IJMMS 32:5 (2002) 313–318
PII. S0161171202202288
http://ijmms.hindawi.com

SP-CLOSEDNESS IN L-FUZZY TOPOLOGICAL SPACES

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Received 20 February 2002

We introduce and study SP-closedness in L-fuzzy topological spaces, where L is a fuzzy lattice. SP-closedness is defined for arbitrary L-fuzzy subsets.

2000 Mathematics Subject Classification: 54Axx.

1. Introduction. Andrijević [1] introduced the definition of semi-preopen sets in general topological spaces. Thakur and Singh [8] extended this definition to fuzzy topological spaces. In [4], using semi-preopen sets, we have introduced and studied a good definitions of semi-precompactness in L-fuzzy topological spaces.

In this note, along the lines of this semi-precompactness, we introduce a definition of SP-closedness in L-fuzzy topological spaces. Also, we obtain some of its properties. SP-closedness is defined for arbitrary L-fuzzy subsets. It is a weaker form of semi-precompactness, but it is a stronger form of P-closedness [3] and S^*-closedness [7].

2. Preliminaries. Throughout this note, X and Y will be nonempty ordinary sets, and $L = L(\leq, \vee, \wedge, ')$ will denote a fuzzy lattice, that is, a completely distributive lattice with a smallest element 0 and largest element 1 and with an order reversing involution $a \to a'$ $(a \in L)$. We will denote by L^X the lattice of all L-fuzzy subsets of X.

Definition 2.1 (Gierz et al. [6])**.** An element p of L is called prime if and only if $p \neq 1$, and whenever $a, b \in L$ with $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $\mathrm{pr}(L)$.

Definition 2.2 (Gierz et al. [6])**.** An element α of L is called union irreducible if and only if whenever $a, b \in L$ with $\alpha \leq a \vee b$, then $\alpha \leq a$ or $\alpha \leq b$. The set of all nonzero union-irreducible elements of L will be denoted by $M(L)$. It is obvious that $p \in \mathrm{pr}(L)$ if and only if $p' \in M(L)$.

Warner [9] has determined the prime element of the fuzzy lattice L^X. We have $\mathrm{pr}(L^X) = \{x_p : x \in X \text{ and } p \in \mathrm{pr}(L)\}$, where, for each $x \in X$ and each $p \in \mathrm{pr}(L)$, $x_p : X \to L$ is the L-fuzzy set defined by

$$x_p(y) = \begin{cases} p & \text{if } y = x, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

These x_p are called the *L-fuzzy points* of X, and we say that x_p is a member of an L-fuzzy set f and write $x_p \in f$ if and only if $f(x) \not\leq p$.

Thus, the union-irreducible elements of L^X are the function $x_\alpha : X \to L$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where $x \in X$ and $\alpha \in M(L)$. Hence, we have $M(L^X) = \{x_\alpha : x \in X \text{ and } \alpha \in M(L)\}$. As these x_α are identified with the L-fuzzy points x_p of X, we will refer to them as fuzzy points. When $x_\alpha \in M(L^X)$, we will call x and α the support of x_α ($x = \operatorname{Supp} x_\alpha$) and the height of x_α ($\alpha = h(x_\alpha)$), respectively. We will denote L-fuzzy topological space by L-fts.

Definition 2.3 (Zhao [10])**.** Let (X,δ) be an L-fts. A net in (X,δ) is a mapping $S : D \to M(L^X)$, where D is a directed set. For $m \in D$, we will denote $S(m)$ by S_m, and the net S by $(S_m)_{m\in D}$. If $A \in L^X$ and for each $m \in D$, $S_m \le A$, then S is called a net in A. A net $(S_m)_{m\in D}$ is called an α-net ($\alpha \in M(L)$) if, for each $\lambda \in \beta^*(\alpha)$ (where $\beta^*(\alpha)$ denotes the union of all minimal sets relative to α), the net $h(S) = (h(S_m))_{m\in D}$ is eventually greater than λ, that is, for each $\lambda \in \beta^*(\alpha)$, there is $m_0 \in D$ such that $h(S_m) \ge \lambda$ whenever $m \ge m_0$, where $h(S_m)$ is the height of L-fuzzy point $S_m \in M(L^X)$. If $h(S_m) = \alpha$ for all $m \in D$, then we will say that $(S_m)_{m\in D}$ is a constant α-net.

