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# THE TRAJECTORY-COHERENT APPROXIMATION AND THE SYSTEM OF MOMENTS FOR THE HARTREE TYPE EQUATION

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The general construction of semiclassically concentrated solutions to the Hartree type equation, based on the complex WKB-Maslov method, is presented. The formal solutions of the Cauchy problem for this equation, asymptotic in small parameter $\hbar$ ($\hbar \to 0$), are constructed with a power accuracy of $O(\hbar^{N/2})$, where $N$ is any natural number. In constructing the semiclassically concentrated solutions, a set of Hamilton-Ehrenfest equations (equations for centered moments) is essentially used. The nonlinear superposition principle has been formulated for the class of semiclassically concentrated solutions of Hartree type equations. The results obtained are exemplified by a one-dimensional Hartree type equation with a Gaussian potential.

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**1. Introduction.** The nonlinear Schrödinger equation

$$\{-i\partial_t + \hat{\mathscr{H}}(t, |\Psi|^2)\}\Psi = 0, \tag{1.1}$$

where $\hat{\mathscr{H}}(t, |\Psi|^2)$ is a nonlinear operator, arises in describing a broad spectrum of physical phenomena. In statistical physics and quantum field theory, the generalized model of the evolution of bosons is described in terms of the second quantization formalism by the Schrödinger equation [24] which, in Hartree's approximation, leads to the classical multidimensional Schrödinger equation with a nonlocal nonlinearity for one-particle functions, that is, a Hartree type equation.

The quantum effects associated with the propagation of an optical pulse in a nonlinear medium are also described in the second quantization formalism by the one-dimensional Schrödinger equation with a delta-shaped interaction potential. In this case, the Hartree approximation results in the classical nonlinear Schrödinger equation (one-dimensional with local cubic nonlinearity) [31, 32], which is integrated by the Inverse Scattering Transform (IST) method and has soliton solutions [51]. Solitons are localized wave packets propagating without distortion and interacting elastically in mutual collisions. The soliton theory has found wide application in various fields of nonlinear physics [1, 14, 42, 50].

Investigations of the statistical properties of optical fields have led to the concept of compressed states of a field in which quantum fluctuations are minimized and the highest possible accuracy of optical measurements is achieved. An important problem of the correspondence between the stressed states describing the quantum properties of radiation and the optical solitons is analyzed in [31, 32].

The Hartree type equation is nonintegrable by the IST method. Nevertheless, approximate solutions showing some properties characteristic of solitons can be constructed. Solutions of this type are referred to as solitary waves or "quasi-solitons" to differentiate them from the solitons (in the strict sense) arising in the IST integrable models.

An efficient method for constructing solutions of this type is offered by the technique of semiclassical asymptotics. Thus, for nonlinear operators of the self-consistent field type, the theory of canonical operators with a real phase has been constructed for a Cauchy problem [36, 38] and for spectral problems, including those with singular potentials [25, 27] (see also [2, 39, 49]). Soliton-like solutions of a Hartree type equation and some types of interaction potentials have also been constructed [18].

In this paper, localized solutions of a (nonlinear) Hartree type equation asymptotic in small parameter $\hbar$ ($\hbar \to 0$) are constructed using the so-called WKB method or the Maslov complex germ theory [5, 37]. The constructed solutions are a generalization of the well-known quantum mechanical coherent and compressed states for linear equations [9, 34] for the case of nonlinear Hartree type equations with variable coefficients. We refer to the corresponding asymptotic solutions, like in the linear case [5], as *semiclassically concentrated* solutions (or states).

