IJMMS 32:6 (2002) 383–386
PII. S0161171202007901
http://ijmms.hindawi.com

COHOMOLOGY WITH BOUNDS AND CARLEMAN ESTIMATES FOR THE $\bar{\partial}$-OPERATOR ON STEIN MANIFOLDS

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Received 20 June 2001

Cohomology with bounds are used to globalize a result of Hörmander obtaining Carleman estimates for the Cauchy-Riemann operator on Stein manifolds.

2000 Mathematics Subject Classification: 32C35, 32E10, 35N15.

1. Introduction. In [7] Hörmander proved the following theorems.

Theorem 1.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $f \in L^2_{(p,q)}(\Omega)$ be a $\bar{\partial}$-closed (p,q)-form, $q \geq 1$, then there is a $(p,q-1)$-form $u \in L^2_{(p,q-1)}(\Omega)$ such that $\bar{\partial}u = f$ and*

$$\|u\|_{L^2_{(p,q-1)}(\Omega)} \leq K\|f\|_{L^2_{(p,q)}(\Omega)}, \tag{1.1}$$

where K is a constant depending on the diameter of Ω.

Actually the above theorem was contained in the following.

Theorem 1.2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain, φ any plurisubharmonic function on Ω, and $f \in L^2_{(p,q)}(\Omega,\varphi)$ a $\bar{\partial}$-closed (p,q)-form, $q \geq 1$, then there is a $(p,q-1)$-form $u \in L^2_{(p,q-1)}(\Omega,\varphi)$ such that $\bar{\partial}u = f$ and*

$$\|u\|_{L^2_{(p,q-1)}(\Omega,\varphi)} \leq K\|f\|_{L^2_{(p,q-1)}(\Omega,\varphi)}, \tag{1.2}$$

where again K depends on the diameter of Ω.

These theorems turned out to be very useful in complex analysis and their applications include the Levi problem with bounds and cohomology with bounds. It is, therefore, natural to seek to generalize these theorems to manifolds. In [5], Theorem 1.1 was so generalized and we generalize Theorem 1.2 in this paper.

The term *Carleman estimates* refers to the estimates in Theorem 1.2 and we use Theorem 1.2 to obtain a Leray's isomorphism theorem with bounds on Stein manifolds, combining with weak elliptic estimates to get the generalization of Theorem 1.2 to Stein manifolds.

2. Preliminaries

2.1. Let X be an n-dimensional complex manifold with a C^∞-Hermitian metric and $\Omega \Subset X$ a relatively compact Stein subdomain of X. Where φ is any plurisubharmonic function on Ω, the scalar product

$$(f,g)=\int_{\Omega}e^{-\varphi}f\Lambda * \bar{g} \tag{2.1}$$

makes the space $L^2_{(p,q)}(\Omega,\varphi)=\{f \text{ measurable on } \Omega : \int_{\Omega}e^{-\varphi}f\Lambda * \bar{f}<\infty\}$ a Hilbert space, where $*$ is the Hodge $*$-operator associated with the metric and the orientation on X.

Our result is as follows.

Theorem 2.1. *Let $f\in L^2_{(p,q)}(\Omega,\varphi)$ be $\bar{\partial}$-closed in the sense of distributions. Then there is a $u\in L^2_{(p,q-q)}(\Omega,\varphi)$ such that $\bar{\partial}u=f$ in the sense of distributions and*

$$\|u\|_{L^2_{(p,q-1)}(\Omega,\varphi)}\le K\|f\|_{L^2_{(p,q)}(\Omega,\varphi)},\quad q>0, \tag{2.2}$$

where K depends on Ω.

2.2. Let U be a bounded open set in $\mathbb{C}^n$, $\mathbb{O}$ the structure sheaf of $\mathbb{C}^n$. A section $f=(f_1,\dots,f_p)\in\Gamma(U,\mathbb{O}^p)$, where p is an integer, is L^2_{φ}-bounded if

$$\|f\|_{L^2(U,\varphi)}=||f_1||_{L^2(U,\varphi)}+\cdots+||f_p||_{L^2(U,\varphi)}<\infty, \tag{2.3}$$

where φ is a plurisubharmonic function in U. We then denote all sections of $\mathbb{O}^p$ over U that are L^2_{φ}-bounded by $\Gamma_{\varphi}(U,\mathbb{O}^p)$.