Definition 2.4 (Thakur and Singh [8])**.** Let (X,δ) be an L-fts and $f \in L^X$. Then, f is called semi-preopen if and only if there is a preopen set g [3, 5] such that $g \le f \le g^-$ and semi-preclosed if and only if f' is semi-preopen. $f_\square = \bigvee\{g : g \text{ is semi-preopen}, g \le f\}$ and $f_\frown = \bigwedge\{g : g \text{ is semi-preclosed}, g \ge f\}$ are called the semi-preinterior and semi-preclosure of f, respectively.

It is clear that every semi-open L-fuzzy set is semi-preopen and every preopen L-fuzzy set is semi-preopen. None of the converses needs to be true [9].

Definition 2.5 (Aygün [2])**.** Let (X,δ) be an L-fts and $g \in L^X$, $r \in L$. A collection $\mu = \{f_i\}_{i\in J}$ of L-fuzzy sets is called an r-level cover of g if and only if $(\bigvee_{i\in J} f_i)(x) \not\le r$ for all $x \in X$ with $g(x) \ge r'$. If each f_i is open, then μ is called an r-level open cover of g. If g is the whole space X, then μ is called an r-level cover of X if and only if $(\bigvee_{i\in J} f_i)(x) \not\le r$ for all $x \in X$. An r-level cover $\mu = \{f_i\}_{i\in J}$ of g is said to have a finite r-level subcover if there exists a finite subset F of J such that $(\bigvee_{i\in F} f_i)(x) \not\le r$ for all $x \in X$ with $g(x) \ge r'$.

Definition 2.6 (Bai [4])**.** Let (X,δ) be an L-fts and $g \in L^X$. We call g semi-precompact if and only if every p-level semi-preopen cover of g has a finite p-level subcover, where $p \in \operatorname{pr}(L)$. If g is the whole space, then we say that the L-fts (X,δ) is semi-precompact.

3. SP-closedness

Definition 3.1. Let (X,δ) be an L-fts and let $g \in L^X$, $r \in L$. An r-level cover $\mu = \{f_i\}_{i\in J}$ of g is said to have a finite $r_\frown$-level subcover if there exists a finite subset F of J such that $(\bigvee_{i\in F} (f_i)_\frown)(x) \not\le r$ for all $x \in X$ with $g(x) \ge r'$.

Definition 3.2. Let (X,δ) be an L-fts and let $g \in L^X$. We call g SP-closed if and only if every p-level semi-preopen cover of g has a finite $p_{\frown}$-level subcover, where $p \in \mathrm{pr}(L)$. If g is the whole space, then we say that the L-fts (X,δ) is SP-closed.

Theorem 3.3. *Every semi-precompact set is SP-closed in an L-fts.*

Proof. This immediately follows from Definitions 2.6 and 3.2. □

Theorem 3.4. *Every SP-closed set is not only P-closed* [3] *but also S^*-closed* [7] *in an L-fts.*

Proof. Since every preopen L-fuzzy set is semi-preopen and every semiopen L-fuzzy set is semi-preopen, and since for every L-fuzzy set f we have $f_{\frown} \leq f^{\frown}$ and $f_{\frown} \leq f_{-}$, where $f^{\frown} = \bigwedge\{g : g \text{ is preclosed}, g \geq f\}$ and $f_{-} = \bigwedge\{g : g \text{ is semiclosed}, g \geq f\}$, this directly follows from the definitions of SP-closedness, P-closedness, and S^*-closedness. □

Theorem 3.5. *Let (X,δ) be an L-fts. Then, $g \in L^X$ is SP-closed if and only if, for every $\alpha \in M(L)$ and every collection $(h_i)_{i\in J}$ of semi-preclosed L-fuzzy sets with $(\bigwedge_{i\in J} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there is a finite subset F of J such that $(\bigwedge_{i\in F}(h_i)_{\square})(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.*

Proof. This follows immediately from Definition 3.2. □

Definition 3.6. Let (X,δ) be an L-fts, x_α be an L-fuzzy point in $M(L^X)$, and $S = (S_m)_{m\in D}$ be a net. We call x_α an SP-cluster point of S if and only if, for each semi-preclosed L-fuzzy set f with $f(x) \not\geq \alpha$ and for all $n \in D$, there is $m \in D$ such that $m \geq n$ and $S_m \not\leq f_{\square}$, that is, $h(S_m) \not\leq f_{\square}(\mathrm{Supp}\, S_m)$.