The most typical of solitary waves ("quasi-solitons") is that they show some properties characteristic of particles. For the "quasi-solitons" being semiclassically concentrated states of a Hartree type equation, these properties are represented by a dynamic set of ordinary differential equations for the "quantum" means $\vec{X}(t,\hbar)$ and $\vec{P}(t,\hbar)$ of the operators of coordinates $\hat{x}$ and momenta $\hat{p}$ and for the centered higher-order moments. In the limit of $\hbar \to 0$, the centroid of such a quasi-soliton moves in the phase space along the trajectory of this dynamic system: at each point in time, the semiclassically concentrated state is efficiently concentrated in the neighborhood of the point $\vec{X}(t,0)$ (in the $x$ representation) and in the neighborhood of the point $\vec{P}(t,0)$ (in the $p$ representation). Note that a similar set of equations in quantum means has been obtained in [3, 4] for the linear case (Schrödinger equation), and in [5] for a more general case. It has been shown [7, 8] that these equations are Poisson equations with respect to the (degenerate) nonlinear Dirac bracket. Therefore, we call the equations in quantum means for a Hartree type equation, like in the linear case [5], Hamilton-Ehrenfest equations. The Hamiltonian character of these equations is the subject of a special study. Nevertheless, it should be noted that, as distinct from the linear case, the construction of the semiclassically concentrated states for a Hartree type equation essentially uses the solutions of the correspondent Hamilton-Ehrenfest equations.

The specificity of the Hartree type equation, where nonlinear terms are only under the integral sign, is that it shows some properties inherent in linear equations. In particular, it has been demonstrated that for the class of semiclassically concentrated solutions of this type of equation (with a given accuracy $\hbar$, $\hbar \to 0$), the *nonlinear superposition principle* is valid.

In terms of the approach under consideration, the formal asymptotic solutions of the Cauchy problem for this equation and the evolution operator have been constructed in the class of trajectory-concentrated functions, allowing any accuracy in small parameter $\hbar$, $\hbar \to 0$.

It should be stressed that throughout this paper we deal with the construction of the formal asymptotic solutions to a Hartree type equation with the residual whose norm has a small estimate in parameter $\hbar$, $\hbar \to 0$. To substantiate these asymptotics for finite times $t \in [0,T]$, $T = \text{const}$, is a special nontrivial mathematical problem. This problem is concerned with obtaining a priori estimates uniform in parameter $\hbar \in ]0,1]$ for the solution of nonlinear equation (1.1), and is beyond the scope of the present work. Note that, in view of the heuristic considerations given in [25], it seems that the difference between an exact solution and the constructed formal asymptotic solution can be found with the use of the method developed in [25, 35]. Asymptotic solutions in $T \to \infty$ of the scattering problem were constructed for some special cases of the Hartree type equation in a number of papers (see, e.g., [19, 20, 23] and the references therein). The existence of semiclassical wave packets for the linear Schrödinger equation was studied in [10, 12, 35, 46, 47, 53] and their time evolution was discussed in [5, 11, 33, 40]. Finally, we mention a class of nonlinear equations in which nonlinear terms are local and nonlocal terms are linear [41]. These equations are different from the Hartree type equation under consideration.

This paper is arranged as follows. Section 2 gives principal notions and definitions. In Section 3, a class of trajectory-concentrated functions is specified and the simplest properties of these functions are considered. In Section 4, Hamilton-Ehrenfest equations are constructed which describe the “particle-like” properties of the semiclassically concentrated solutions of the Hartree type equation. In Section 5, the Hartree type equation is linearized for the solutions of Hamilton-Ehrenfest equations, and a set of associated linear equations which determine the asymptotic solution of the starting problem is obtained. In Section 6, we construct, accurate to $O(\hbar^{3/2})$, semiclassical coherent solutions to the Hartree type equation. In Section 7, the principal term of the semiclassical asymptotic of this equation is obtained in a class of semiclassically concentrated functions. The semiclassically concentrated solutions to the Hartree type equation are constructed with an arbitrary accuracy in $\sqrt{\hbar}$ in Section 8, while the kernel of the evolution operator (Green function) of the Hartree type equation is constructed in Section 9. Herein, the nonlinear superposition principle is substantiated for the class of semiclassically concentrated solutions. In Section 10, a Hartree type equation with a Gaussian potential is considered as an example. The appendix presents the properties of the solutions of equations in variations necessary to construct the asymptotic solutions and the approximate evolution operator to the Hartree type equation.

