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NOTES ON STABILITY OF THE GENERALIZED GAMMA FUNCTIONAL EQUATION

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The Hyers-Ulam stability in three senses is discussed by Kim (2001) for the generalized gamma functional equation $g(x+p) = a(x)g(x)$ under some conditions which involve convergence of complicated series. In this note, those conditions are simplified to be checked easily and more interesting examples other than the classical gamma functional equation are displayed.

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1. Introduction. The functional equation

$$E_1(g) = E_2(g) \tag{1.1}$$

is said to have the *Hyers-Ulam stability* if for an approximate solution f, such that

$$\left|E_1(f)(x) - E_2(f)(x)\right| \le \delta \tag{1.2}$$

for some fixed constant $\delta \ge 0$, there exists a solution g of (1.1) such that

$$\left|f(x) - g(x)\right| \le \varepsilon \tag{1.3}$$

for some positive constant ε depending only on δ. Sometimes we call f a δ-approximate solution of (1.1) and g ε-close to f.

Such an idea of stability was given by Ulam [13] for Cauchy equation $f(x+y) = f(x)+f(y)$ and his problem was solved by Hyers [4]. Later, the Hyers-Ulam stability was studied extensively (see, e.g., [6, 8, 10, 11]). Moreover, such a concept is also generalized in [2, 3, 12]. As in [5] we say (1.1) has the *generalized Hyers-Ulam-Rassias stability* if for an approximate solution f, such that

$$\left|E_1(f)(x) - E_2(f)(x)\right| \le \psi(x) \tag{1.4}$$

for some fixed function $\psi(x)$, there exists a solution g of (1.1) such that

$$\left|f(x) - g(x)\right| \le \Phi(x) \tag{1.5}$$

for some fixed function $\Phi(x)$ depending only on $\psi(x)$. We say (1.1) has the *stability in the sense of Ger* if for an approximate solution f, such that

$$\left|\frac{E_1(f)(x)}{E_2(f)(x)} - 1\right| \le \psi(x) \tag{1.6}$$

for some fixed function $\psi(x)$, there exists a solution g of (1.1) such that

$$\alpha(x) \leq \frac{f(x)}{g(x)} \leq \beta(x) \tag{1.7}$$

for some fixed functions $\alpha(x)$ and $\beta(x)$ depending only on $\psi(x)$.

The three senses of the Hyers-Ulam stability are discussed in [5] for the generalized gamma functional equation

$$g(x+p) = a(x)g(x), \tag{1.8}$$

where $p > 0$ is a fixed real constant. It is proved that (1.8) has the Hyers-Ulam stability if

$$\sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{1}{a(x+pk)} < +\infty, \quad \forall x > n_0, \tag{1.9}$$

for a nonnegative constant n_0, has the generalized Hyers-Ulam-Rassias stability if the function $\psi(x)$ in (1.4) satisfies

$$\sum_{j=0}^{\infty} \psi(x+pj) \prod_{k=0}^{j} \frac{1}{a(x+pk)} < +\infty, \quad \forall x > n_0, \tag{1.10}$$

for a nonnegative constant n_0, and has the stability in the sense of Ger if the function $\psi(x)$ in (1.6) satisfies

$$\sum_{j=0}^{\infty} \log\left(1-\psi(x+pj)\right) > -\infty, \quad \sum_{j=0}^{\infty} \log\left(1+\psi(x+pj)\right) < +\infty, \quad \forall x > n_0, \tag{1.11}$$

for a nonnegative constant n_0. In [5] conditions (1.9), (1.10), and (1.11) are checked with the concrete equation $g(x+1) = xg(x)$, which the well-known gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ satisfies.