Theorem 3.7. *Let (X,δ) be an L-fts. Then, $g \in L^X$ is SP-closed if and only if every constant α-net in g, where $\alpha \in M(L)$, has an SP-cluster point in g with height α.*

Proof

Necessity. Let $\alpha \in M(L)$ and $S = (S_m)_{m\in D}$ be a constant α-net in g without any SP-cluster point with height α in g. Then, for each $x \in X$ with $g(x) \geq \alpha$, x_α is not an SP-cluster point of S, that is, there are $n_x \in D$ and a semi-preclosed L-fuzzy set f_x with $f_x(x) \not\geq \alpha$ and $S_m \leq (f_x)_{\square}$ for each $m \geq n_x$. Let $x^1,\dots,x^k$ be elements of X with $g(x^i) \geq \alpha$ for each $i \in \{1,\dots,k\}$. Then, there are $n_{x_1},\dots,n_{x_k} \in D$, semi-preclosed L-fuzzy set f_{x_i} with $f_{x_i}(x^i) \not\geq \alpha$, and $S_m \leq (f_{x_i})_{\square}$ for each $m \geq n_{x_i}$ and for each $i \in \{1,\dots,k\}$. Since D is a directed set, there is $n_o \in D$ such that $n_o \geq n_{x_i}$ for each $i \in \{1,\dots,k\}$ and $S_m \leq (f_{x_i})_{\square}$ for $i \in \{1,\dots,k\}$ and each $m \geq n_o$. Now, consider the family $\mu = \{f_x\}_{x\in X}$ with $g(x) \geq \alpha$. Then, $(\bigwedge_{f_x\in\mu} f_x)(y) \not\geq \alpha$ for all $y \in X$ with $g(y) \geq \alpha$ because $f_y(y) \not\geq \alpha$. Also, for any finite subfamily $\nu = \{f_{x_1},\dots,f_{x_k}\}$ of μ, there is $y \in X$ with $g(y) \geq \alpha$ and $(\bigwedge_{i=1}^{k}(f_{x_i})_{\square})(y) \geq \alpha$ since $S_m \leq \bigwedge_{i=1}^{k}(f_{x_i})_{\square}$ for each $m \geq n_o$ because $S_m \leq (f_{x_i})_{\square}$ for each $i \in \{1,\dots,k\}$ and for each $m \geq n_o$. Hence, by Theorem 3.5, g is not SP-closed.

Sufficiency. Suppose that g is not SP-closed. Then by Theorem 3.5, there exist $\alpha \in M(L)$ and a collection $\mu = (f_i)_{i\in J}$ of semi-preclosed L-fuzzy sets with $(\bigwedge_{i\in J} f_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, but for any finite subfamily ν of μ, there is $x \in X$ with

$g(x) \geq \alpha$ and $(\bigwedge_{f\in\nu}(f_i)_{\square})(x) \geq \alpha$. Consider the family of all finite subsets of μ, $2^{(\mu)}$, with the order $\nu_1 \leq \nu_2$ if and only if $\nu_1 \subset \nu_2$. Then $2^{(\mu)}$ is a directed set. So, writing x_α as S_ν for every $\nu \in 2^{(\mu)}$, $(S_\nu)_{\nu\in 2^{(\mu)}}$ is a constant α-net in g because the height of S_ν for all $\nu \in 2^{(\mu)}$ is α and $S_\nu \leq g$ for all $\nu \in 2^{(\mu)}$, that is, $g(x) \geq \alpha$. Also, $(S_\nu)_{\nu\in 2^{(\mu)}}$ satisfies the condition that for each semi-preclosed L-fuzzy set $f_i \in \nu$ we have $x_\alpha = S_\nu \leq (f_i)_{\square}$. Let $y \in X$ with $g(y) \geq \alpha$. Then $(\bigwedge_{i\in J} f_i)(y) \not\geq \alpha$, that is, there exists $j \in J$ with $f_j(y) \not\geq \alpha$. Let $\nu_o = \{f_j\}$. So, for any $\nu \geq \nu_o$,

$$S_\nu \leq \bigwedge_{f_i\in\nu} (f_i)_{\square} \leq \bigwedge_{f_i\in\nu_o} (f_i)_{\square} = (f_j)_{\square}. \tag{3.1}$$

Thus, we get a semi-preclosed L-fuzzy set f_j with $f_j(y) \geq \alpha$ and $\nu_o \in 2^{(\mu)}$ such that for any $\nu \geq \nu_o$, $S_\nu \leq (f_j)_{\square}$. That means that $y_\alpha \in M(L^X)$ is not an SP-cluster point $(S_\nu)_{\nu\in 2^{(\mu)}}$ for all $y \in X$ with $g(y) \geq \alpha$. Hence, the constant α-net $(S_\nu)_{\nu\in 2^{(\mu)}}$ has no SP-cluster point in g with height α. □