## 2. The Hartree type equation.

In this paper, by the Hartree type equation is meant the equation

$$\{-i\hbar\partial_t + \hat{\mathcal{H}}(t) + \varkappa \hat{V}(t,\Psi)\}\Psi = 0, \quad \Psi \in L_2(\mathbb{R}^n_x). \tag{2.1}$$

Here, the operators

$$\hat{\mathcal{H}}(t) = \mathcal{H}(\hat{z},t), \tag{2.2}$$

$$\hat{V}(t,\Psi) = \int_{\mathbb{R}^n} d\vec{y}\, \Psi^*(\vec{y},t) V(\hat{z},\hat{w},t)\Psi(\vec{y},t) \tag{2.3}$$

are functions of the noncommuting operators

$$\hat{z} = \Big(-i\hbar\frac{\partial}{\partial\vec{x}},\vec{x}\Big), \quad \hat{w} = \Big(-i\hbar\frac{\partial}{\partial\vec{y}},\vec{y}\Big), \quad \vec{x},\vec{y}\in\mathbb{R}^n, \tag{2.4}$$

the function $\Psi^*$ is complex conjugate to $\Psi$, $\varkappa$ is a real parameter, and $\hbar$ is a small parameter, $\hbar\in[0,1[$. For the operators $\hat{z}$ and $\hat{w}$, the following commutative relations are valid:

$$\begin{gathered}[\hat{z}_k,\hat{z}_j]_- = [\hat{w}_k,\hat{w}_j]_- = i\hbar J_{kj},\\ [\hat{z}_k,\hat{w}_j]_- = 0, \quad k,j=\overline{1,2n},\end{gathered} \tag{2.5}$$

where $J = \|J_{kj}\|_{2n\times 2n}$ is a unit symplectic matrix

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0\end{pmatrix}_{2n\times 2n}. \tag{2.6}$$

For the functions of noncommuting variables, we use the Weyl ordering [13, 26]. In this case, we can write, for instance, for the operator $\hat{\mathcal{H}}$

$$\begin{aligned}&\hat{\mathcal{H}}(t)\Psi(\vec{x},t,\hbar)\\ &\quad = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} d\vec{y}\,d\vec{p}\exp\Big(\frac{i}{\hbar}\langle(\vec{x}-\vec{y}),\vec{p}\rangle\Big)\mathcal{H}\Big(\vec{p},\frac{\vec{x}+\vec{y}}{2},t\Big)\Psi(\vec{y},t,\hbar),\end{aligned} \tag{2.7}$$

where $\mathcal{H}(z,t) = \mathcal{H}(\vec{p},\vec{x},t)$ is the Weyl symbol of the operator $\hat{\mathcal{H}}(t)$ and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product of the vectors

$$\langle\vec{p},\vec{x}\rangle = \sum_{j=1}^{n} p_j x_j, \quad \vec{p},\vec{x}\in\mathbb{R}^n, \qquad \langle z,w\rangle = \sum_{j=1}^{2n} z_j w_j, \quad z,w\in\mathbb{R}^{2n}. \tag{2.8}$$

Here we are interested in localized solutions of (2.1), for each fixed $\hbar\in[0,1[$ and $t\in\mathbb{R}$, belonging to the Schwartz space with respect to the variable $\vec{x}\in\mathbb{R}^n$. For the operators $\hat{\mathcal{H}}(t)$ and $\hat{V}(t,\Psi)$ to be at work in this space, it is sufficient that their Weyl symbols $\mathcal{H}(z,t)$ and $V(z,w,t)$ be smooth functions and grow, together with their derivatives, with $|z|\to\infty$ and $|w|\to\infty$ no more rapidly as the polynomial and uniformly in $t\in\mathbb{R}^1$. (In what follows we assume that for all the operators under consideration, $\hat{A} = A(\hat{z},t)$, their Weyl symbols satisfy Supposition 2.1.) Therefore, we believe that the following conditions for the functions $\mathcal{H}(z)$ and $V(z,w,t)$ are satisfied.

