IJMMS 32:10 (2002) 635–640
PII. S0161171202112221
http://ijmms.hindawi.com

A NOTE ON THE SPECTRAL OPERATORS OF SCALAR TYPE AND SEMIGROUPS OF BOUNDED LINEAR OPERATORS

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Received 20 December 2001

It is shown that, for the spectral operators of scalar type, the well-known characterizations of the generation of C_0- and analytic semigroups of bounded linear operators can be reformulated exclusively in terms of the spectrum of such operators, the conditions on the resolvent of the generator being automatically met and the corresponding semigroup being that of the exponentials of the operator.

2000 Mathematics Subject Classification: 47B40, 47D03, 34G10, 47B15, 47D06.

1. Introduction. As known, the celebrated criteria of the generation of C_0- and analytic semigroups of bounded linear operators contain conditions on the geometry of the *spectrum* of the *generator* along with rather stringent restrictions on its *resolvent* behavior [6, 7, 12, 15].

For a *normal operator* in a complex Hilbert space, the restrictions on the resolvent can be dropped, being automatically satisfied when the conditions on the spectrum of the generator are met [6, 13].

If A is such an operator and $E_A(\cdot)$ is its *spectral measure* (*resolution of the identity*), the generated semigroup consists of its exponentials in the sense of the corresponding *operational calculus* [4, 13]

$$e^{tA} = \int_{\sigma(A)} e^{t\lambda}\, dE_A(\lambda), \quad t \geq 0. \tag{1.1}$$

It is the purpose of the present note to highlight the fact that the criteria acquire purely geometrical form in the more general case of a *spectral operator of scalar type* (*scalar operators*) in a complex Banach space [2, 5].

Observe for that matter that, in a Hilbert space, the *scalar operators* are the operators similar to *normal* ones [14].

2. Preliminaries. Henceforth, unless specifically stated otherwise, A is a scalar operator in a complex Banach space X with a norm $\|\cdot\|$ and $E_A(\cdot)$ is its spectral measure.

For such operators, there is an operational calculus for Borel measurable functions on the spectrum [2, 5].

If $F(\cdot)$ is a Borel measurable function on the spectrum of A, $\sigma(A)$, a new scalar operator

$$F(A) = \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda) \tag{2.1}$$

is defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \,\middle|\, \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}, \tag{2.2}$$

where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A) \mid |F(\lambda)|\le n\}}(\cdot), \quad n = 1,2,\ldots, \tag{2.3}$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α), and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1,2,\ldots, \tag{2.4}$$

being the integrals of bounded Borel measurable functions on $\sigma(A)$, are bounded scalar operators on X defined in the same way as for normal operators (see, e.g., [4, 13]).

The properties of the spectral measure $E_A(\cdot)$ and the operational calculus, which underlie the entire argument henceforth, are exhaustively delineated in [2, 5]. We just note here that, due to its strong countable additivity, the spectral measure $E_A(\cdot)$ is bounded, that is, there is an $M > 0$ such that

$$\|E_A(\delta)\| \le M, \quad \text{for any Borel set } \delta. \tag{2.5}$$

Observe that, in (2.5), the same notation as for the norm in X, $\|\cdot\|$, was used to designate the norm in the space of bounded linear operators on X, $\mathscr{L}(X)$. We will also adhere to this rather common economy of symbols in what follows for the norm in the dual space X^*.

On account of compactness, the terms *spectral measure* and *operational calculus* for spectral operators, repeatedly referred to, will be abbreviated to *s.m.* and *o.c.*, respectively.

3. C_0-semigroups

Proposition 3.1. *A scalar operator A in a complex Banach space X generates a C_0-semigroup of bounded linear operators if and only if, for some real ω,*

$$\operatorname{Re}\lambda \le \omega, \quad \lambda \in \sigma(A), \tag{3.1}$$

in which case the semigroup consists of the exponentials

$$e^{tA} = \int_{\sigma(A)} e^{t\lambda}\, dE_A(\lambda), \quad t \ge 0. \tag{3.2}$$

Proof. Condition (3.1), being a constituent of the general C_0-semigroup generation criterion [6, 7, 15], obviously remains necessary. Thus, we are only to prove its sufficiency.

From (3.1), we immediately infer that $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega\} \subseteq \rho(A)$, where $\rho(A)$ is the *resolvent set* of A.

Furthermore, for $z > \omega$, we have

$$\begin{aligned} \|R(z,A)^n\| &= \left\| \int_{\sigma(A)} (z-\lambda)^{-n} \, dE_A(\lambda) \right\| \quad \text{by the properties of the } \textit{o.c.} \\ &\le 4M \sup_{\lambda \in \sigma(A)} |z-\lambda|^{-n} \quad \text{where } M \text{ is a constant from (2.5)} \\ &= 4M[\operatorname{dist}(z,\sigma(A))]^{-n} \quad \text{by (3.1)} \\ &\le \frac{4M}{(z-\omega)^n}. \end{aligned} \tag{3.3}$$

Whence, by the C_0-semigroup generation criterion (see, e.g., [6]), it follows that A generates a certain C_0-semigroup of bounded linear operators $\{T(t) \mid t \ge 0\}$.