Corollary 3.8. *An L-fts (X,δ) is SP-closed if and only if every constant α-net in (X,δ) has an SP-cluster point with height α, where $\alpha \in M(L)$.*

Theorem 3.9. *Let (X,δ) be an L-fts and $g,h \in L^X$. If g and h are SP-closed, then $g \vee h$ is SP-closed as well.*

Proof. Let $\{f_i\}_{i\in J}$ be a p-level semi-preopen cover of $g\vee h$, where $p \in \mathrm{pr}(L)$. Then, $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x \in X$ with $(g \vee h)(x) \geq p'$. Since p is prime, we have $(g \vee h)(x) \geq p'$ if and only if $g(x) \geq p'$ or $h(x) \geq p'$. So, by the SP-closedness of g and h, there are finite subsets E, F of J such that $(\bigvee_{i\in E}(f_i)_{\frown})(x) \not\leq P$ for all $x \in X$ with $g(x) \geq p'$ and $(\bigvee_{i\in F}(f_i)_{\frown})(x) \not\leq P$ for all $x \in X$ with $h(x) \geq p'$. Then, $(\bigvee_{i\in E\cup F}(f_i)_{\frown})(x) \not\leq P$ for all $x \in X$ with $g(x) \geq p'$ or $h(x) \geq p'$, that is, $(\bigvee_{i\in E\cup F}(f_i)_{\frown})(x) \not\leq P$ for all $x \in X$ with $(g\vee h)(x) \geq p'$. Thus, $g \vee h$ is SP-closed. □

Theorem 3.10. *Let (X,δ) be an L-fts and $g,h \in L^X$. If g is SP-closed and h is semi-preclopen, then $g \wedge h$ is SP-closed.*

Proof. Let $\{f_i\}_{i\in J}$ be a p-level semi-preopen cover of $g\wedge h$, where $p \in \mathrm{pr}(L)$. Then, $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x \in X$ with $(g \wedge h)(x) \geq p'$. Thus, $\mu = \{f_i\}_{i\in J} \cup \{h'\}$ is a p-level semi-preopen cover of g. In fact, for each $x \in X$ with $g(x) \geq p'$, if $h(x) \geq p'$, then $(g \wedge h)(x) \geq p'$, which implies that $(\bigvee_{i\in J} f_i)(x) \not\leq p$, thus $(\bigvee_{k\in\mu} k)(x) \not\leq p$. If $h(x) \not\geq p'$ then $h'(x) \not\leq p$ which implies $(\bigvee_{k\in\mu} k)(x) \not\leq p$. From the SP-closedness of g, there is a finite subfamily ν of μ, say $\nu = \{f_1,\dots,f_n,h'\}$ with $(\bigvee_{k\in\nu} k_{\frown})(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. Then, $(\bigvee_{i=1}^{n}(f_i)_{\frown})(x) \not\leq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. In fact, if $(g\wedge h)(x) \geq p'$, then $g(x) \geq p'$, hence $(\bigvee_{k\in\nu} k_{\frown})(x) \not\leq p$. So, there is $k \in \nu$ such that $k_{\frown}(x) \not\leq p$. Moreover, $h(x) \geq p'$ as well, that is, $h'(x) \leq p$. Since h is semi-preopen, then h' is semi-preclosed, that is, $h' = (h')_{\frown}$. So, $h'(x) \leq p$ implies that $(h')_{\frown}(x) \leq p'$. Consequently, $(\bigvee_{i=1}^{n}(f_i)_{\frown})(x) \not\leq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. Hence, $g \wedge h$ is SP-closed. □

Corollary 3.11. *Let (X,δ) be an SP-closed space and g be a semi-preclopen L-fuzzy set. Then g is SP-closed.*

Definition 3.12. Let (X,δ) and (Y,τ) be L-fts's. A function $f:(X,\delta)\to(Y,\tau)$ is called

(1) semi-preirresolute if and only if $f^{-1}(g)$ is semi-preopen in (X,δ) for each semi-preopen L-fuzzy set g in (Y,τ);

(2) weakly semi-preirresolute if and only if $f^{-1}(g)\le(f^{-1}(g_{\frown}))_{\square}$ for each semi-preopen L-fuzzy set g in (Y,τ).