**Supposition 2.1.** *For any multi-indices $\alpha$, $\beta$, $\mu$, and $\nu$ there exist constants $C^{\alpha}_{\beta}(T)$ and $C^{\alpha\mu}_{\beta\nu}(T)$, such that the inequalities*

$$\left|z^{\alpha}\frac{\partial^{|\beta|}\mathcal{H}(z,t)}{\partial z^{\beta}}\right| \le C^{\alpha}_{\beta}(T), \quad \left|z^{\alpha}w^{\mu}\frac{\partial^{|\beta+\nu|}V(z,w,t)}{\partial z^{\beta}\partial w^{\nu}}\right| \le C^{\alpha\mu}_{\beta\nu}(T), \quad z,w\in\mathbb{R}^{2n},\ 0\le t\le T \tag{2.9}$$

*are fulfilled.*

Here, $\alpha$, $\beta$, $\mu$, and $\nu$ are multi-indices ($\alpha,\beta,\mu,\nu \in \mathbb{Z}_+^{2n}$) defined as

$$\begin{gathered}\alpha = (\alpha_1,\alpha_2,\dots,\alpha_{2n}), \qquad |\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_{2n}, \qquad z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_{2n}^{\alpha_{2n}},\\ \frac{\partial^{|\alpha|}V(z)}{\partial z^\alpha} = \frac{\partial^{|\alpha|}V(z)}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\cdots\partial z_{2n}^{\alpha_{2n}}}, \quad \alpha_j = \overline{0,\infty},\ j=\overline{1,2n}.\end{gathered} \tag{2.10}$$

We are coming now to the description of the class of functions for which we will seek asymptotic solutions to (2.1).

**3. The class of trajectory-concentrated functions.** We introduce a class of functions singularly depending on a small parameter $\hbar$, which is a generalization of the notion of a solitary wave. It appears that asymptotic solutions of (2.1) can be constructed based on functions of this class, which depend on the phase trajectory $z = Z(t,\hbar)$, the real function $S(t,\hbar)$ (analogous to the classical action at $\varkappa = 0$ in the linear case), and the parameter $\hbar$. For $\hbar \to 0$, the functions of this class are concentrated in the neighborhood of a point moving along a given phase curve $z = Z(t,0)$. Functions of this type are well known in quantum mechanics. In particular, among these are coherent and compressed states of quantum systems with a quadric Hamiltonian [9, 21, 22, 28, 29, 30, 34, 43, 45, 48]. Note that a soliton solution localized only with respect to spatial (but not momentum) variables does not belong to this class.

We denote this class of functions as $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, and define it as

$$\begin{aligned}\mathcal{P}_\hbar^t &= \mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))\\ &= \left\{\Phi : \Phi(\vec{x},t,\hbar) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}},t,\hbar\right)\exp\left[\frac{i}{\hbar}\left(S(t,\hbar)+\langle\vec{P}(t,\hbar),\Delta\vec{x}\rangle\right)\right]\right\},\end{aligned} \tag{3.1}$$

where the function $\varphi(\vec{\xi},t,\hbar)$ belongs to the Schwartz space $\mathbb{S}$ in the variable $\vec{\xi} \in \mathbb{R}^n$, and depends smoothly on $t$ and regularly on $\sqrt{\hbar}$ for $\hbar \to 0$. Here, $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$, and the real function $S(t,\hbar)$, and the $2n$-dimensional vector function $Z(t,\hbar) = (\vec{P}(t,\hbar),\vec{X}(t,\hbar))$, which characterize the class $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, depend regularly on $\sqrt{\hbar}$ in the neighborhood of $\hbar = 0$ and *are to be determined*. In the cases where this does not give rise to ambiguity, we use a shorthand symbol of $\mathcal{P}_\hbar^t$ for $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$.

The functions of the class $\mathcal{P}_\hbar^t$ are normalized to

$$||\Phi(t)||^2 = \langle\Phi(t)|\Phi(t)\rangle \tag{3.2}$$

in the space $L_2(\mathbb{R}_x^n)$ with the scalar product

$$\langle\Psi(t)|\Phi(t)\rangle = \int_{\mathbb{R}^n} d\vec{x}\,\Psi^*(\vec{x},t,\hbar)\Phi(\vec{x},t,\hbar). \tag{3.3}$$

In the subsequent manipulation, the argument $t$ in the expression for the norm may be omitted, $\|\Phi(t)\|^2 = \|\Phi\|^2$.