For the definition of L^2_{φ}-bounded sections of coherent analytic sheaves, we require the coherent analytic sheaf $\mathcal{F}$ to be defined on a simply connected polycylinder neighborhood V of the closure of U. Then there is an $\mathbb{O}$-homomorphism in another simply connected polycylinder neighborhood V^1 of the closure of U

$$\mathbb{O}^p\xrightarrow{\lambda}\mathcal{F}\longrightarrow 0, \tag{2.4}$$

where $p>0$ is some integer, and $f\in\Gamma(U,\mathcal{F})$ is L^2_{φ}-bounded if $f\in\Gamma_{\varphi}(U,\mathcal{F}):=\lambda(\Gamma_{\varphi}(U,\mathbb{O}^p))$. It can be shown, as is done in [2], that $\Gamma_{\varphi}(U,\mathcal{F})$ is independent of λ and p, so that $\Gamma_{\varphi}(U,\mathcal{F})$ is well defined.

Now, let Ω be a relatively compact Stein subdomain of an n-dimensional complex manifold X, and φ a plurisubharmonic function defined on Ω. An open subset Y of Ω is said to be admissible for the coherent analytic sheaf $\mathcal{F}$ defined in a neighborhood of the closure of Ω in X, if Y is Stein, there is a coordinate neighborhood V in X of the closure $\bar{Y}$ of Y such that V is biholomorphic to a simply connected polycylinder V^1 in $\mathbb{C}^n$. The section $f\in\Gamma(Y,\mathcal{F})$ is L^2_{φ}-bounded if

$$f\in\Gamma_{\varphi}(Y,\mathcal{F}):=\left\{g\in\Gamma(Y,\mathcal{F}):\eta_*(g)\in\Gamma_{\varphi\cdot\eta^{-1}}(\eta(Y),\eta_*(\mathcal{F}))\right\}, \tag{2.5}$$

where η is the restriction of the biholomorphic map $V\to V^1$ to Y and $\eta_*(\mathcal{F})$ is the zeroth direct image of $\mathcal{F}$ on Y.

2.3. Let Ω, X, φ, and $\mathcal{F}$ be as above. Then it is clear that Ω is a finite union $\Omega = \bigcup_{j=1}^{m}\Omega_j$, where each Ω_j is admissible for $\mathcal{F}$. If $\mathcal{V} = \{\Omega_j\}_{j\in I}$, $I = \{1,\dots,m\}$, where each Ω_j is as above, then $\mathcal{V}$ is a finite admissible cover of Ω for $\mathcal{F}$ and we define the L^2_φ (alternate) q-cochains of $\mathcal{V}$ with values in $\mathcal{F}$ as those cochains

$$c = (c_\alpha) \in C^q(\mathcal{V},\mathcal{F}) = \prod_{\alpha\in I^{q+1}} \Gamma(\Omega_\alpha,\mathcal{F}), \tag{2.6}$$

where $\Omega_\alpha = \Omega_{i_0} \cap \cdots \cap \Omega_{i_q}$, $\alpha = (i_0,\dots,i_q)$, which are alternate and satisfy $c_\alpha \in \Gamma_\varphi(\Omega_\alpha,\mathcal{F})$ for all $\alpha \in I^{q+1}$. Denote by $C^q_\varphi(\mathcal{V},\mathcal{F})$ the space of L^2_φ-bounded cochains. The coboundary operator

$$\delta : C^q(\mathcal{V},\mathcal{F}) \longrightarrow C^{q+1}(\mathcal{V},\mathcal{F}) \tag{2.7}$$

maps $C^q_\varphi(\mathcal{V},\mathcal{F})$ into $C^{q+1}_\varphi(\mathcal{V},\mathcal{F})$. If $Z^q_\varphi(\mathcal{V},\mathcal{F}) = \{c \in C^q_\varphi(\mathcal{V},\mathcal{F}) : \delta c = 0\}$ and $B^q_\varphi(\mathcal{V},\mathcal{F}) = \delta C^{q-1}_\varphi(\mathcal{V},\mathcal{F})$, then as usual $B^q_\varphi(\mathcal{V},\mathcal{F}) \subseteq Z^q_\varphi(\mathcal{V},\mathcal{F})$ and we define

$$H^q_\varphi(\mathcal{V},\mathcal{F}) := Z^q_\varphi(\mathcal{V},\mathcal{F})/B^q_\varphi(\mathcal{V},\mathcal{F}) \tag{2.8}$$

and call it the L^2_φ-bounded cohomology of $\mathcal{V}$ with values in $\mathcal{F}$. We then have the following theorem.