Theorem 3.13. *Let $f:(X,\delta)\to(Y,\tau)$ be a semi-preirresolute mapping with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is SP-closed in (X,δ), then $f(g)$ is SP-closed in (Y,τ) as well.*

Proof. Let $\{f_i\}_{i\in J}$ be a p-level semi-preopen cover of $f(g)$, where $p\in\mathrm{pr}(L)$. Because f is semi-preirresolute, $\{f^{-1}(f_i)\}_{i\in J}$ is a p-level semi-preopen cover of g. By the SP-closedness of g, $\{f^{-1}(f_i)\}_{i\in J}$ has a finite $p_{\frown}$-level subcover, that is, there is a finite subset F of J such that $(\bigvee_{i\in F}(f^{-1}(f_i))_{\frown})(x)\not\le p$ for all $x\in X$ with $g(x)\ge p'$. We are going to show that $\{f_i\}_{i\in J}$ has a finite $p_{\frown}$-level subcover of $f(g)$, that is, $(\bigvee_{i\in F}(f_i)_{\frown})(y)\not\le p$ for all $y\in Y$ with $f(g)(y)\ge p'$. Since $f^{-1}(y)$ is finite for every $y\in Y$, $f(g)(y)\ge p'$ implies that there is $x\in X$ with $g(x)\ge p'$ and $f(x)=y$. Again, f is semi-preirresolute. Thus, we have

$$\begin{aligned}\Big(\bigvee_{i\in F}(f_i)_{\frown}\Big)(y)&=\Big(\bigvee_{i\in F}(f_i)_{\frown}\Big)(f(x))=\Big(\bigvee_{i\in F}f^{-1}((f_i)_{\frown})\Big)(x)\\&=\Big(\bigvee_{i\in F}(f^{-1}((f_i)_{\frown}))_{\frown}\Big)(x)\ge\Big(\bigvee_{i\in F}(f^{-1}(f_i))_{\frown}\Big)(x)\not\le p.\end{aligned}\tag{3.2}$$

This has proved that $\{f_i\}_{i\in J}$ has a finite $p_{\frown}$-level subcover of $f(g)$. Hence, $f(g)$ is SP-closed. □

Theorem 3.14. *Let $f:(X,\delta)\to(Y,\tau)$ be a weakly semi-preirresolute mapping with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is semi-precompact in (X,δ), then $f(g)$ is SP-closed in (Y,τ).*

Proof. Let $\{f_i\}_{i\in J}$ be a p-level semi-preopen cover of $f(g)$, where $p\in\mathrm{pr}(L)$. Because f is weakly semi-preirresolute, for every $i\in J$, $f^{-1}(f_i)\le(f^{-1}((f_i)_{\frown}))_{\square}$. Then, $\{(f^{-1}((f_i)_{\frown}))_{\square}\}_{i\in J}$ is a p-level semi-preopen cover of g. By the semi-precompactness of g, $\{(f^{-1}((f_i)_{\frown}))_{\square}\}_{i\in J}$ has a finite p-level subcover, that is, there is a finite subset F of J such that $(\bigvee_{i\in F}(f^{-1}((f_i)_{\frown}))_{\square})(x)\not\le p$ for all $x\in X$ with $g(x)\ge p'$.

We are going to show that $\{f_i\}_{i\in J}$ has a finite $p_{\frown}$-level subcover of $f(g)$, that is, $(\bigvee_{i\in F}(f_i)_{\frown})(y)\not\le p$ for all $y\in Y$ with $f(g)(y)\ge p'$. In fact, if $f(g)(y)\ge p'$ and since $f^{-1}(y)$ is finite for every $y\in Y$, there is $x\in X$ with $g(x)\ge p'$ and $f(x)=y$. So,

$$\begin{aligned}\Big(\bigvee_{i\in F}(f_i)_{\frown}\Big)(y)&=\Big(\bigvee_{i\in F}(f_i)_{\frown}\Big)(f(x))=\Big(\bigvee_{i\in F}f^{-1}((f_i)_{\frown})\Big)(x)\\&\ge\Big(\bigvee_{i\in F}(f^{-1}((f_i)_{\frown}))_{\square}\Big)(x)\not\le p.\end{aligned}\tag{3.3}$$

Hence, $f(g)$ is SP-closed. □

Acknowledgment. This work is supported by the National Natural Science Foundation of China and the Provincial Natural Science Foundation of Guangdong.

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