In constructing asymptotic solutions, it is useful to define, along with the class of functions $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, the following class of functions:

$$\mathscr{C}_\hbar^t(Z(t,\hbar),S(t,\hbar)) \\ = \left\{\Phi : \Phi(\vec{x},t,\hbar) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}},t\right)\exp\left[\frac{i}{\hbar}\left(S(t,\hbar)+\langle\vec{P}(t,\hbar),\Delta\vec{x}\rangle\right)\right]\right\}, \tag{3.4}$$

where the functions $\varphi$, as distinct from (3.1), are independent of $\hbar$.

At any fixed point in time $t \in \mathbb{R}^1$, the functions belonging to the class $\mathcal{P}_\hbar^t$ *are concentrated*, in the limit of $\hbar \to 0$, in the neighborhood of a point lying on the phase curve $z = Z(t,0)$, $t \in \mathbb{R}^1$ (the sense of this property is established exactly in Theorems 3.1, 3.2, and 3.4). Therefore, it is natural to refer to the functions of the class $\mathcal{P}_\hbar^t$ as *trajectory-concentrated functions*. The definition of the class of trajectory-concentrated functions includes the phase trajectory $Z(t,\hbar)$ and the scalar function $S(t,\hbar)$ as free "parameters." It appears that these "parameters" are determined unambiguously from the Hamilton-Ehrenfest equations (see Section 4) fitting the nonlinear ($\varkappa \neq 0$) Hamiltonian of (2.1). Note that for a linear Schrödinger equation, in the limiting case of $\varkappa = 0$, the principal term of the series in $\hbar \to 0$ determines the phase trajectory of the Hamilton system with the Hamiltonian $\mathscr{H}(\vec{p},\vec{x},t)$, and the function $S(t,0)$ is the classical action along this trajectory. In particular, in this case, the class $\mathcal{P}_\hbar^t$ includes the well-known dynamic (compressed) coherent states of quantum systems with quadric Hamiltonians when the amplitude of $\varphi$ in (3.1) is taken as a Gaussian exponential

$$\varphi(\vec{\xi},t) = \exp\left[\frac{i}{2}\left\langle\vec{\xi},Q(t)\vec{\xi}\right\rangle\right]f(t), \tag{3.5}$$

where $Q(t)$ is a complex symmetric matrix with a positive imaginary part, and the time factor is given by

$$f(t) = \sqrt[4]{\det \operatorname{Im} Q(t)}\exp\left[-\frac{i}{2}\int_0^t \operatorname{Sp}\operatorname{Re} Q(\tau)d\tau\right] \tag{3.6}$$

(see [5], for details).

Consider the principal properties of the functions of the class $\mathcal{P}_\hbar^t(Z(t,\hbar), S(t,\hbar))$, which are also valid for those of the class $\mathscr{C}_\hbar^t(Z(t,\hbar),S(t,\hbar))$.

**Theorem 3.1.** *For the functions of the class $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, the following asymptotic estimates are valid for centered moments $\Delta_\alpha(t,\hbar)$ of order $|\alpha|$, $\alpha \in \mathbb{Z}_+^{2n}$:*

$$\Delta_\alpha(t,\hbar) = \frac{\langle\Phi\,|\,\{\Delta\hat{z}\}^\alpha\,|\,\Phi\rangle}{\|\Phi\|^2} = O(\hbar^{|\alpha|/2}), \quad \hbar \longrightarrow 0. \tag{3.7}$$

*Here, $\{\Delta\hat{z}\}^\alpha$ denotes the operator with the Weyl symbol $(\Delta z)^\alpha$,*

$$\Delta z = z - Z(t,\hbar) = (\Delta\vec{p},\Delta\vec{x}), \quad \Delta\vec{p} = \vec{p}-\vec{P}(t,\hbar),\ \Delta\vec{x} = \vec{x}-\vec{X}(t,\hbar). \tag{3.8}$$