2. On Hyers-Ulam stability

Theorem 2.1. *Consider approximate solutions $f : (0,+\infty) \to \mathbb{R}$ of (1.8) which satisfy that $|f(x+p) - a(x)f(x)| \leq \delta$ for all $x > n_0$ where $\delta \geq 0$ is a fixed constant and n_0 is a nonnegative constant. If the function $a(x)$ satisfies*

$$\liminf_{k\to\infty} a(x+pk) > 1, \quad \forall x > n_0, \tag{2.1}$$

then (1.8) has the Hyers-Ulam stability.

Proof. Consider the sequence $\{u_j(x)\}$ defined by

$$u_j(x) = \prod_{k=0}^{j} \frac{1}{a(x+pk)}. \tag{2.2}$$

Note that

$$\begin{aligned}\limsup_{k\to\infty}\frac{u_k}{u_{k-1}} &= \limsup_{k\to\infty}\frac{1}{a(x+pk)}\\ &= \frac{1}{\liminf_{k\to\infty}a(x+pk)}\\ &< 1, \quad \forall x > n_0,\end{aligned} \tag{2.3}$$

by (2.1). By ratio test we see that the series (1.9) converges for all $x > n_0$. By [5, Theorem 2.1] we obtain the Hyers-Ulam stability. □

A similar idea to give conditions of stability by use of inferior limit was once taken in [7].

Example 2.2. It is easier to see that the gamma functional equation

$$g(x+1) = xg(x) \tag{2.4}$$

has the Hyers-Ulam stability because in this case $a(x) = x$ satisfies

$$\lim_{x\to+\infty} a(x) = +\infty \tag{2.5}$$

and condition (2.1) in Theorem 2.1 is satisfied.

Example 2.3. As in [9], the G-functional equation

$$g(x+1) = \Gamma(x)g(x) \tag{2.6}$$

has the Hyers-Ulam stability because we consider $a(x) = \Gamma(x)$, which obviously satisfies the same as in (2.5).

Similarly, (1.8) also has the Hyers-Ulam stability when $a(x) = x^r$ where the real $r > 0$ or $a(x) = \log x, \sinh x$, which are not power functions, because (2.5) holds in these cases.

Example 2.4. The functional equation

$$g(x+1) = \arctan x\, g(x) \tag{2.7}$$

has the Hyers-Ulam stability because in this case $a(x) = \arctan x$ satisfies

$$\lim_{x\to+\infty} a(x) = \frac{\pi}{2} > 1 \tag{2.8}$$

and condition (2.1) in Theorem 2.1 is satisfied.

Example 2.5. With notations that

$$[x] = \frac{1-q^x}{1-q}, \quad (x;q)_\infty = \prod_{n\geq 0}(1-xq^n), \tag{2.9}$$

where $q \in (0,1)$, the equation

$$g(x+1) = [x]g(x), \tag{2.10}$$

called q-Gamma functional equation, is considered in [1, 14]. On $\{x \in \mathbb{C} : \Re x > 0\}$ it has solutions

$$\begin{aligned}
\Gamma_q(x) &= \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty},\\
g_q(x) &= \int_0^{+\infty} \frac{(-t;q)_\infty (-qt^{-1};q)_\infty}{(-q^x t;q)_\infty (-q^{1-x}t^{-1};q)_\infty ((q-1)t;q)_\infty} \frac{dt}{t}.
\end{aligned} \tag{2.11}$$

In particular, the first one is called Jackson's q-Gamma function. Restricted to real line, namely to $(0,+\infty)$, this equation has the Hyers-Ulam stability because in this case $a(x) = [x]$ and

$$\lim_{x\to+\infty} [x] = \frac{1}{1-q} > 1, \tag{2.12}$$

which implies that condition (2.1) in Theorem 2.1 is satisfied.

Theorem 2.1 also provides a method to discuss cases of divergent $a(x)$.

Example 2.6. Consider the functional equation

$$g(x+1) = (b_0 + b_1 \sin x) g(x). \tag{2.13}$$

Although $a(x) = b_0 + b_1 \sin x$ oscillates when $x \to +\infty$, we still see that

$$\liminf_{x\to+\infty} a(x) = b_0 - b_1. \tag{2.14}$$

By Theorem 2.1, this equation has the Hyers-Ulam stability when $b_0 - b_1 > 1$.

