alpha = 0 \to$ $\hat{B}\phi'_\alpha(\psi\phi_1,\dots,\psi\phi_6) = 0,\ \square\phi_\beta = 0$	$\phi'_\alpha = \psi M_{\alpha\beta}\phi_\beta$ $\xi_\alpha = \psi\phi_\alpha$ $\alpha,\beta = 1,\dots,6$

in [6, 7, 8]. The publications corresponding to the nonlinear conditions (2.3) are completely absent. The difficulties arising here are connected with the analysis of compatibility of the set (2.3) containing the nonlinear partial differential equation.

4. Conclusion. It is shown that with the generalized understanding of the symmetry according to Definition 2.1, the D'Alembert equation for one component field is invariant with respect to any arbitrary reversible coordinate transformations $x' = x'(x)$. In particular, they contain the transformations of the conformal and Galilei groups realizing the type of standard and generalized symmetry for $\Phi(x) = \phi'(x' \to x)/\phi(x)$. The concept of partial differential equations symmetry is conventional.

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