**Proof.** The operator symbol $\{\Delta\hat{z}\}^{\alpha}$ can be written as

$$(\Delta z)^{\alpha} = (\Delta\vec{p})^{\alpha_p}(\Delta\vec{x})^{\alpha_x}, \quad (\alpha_p,\alpha_x) = \alpha, \tag{3.9}$$

and, hence, according to the definition of Weyl-ordered pseudodifferential operators (2.7), we have for the mean value $\sigma_\alpha(t,\hbar)$ of the operator $\{\Delta\hat{z}\}^{\alpha}$

$$\begin{aligned}\sigma_\alpha(t,\hbar) &= \langle\Phi\,|\,\{\Delta\hat{z}\}^{\alpha}\,|\,\Phi\rangle \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{3n}} d\vec{x}\,d\vec{y}\,d\vec{p}\,\Phi^*(\vec{x},t,\hbar) \\ &\qquad\times\exp\left(\frac{i}{\hbar}\langle(\vec{x}-\vec{y}),\vec{p}\rangle\right)[\Delta\vec{p}]^{\alpha_p}\left(\frac{\Delta\vec{x}+\Delta\vec{y}}{2}\right)^{\alpha_x}\Phi(\vec{y},t,\hbar).\end{aligned} \tag{3.10}$$

Here, we have denoted

$$\Delta\vec{y} = \vec{y} - \vec{X}(t,\hbar). \tag{3.11}$$

After the change of variables,

$$\Delta\vec{x} = \sqrt{\hbar}\vec{\xi}, \qquad \Delta\vec{y} = \sqrt{\hbar}\vec{\zeta}, \qquad \Delta\vec{p} = \sqrt{\hbar}\vec{\omega}, \tag{3.12}$$

and taking into consideration the implicit form of the functions

$$\Phi(\vec{x},t,\hbar) = \exp\{i/\hbar(S(t,\hbar)+\langle\vec{P}(t,\hbar),\Delta\vec{x}\rangle)\}\varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}},t,\hbar\right), \tag{3.13}$$

belonging to the class $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, we find that

$$\begin{aligned}\sigma_\alpha(t,\hbar) &= \frac{1}{(2\pi\hbar)^n}\hbar^{3n/2}\hbar^{|\alpha|/2}2^{-|\alpha_p|} \\ &\quad\times\int_{\mathbb{R}^{3n}} d\vec{\xi}\,d\vec{\zeta}\,d\vec{\omega}\,\varphi^*(\vec{\xi},t,\hbar)\exp\{i\langle(\vec{\xi}-\vec{\zeta}),\vec{\omega}\rangle\}\vec{\omega}^{\,\alpha_x}(\vec{\xi}+\vec{\zeta})^{\alpha_p}\varphi(\vec{\zeta},t,\hbar) \\ &= \hbar^{(n+|\alpha|)/2}M_\alpha(t,\hbar), \\ \|\Phi\|^2 &= \hbar^{n/2}\int_{\mathbb{R}^n} d\vec{\xi}\,\varphi^*(\vec{\xi},t,\hbar)\varphi(\vec{\xi},t,\hbar) \\ &= \hbar^{n/2}M_0(t,\hbar).\end{aligned} \tag{3.14}$$

Since $\varphi(\vec{\xi},t,\hbar)$ depends on $\sqrt{\hbar}$ regularly and $M_0(t,\hbar) > 0$, we get

$$\begin{aligned}\Delta_\alpha(t,\hbar) &= \frac{\sigma_\alpha(t,\hbar)}{\|\Phi\|^2} \\ &= \hbar^{|\alpha|/2}\frac{M_\alpha(t,\hbar)}{M_0(t,\hbar)} \le \hbar^{|\alpha|/2}\max_{t\in[0,T]}\frac{M_\alpha(t,\hbar)}{M_0(t,\hbar)} \\ &= O(\hbar^{|\alpha|/2}),\end{aligned} \tag{3.15}$$

and the theorem is proved. □

Denote an operator $\hat{F}$, such that, for any function $\Phi$ belonging to the space $\mathcal{P}_\hbar^t(z(t,\hbar),S(t,\hbar))$, the asymptotic estimate

$$\frac{\|\hat{F}\Phi\|}{\|\Phi\|}=O(\hbar^\nu),\quad \hbar\longrightarrow 0, \tag{3.16}$$

is valid, by the symbol $\hat{O}(\hbar^\nu)$.