Different from Example 2.6, in some cases the fact $\liminf_{x\to+\infty} a(x) > 1$ does not hold, but we can still discuss the Hyers-Ulam stability with Theorem 2.1.

Example 2.7. Consider the functional equation

$$g\left(x + \sqrt{2}\right) = a(x) g(x), \tag{2.15}$$

where

$$a(x) = \begin{cases} \dfrac{1}{2}, & x \in \mathbb{N}, \\ 2, & x \notin \mathbb{N}. \end{cases} \tag{2.16}$$

Clearly $\liminf_{x\to+\infty} a(x) = 1/2$, but

$$\liminf_{k\to\infty} a\left(x + \sqrt{2}k\right) = 2, \quad \forall x > 0. \tag{2.17}$$

By Theorem 2.1, this equation has the Hyers-Ulam stability.

3. On generalized Hyers-Ulam-Rassias stability

Theorem 3.1. *Consider the approximate solutions* $f : (0,+\infty) \to \mathbb{R}$ *of (1.8) which satisfy that* $|f(x+p) - a(x)f(x)| \leq \psi(x)$ *for all* $x > n_0$*, where* $\psi : (0,+\infty) \to (0,+\infty)$ *is a fixed function and* n_0 *is a nonnegative constant. If*

$$\liminf_{k\to\infty} \frac{\psi(x+p(k-1))}{\psi(x+pk)} a(x+pk) > 1, \quad \forall x > n_0, \tag{3.1}$$

then (1.8) has the generalized Hyers-Ulam-Rassias stability.

We omit the proof of Theorem 3.1 (it can be given similarly by ratio test as done for Theorem 2.1). Here we focus on various cases of $\psi(x)$:

(i) $\psi(x)$ is a polynomial,
(ii) $\psi(x) = \log_r x$, where $0 < r \neq 1$,
(iii) $\psi(x) = r^x$, where $0 < r \neq 1$,
(iv) $\psi(x)$ is bounded.

Corollary 3.2. *In cases (i) and (ii), (1.8) has the generalized Hyers-Ulam-Rassias stability if (2.1) holds. In case (iii), (1.8) has the generalized Hyers-Ulam-Rassias stability if* $\liminf_{k\to\infty} a(x+pk) > r^p$ *for all* $x > n_0$*. In case (iv), (1.8) has the generalized Hyers-Ulam-Rassias stability if* $\lim_{x\to+\infty} a(x) = +\infty$*.*

Proof. In fact, $\lim_{k\to\infty} \psi(x+p(k-1))/\psi(x+pk) = 1$ in case (i). In case (ii), we obtain the same by L'Hospital's rule. In case (iii), we note that $\psi(x+p(k-1))/\psi(x+pk) \equiv r^{-p}$ and the corresponding result follows. The result in case (iv) is obvious from Theorem 3.1. □

Remark that in the first three cases $\lim_{k\to\infty} \psi(x+p(k-1))/\psi(x+pk)$ converges but in case (iv) this limit may not exist.

Example 3.3. Consider $\psi(x) = \sin x$ for (1.8) where $a(x) = x$ and $a(x) = \Gamma(x)$ separately. They are in case (iv) of the corollary although $\lim_{k\to\infty} \psi(x+p(k-1))/\psi(x+pk)$ does not converge. Therefore both gamma functional equation and G-functional equation have the generalized Hyers-Ulam-Rassias stability with such an $\psi(x)$. Besides, the q-Gamma functional equation (2.10) can be considered in cases (i), (ii), and (iii), so it has the generalized Hyers-Ulam-Rassias stability with $\psi(x)$ in the forms of polynomial, logarithm, and exponential function r^x where $r < 1/(1-q)$.