From condition (3.1), we also obtain the estimate

$$|e^{t\lambda}| = e^{t \operatorname{Re}\lambda} \le e^{\omega t}, \quad t \ge 0, \ \lambda \in \sigma(A), \tag{3.4}$$

which, by the properties of the *o.c.*, implies that exponentials (3.2) form a semigroup of bounded linear operators.

To prove that $\{e^{tA} \mid t \ge 0\}$ is a C_0-semigroup, it is enough to demonstrate its continuity at 0 in the *weak sense* [15].

For any $f \in X$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} |\langle e^{tA}f - f, g^*\rangle| &= \left| \int_{\sigma(A)} (e^{t\lambda}-1) \, d\langle E_A(\lambda)f, g^*\rangle \right| \\ &\le \int_{\sigma(A)} |e^{t\lambda}-1| \, dv(f,g^*,\lambda), \end{aligned} \tag{3.5}$$

by the properties of the *o.c.*, $\langle \cdot, \cdot \rangle$ being the *pairing* between the space X and its dual, X^*, where $v(f,g^*,\cdot)$, the *total variation* of the complex-valued Borel measure $\langle E_A(\cdot)f, g^*\rangle$, is a *finite* positive Borel measure.

By the Lebesgue Dominated Convergence theorem, whose conditions as easily seen are readily met, the latter integral approaches 0 as $t \to 0$.

Finally, we need to prove that $T(t) = e^{tA}$, $t \ge 0$.

We first show that, for arbitrary $f \in D(A)$ and $g^* \in X^*$,

$$\frac{d}{dt}\langle e^{tA}, f^*\rangle = \langle Ae^{tA}f, g^*\rangle, \quad t \ge 0. \tag{3.6}$$

Note that, by the properties of the *o.c.*, for all $f \in D(A)$,

$$e^{At}f \in D(A), \quad Ae^{tA}f = e^{tA}Af, \quad t \ge 0. \tag{3.7}$$

Fix a $t \ge 0$ and choose a small segment $[a,b] \subset [0,\infty)$ so that t is its left endpoint if $t = 0$ and the midpoint otherwise. For all sufficiently small increments $\Delta t \ne 0$ such that $t+\Delta t \in [a,b]$, and arbitrary $f \in D(A)$ and $g^* \in X^*$, we have

$$\begin{aligned}
&\left|\left\langle \frac{e^{(t+\Delta t)A}f - e^{tA}f}{\Delta t} - Ae^{tA}f, g^*\right\rangle\right| \quad \text{by the properties of the } \textit{o.c.} \\
&\qquad = \left|\int_{\sigma(A)}\left[\frac{e^{(t+\Delta t)\lambda}-e^{t\lambda}}{\Delta t} - \lambda e^{t\lambda}\right] d\langle E_A(\lambda)f, g^*\rangle\right| \\
&\qquad \le \int_{\sigma(A)}\left|\frac{e^{(t+\Delta t)\lambda}-e^{t\lambda}}{\Delta t} - \lambda e^{t\lambda}\right| dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0,
\end{aligned} \tag{3.8}$$

by the Lebesgue Dominated Convergence theorem,

$$\left|\frac{(e^{(t+\Delta t)\lambda}-e^{t\lambda})}{\Delta t} - \lambda e^{t\lambda}\right| \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0,\ \lambda \in \sigma(A). \tag{3.9}$$

For $\lambda \in \sigma(A)$,

$$\begin{aligned}
&\left|\frac{e^{(t+\Delta t)\lambda}-e^{t\lambda}}{\Delta t} - \lambda e^{t\lambda}\right| \quad \text{by the total change theorem} \\
&\quad \le 2\max_{a\le s\le b}\left|\lambda e^{s\lambda}\right| = 2\max_{a\le s\le b} e^{s\operatorname{Re}\lambda}|\lambda| \quad \text{by (3.1)} \\
&\quad \le 2\max_{a\le s\le b} e^{s\omega}|\lambda| \quad \text{without restricting generality, } \omega \text{ can be regarded to be } \textit{positive} \\
&\quad \le 2e^{b\omega}|\lambda|.
\end{aligned} \tag{3.10}$$

Since $f \in D(A)$, $\int_{\sigma(A)} |\lambda|\, dv(f,g^*,\lambda) < \infty$ for any $g^* \in X^*$ [3, 5]. Thus, for any $f \in D(A)$ and $g^* \in D(A^*)$, where A^* is the *adjoint* of A ($D(A)$ is *dense* in X),

$$\frac{d}{dt}\langle e^{tA}f, g^*\rangle = \langle Ae^{tA}f, g^*\rangle = \langle e^{tA}f, A^*g^*\rangle, \quad t \ge 0, \tag{3.11}$$

that is, $e^{\cdot A}f$ is a *weak solution* of the equation

$$y'(t) = Ay(t) \tag{3.12}$$

on $[0,\infty)$ in the sense of [1].