**Theorem 3.2.** *For the functions belonging to $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, the following asymptotic estimates are valid:*

$$\{\Delta\hat{z}\}^\alpha=\hat{O}(\hbar^{|\alpha|/2}),\quad \alpha\in\mathbb{Z}_+^{2n},\ \hbar\longrightarrow 0. \tag{3.17}$$

**Proof.** The proof is similar to that of relation (3.7). □

**Corollary 3.3.** *For the functions belonging to $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, the following asymptotic estimates are valid:*

$$\{-i\hbar\partial_t-\dot{S}(t,\hbar)+\langle\vec{P}(t,\hbar),\dot{\vec{X}}(t,\hbar)\rangle+\langle\dot{Z}(t,\hbar),J\Delta\hat{z}\rangle\}=\hat{O}(\hbar), \tag{3.18}$$

$$\Delta\hat{x}_k=\hat{O}(\sqrt{\hbar}),\quad \Delta\hat{p}_j=\hat{O}(\sqrt{\hbar}),\quad k,j=\overline{1,n}. \tag{3.19}$$

**Proof.** Follows from the explicit form (3.13) of the trajectory-concentrated functions [$\Phi(\vec{x},t,\hbar)\in\mathcal{P}_\hbar^t$, (3.4)] and from the estimates (3.17). □

**Theorem 3.4.** *For any function $\Phi(\vec{x},t,\hbar)\in\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$, the limiting relations*

$$\lim_{\hbar\to 0}\frac{1}{\|\Phi\|^2}|\Phi(\vec{x},t,\hbar)|^2=\delta(\vec{x}-\vec{X}(t,0)), \tag{3.20}$$

$$\lim_{\hbar\to 0}\frac{1}{\|\tilde{\Phi}\|^2}|\tilde{\Phi}(\vec{p},t,\hbar)|^2=\delta(\vec{p}-\vec{P}(t,0)), \tag{3.21}$$

*where $\tilde{\Phi}(\vec{p},t,\hbar)=F_{\hbar,\vec{x}\to\vec{p}}\Phi(\vec{x},t,\hbar)$, $F_{\hbar,\vec{x}\to\vec{p}}$ is the $\hbar^{-1}$ Fourier transform* [37]*, are valid.*

**Proof.** Consider an arbitrary function $\phi(\vec{x})\in\mathbb{S}$. Then, for any function $\Phi(\vec{x},t,\hbar)\in\mathcal{P}_\hbar^t$, the integral

$$\begin{aligned}\left\langle\frac{|\Phi(t,\hbar)|^2}{\|\Phi(t,\hbar)\|^2}\middle|\phi\right\rangle&=\frac{1}{\|\Phi(t,\hbar)\|^2}\int_{\mathbb{R}_x^n}\phi(\vec{x})|\Phi(\vec{x},t,\hbar)|^2d\vec{x}\\&=\frac{1}{\|\varphi(t,\hbar)\|^2}\int_{\mathbb{R}_x^n}\phi(\vec{x})\left|\varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}},t\right)\right|^2d\vec{x},\end{aligned} \tag{3.22}$$

after the change of variables $\vec{\xi}=\Delta\vec{x}/\sqrt{\hbar}$, becomes

$$\langle|\Phi(t,\hbar)|^2|\phi\rangle=\frac{\hbar^{n/2}}{\|\varphi(t,\hbar)\|^2}\int_{\mathbb{R}_\xi^n}\phi(\vec{X}(t,\hbar)+\sqrt{\hbar}\vec{\xi})|\varphi(\vec{\xi},t,\hbar)|^2d\vec{\xi}. \tag{3.23}$$

We pass in the last equality to the limit of $\hbar \to 0$, and, in view of

$$\|\varphi(t,\hbar)\|^2 = \hbar^{n/2} \int_{\mathbb{R}^n_\xi} |\varphi(\vec{\xi},t,\hbar)|^2 d\vec{\xi} \tag{3.24}$$

and a regular dependence of the function $\varphi(\vec{\xi},t,\hbar)$ on $\sqrt{\hbar}$, we arrive at the required statement.