4. On stability in the sense of Ger

Theorem 4.1. *Consider the approximate solutions* $f : (0,+\infty) \to \mathbb{R}$ *of (1.8) which satisfy that* $|f(x+p)/a(x)f(x) - 1| \leq \psi(x)$ *for all* $x > n_0$ *where* $\psi : (0,+\infty) \to (0,1)$ *is a fixed function and* n_0 *is a nonnegative constant. If*

$$\sum_{k=0}^{\infty} \psi(x+pk) < +\infty, \quad \forall x > n_0, \tag{4.1}$$

then (1.8) has the stability in the sense of Ger.

Proof. Condition (4.1) implies that $\prod_{j=0}^{+\infty}(1\pm\psi(x+pj))$ converges. Thus

$$\sum_{j=0}^{\infty}\log(1-\psi(x+pj))>-\infty,\quad \sum_{j=0}^{\infty}\log(1+\psi(x+pj))<+\infty,\quad \forall x>n_0,\tag{4.2}$$

that is, (1.11) holds. □

Remark that in Theorem 4.1 we do not require condition (1.9). This condition, required in [5, Theorem 3.2], is in fact unnecessary. In the proof of [5, Theorem 3.2] the convergence in (1.11) guarantees that $\{\log P_n(x)\}$ is a Cauchy sequence. Thus $L(x):=\lim_{n\to\infty}\log P_n(x)$ exists and so does $\lim_{n\to\infty}P_n(x)$. The restriction of $a(x)$ is given by the convergence in (1.11) and the range of ψ in $(0,1)$ because it is required that $|f(x+p)/a(x)f(x)-1|\le\psi(x)$.

Corollary 4.2. *Suppose that the function $\psi:(0,+\infty)\to(0,1)$ is continuous and decreasing such that*

$$\lim_{x\to+\infty}x^{\eta}\psi(x)=l\in[0,+\infty)\tag{4.3}$$

for some constant $\eta>1$. Then (1.8) has the stability in the sense of Ger.

Proof. Obviously,

$$\psi(x+p(k-1))\ge\int_{k-1}^{k}\psi(x+pt)\,dt\ge\psi(x+pk).\tag{4.4}$$

Taking summation, we obtain

$$\sum_{k=1}^{+\infty}\psi(x+p(k-1))\ge\int_{0}^{+\infty}\psi(x+pt)\,dt\ge\sum_{k=1}^{+\infty}\psi(x+pk).\tag{4.5}$$

It follows that the series $\sum_{k=1}^{+\infty}\psi(x+pk)$ and the integral $\int_0^{+\infty}\psi(x+pt)\,dt$ converge or diverge simultaneously. Clearly, (4.3) implies that the integral $\int_0^{+\infty}\psi(x+pt)\,dt$ converges and so does the series $\sum_{k=0}^{\infty}\psi(x+pk)$. Consequently, the result can be deduced from Theorem 4.1. □

Example 4.3. Consider the Gamma equation (2.4) and the function $f:(0,+\infty)\to(0,+\infty)$ satisfies the inequality

$$\left|\frac{f(x+1)}{xf(x)}-1\right|\le\frac{\delta}{x^{s}},\quad \forall x>\max\{n_0,\delta^{1/s}\},\tag{4.6}$$

where $s>1$, $n_0\ge0$, and $\delta>0$. Clearly $\psi(x):=\delta/x^{s}$ satisfies (4.3). Thus the gamma equation (2.4) has the stability in the sense of Ger with such a $\psi(x)$.

Example 4.4. Consider (2.7) as in Example 2.4 and the function $f:(0,+\infty)\to(0,+\infty)$ satisfies the inequality

$$\left|\frac{f(x+1)}{\arctan x f(x)}-1\right|\le r^{x},\quad \forall x>n_0,\tag{4.7}$$

where $0 < r < 1$ and $n_0 \geq 0$. Clearly, $\lim_{x\to+\infty} x^2 r^x = 0$. Hence (2.7) has the stability in the sense of Ger with the $\psi(x) := r^x$.

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