Since A generates the C_0-semigroup $\{T(t) \mid t \ge 0\}$, such a solution is unique for any $f \in X$ and is the orbit $T(t)f$, $t \ge 0$ [1].

Therefore, for any $f \in D(A)$: $e^{tA}f = T(t)f$, $t \ge 0$. Whence, by the density of $D(A)$ in X, we conclude that

$$e^{tA} = T(t), \quad t \ge 0. \tag{3.13}$$

□

4. Analytic semigroups

Proposition 4.1. *A scalar operator A in a complex Banach space X generates an analytic semigroup of bounded linear operators if and only if, for some real ω and $0<\theta<\pi/2$,*

$$\sigma(A) \subseteq \{z \in \mathbb{C} \mid |\arg(z-\omega)| \geq \pi/2+\theta\}, \tag{4.1}$$

where $\arg\cdot$ is the principal value of the argument from the interval $(-\pi,\pi]$.

The semigroup is analytically continuable into the sector $\Sigma_\theta = \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ by the formula

$$e^{zA} = \int_{\sigma(A)} e^{z\lambda}\, dE_A(\lambda), \quad z \in \Sigma_\theta. \tag{4.2}$$

Proof. Condition (4.1) is necessary since it is a constituent of the general criterion of generation of analytic semigroup [6, 7, 15]. Thus, we need to validate its sufficiency only.

First, we infer that $\rho(A) \subseteq \{z \in \mathbb{C} \mid |\arg(z-\omega)| < \pi/2+\theta\}$.

For an arbitrary $0<\varepsilon<\theta$ and any z such that $|\arg(z-\omega)| < \pi/2+\theta-\varepsilon$, there are two possibilities:

(a) $\pi/2-\theta < |\arg(z-\omega)| \leq \pi/2+\theta-\varepsilon$,

(b) $|\arg(z-\omega)| \leq \pi/2-\theta$.

In the first case,

$$\begin{aligned} \|R(z,A)\| &= \left\| \int_{\sigma(A)} (z-\lambda)^{-1}\, dE_A(\lambda) \right\| \quad \text{by the properties of the } \textit{o.c.} \\ &\leq 4M \sup_{\lambda\in\sigma(A)} |z-\lambda|^{-1} \quad \text{where } M \text{ is a constant from (2.5)} \\ &= 4M\left[\operatorname{dist}(z,\sigma(A))\right]^{-1} \leq \frac{4M}{|z-\omega|\sin\varepsilon}. \end{aligned} \tag{4.3}$$

In the second case,

$$\begin{aligned} \|R(z,A)\| &= \left\| \int_{\sigma(A)} (z-\lambda)^{-1}\, dE_A(\lambda) \right\| \quad \text{by the properties of the } \textit{o.c.} \\ &\leq 4M \sup_{\lambda\in\sigma(A)} |z-\lambda|^{-1} = 4M\left[\operatorname{dist}(z,\sigma(A))\right]^{-1} \leq \frac{4M}{|z-\omega|}. \end{aligned} \tag{4.4}$$

Thus, for any $0<\varepsilon<\theta$,

$$\|R(z,A)\| \leq \frac{4M\csc\varepsilon}{|z-\omega|} \quad \text{whenever } |\arg(z-\omega)| < \frac{\pi}{2}+\theta-\varepsilon. \tag{4.5}$$

Condition (4.1) and the latter estimate imply that A generates an analytic semigroup $\{T(t) \mid t \geq 0\}$ (see, e.g., [6]).

The fact that the exponentials

$$e^{zA} = \int_{\sigma(A)} e^{z\lambda}\, dE_A(\lambda), \quad |\arg z| < \theta, \tag{4.6}$$

are bounded linear operators with the semigroup property can be easily inferred from (4.1). And the fact that $e^{tA} = T(t)$, $t \geq 0$, can be substantiated in the same way as in the case of C_0-semigroups above. □

5. Some final remarks. The author intentionally did not entertain the idea of developing here similar arguments for *differentiable* C_0-semigroups [6, 12] and, for that matter, the C_0-semigroups with orbits belonging to the *Gevrey* or, more generally, *Carleman* classes of *ultradifferentiable* vector functions leaving that for a discussion in a more general context (for the case of the *normal operators*, see [8, 9, 10, 11]).

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