Theorem 2.2. *For any $q \geq 1$, the natural map*

$$H^q_\varphi(\mathcal{V},\mathcal{F}) \longrightarrow H^q(\Omega,\mathcal{F}) \tag{2.9}$$

is an isomorphism.

Theorem 2.2 is used to prove Theorem 2.1, but we do not prove Theorem 2.2 here because its proof is easier than the proof of the theorem in [3].

3. Carleman estimates

3.1. Let Ω, X, and φ be as above. If $U \neq \varnothing$ is open in $\bar{\Omega}$, then $B^p_\Omega(U,\varphi)$ is the Hilbert space of holomorphic p-forms h on $\Omega\cap U$ such that $\|h\|_{L^2_{(p,0)}(U\cap\Omega,\varphi)} < \infty$.

If V is open in $\bar{\Omega}$ with $V \subset U$, the restriction map $\gamma^U_V : B^p_\Omega(U,\varphi) \to B^p_\Omega(V,\varphi)$ is defined. Then $B^p_\varphi = \{B^p_\Omega(u,\varphi);\mathcal{Y}^U_V\}$ is then the canonical pre-sheaf of L^2_φ-holomorphic p-forms on Ω. The associated sheaf $\mathcal{B}^p_\varphi$ is the sheaf of germs of L^2_φ-holomorphic p-forms on $\bar{\Omega}$.

We then have the following lemma.

Lemma 3.1. *The cohomology group $H^q(\bar{\Omega},\mathcal{B}^p_\varphi) = 0$ for all $q \geq 1$ and $p \geq 0$.*

Proof. Let $\mathcal{H}^p$ be the sheaf of germs of holomorphic p-forms on X. Note that for Y admissible in Ω, $\Gamma_\varphi(Y,\mathcal{H}^p) = \mathcal{B}^p_\Omega(Y,\varphi)$, since $\mathcal{H}^p$ is a coherent analytic sheaf in a neighborhood of the closure of Ω in X. Now, if $\mathcal{V}$ is any finite admissible cover of Ω for $\mathcal{H}^p$ (Theorem 2.2), $H^q_\varphi(\mathcal{V},\mathcal{H}^p)$ is isomorphic to $H^q_\varphi(\Omega,\mathcal{H}^q)$. But any finite cover of $\bar{\Omega}$ has a refinement $\mathcal{U} = \{V_j\}_{j\in J}$ such that $\mathcal{V}_\Omega = \{V_j\cap\Omega\}_{j\in J}$ is a finite admissible cover of Ω for $\mathcal{H}^p$. Therefore, $H^q_\varphi(\bar{\Omega},\mathcal{B}^p_\varphi)$ is isomorphic to $H^q(\Omega,\mathcal{H}^p)$ for $q \geq 0$ and $p \geq 0$. By Cartan's theorem (Theorem 1.2), we have $H(\Omega,\mathcal{H}^p) = 0$ for all $q \geq 1$ and $p \geq 0$. Therefore, $H^q(\bar{\Omega},\mathcal{B}^p_\varphi) = 0$ for $q \geq 1$ and $p \geq 0$. □

3.2. By the following lemma (whose proof follows from Theorem 1.2), the proof of Theorem 2.1 is concluded.

Lemma 3.2. *The cohomology group $H^q(\bar{\Omega}, \mathcal{B}_\varphi^p)$ is isomorphic to the quotient space*

$$\left\{g : g \in L^2_{(p,q)}(\Omega, \varphi),\ \bar{\partial} g = 0\right\} / \left\{\bar{\partial} h : h \in L^2_{(p,q-1)}(\Omega, \varphi),\ \bar{\partial} h \in L^2_{(p,q)}(\Omega, \varphi)\right\}. \qquad (3.1)$$

Since $H^q(\bar{\Omega}, \mathcal{B}_\varphi^p) = 0$ for $q > 0$ and $p \geq 0$, then we get the following lemma.

Lemma 3.3. *If $f \in L^2_{(p,q)}(\Omega, \varphi)$ is $\bar{\partial}$-closed, $q > 0$, then there is a $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ such that $\bar{\partial} u = f$.*

Now referring to [1, Theorem B, page 750], the proof of Theorem 2.1 is complete.

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