The proof of relation (3.21) is similar to the previous one if we notice that the Fourier transform of the function $\Phi(\vec{x},t,\hbar) \in \mathcal{P}^t_\hbar$ can be represented as

$$\tilde{\Phi}(\vec{p},t,\hbar) = \exp\left\{\frac{i}{\hbar}[S(t,\hbar) - \langle\vec{p},\vec{X}(t,\hbar)\rangle]\right\}\tilde{\varphi}\left(\frac{\vec{p}-\vec{P}(t,\hbar)}{\sqrt{\hbar}},t,\hbar\right), \tag{3.25}$$

where

$$\tilde{\varphi}(\vec{\omega},t,\hbar) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} e^{-i\langle\vec{\omega},\vec{\xi}\rangle}\varphi(\vec{\xi},t,\hbar)d\xi. \tag{3.26}$$

□

Denote by $\langle\hat{L}(t)\rangle$ the mean value of the operator $\hat{L}(t)$, $t \in \mathbb{R}^1$, self-conjugate in $L_2(\mathbb{R}^n_x)$, calculated from the function $\Phi(\vec{x},t,\hbar) \in \mathcal{P}^t_\hbar$. Then the following corollary is valid.

**Corollary 3.5.** *For any function $\Phi(\vec{x},t,\hbar) \in \mathcal{P}^t_\hbar(Z(t,\hbar),S(t,\hbar))$ and any operator $\hat{A}(t,\hbar)$ whose Weyl symbol $A(z,t,\hbar)$ satisfies Supposition 2.1, the equality*

$$\begin{aligned}\lim_{\hbar\to 0}\langle\hat{A}(t,\hbar)\rangle &= \lim_{\hbar\to 0}\frac{1}{\|\Phi\|^2}\langle\Phi(\vec{x},t,\hbar)\,|\,\hat{A}(t,\hbar)\,|\,\Phi(\vec{x},t,\hbar)\rangle \\ &= A\big(Z(t,0),t,0\big)\end{aligned} \tag{3.27}$$

*is valid.*

**Proof.** The proof is similar to that of relations (3.20) and (3.21). □

Following [5], we introduce the following definition.

**Definition 3.6.** We refer to the solution $\Phi(\vec{x},t,\hbar) \in \mathcal{P}^t_\hbar$ of (2.1) as semiclassically concentrated on the phase trajectory $Z(t,\hbar)$ for $\hbar \to 0$, provided that the conditions (3.20) and (3.21) are fulfilled.

**Remark 3.7.** The estimates (3.17) of operators $\{\Delta\hat{z}\}^\alpha$ allow a consistent expansion of the functions of the class $\mathcal{P}^t_\hbar(Z(t,\hbar),S(t,\hbar))$ and the operator of (2.1) in a power series of $\sqrt{\hbar}$. This expansion gives rise to a set of recurrent equations which determine the sought-for asymptotic solution of (2.1).

For any function $\Phi \in \mathcal{P}^t_\hbar(Z(t,\hbar),S(t,\hbar))$, the representation

$$\Phi(\vec{x},t,\hbar) = \sum_{k=0}^{N}\hbar^{k/2}\Phi^{(k)}(\vec{x},t,\hbar) + O(\hbar^{(N+1)/2}), \tag{3.28}$$

where $\Phi^{(k)}(\vec{x},t,\hbar) \in \mathscr{C}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar))$, is valid. Representation (3.28) naturally induces the expansion of the space $\mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar))$ in a direct sum of subspaces,

$$\mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar)) = \bigoplus_{l=0}^{\infty} \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),l). \tag{3.29}$$

Here, the functions $\Phi \in \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),l) \subset \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar))$, according to (3.4), have estimates by the norm

$$\frac{1}{\hbar^{n/2}}\|\Phi\|_{L_2(\mathbb{R}_x^n)} = \hbar^{l/2}\mu(t), \tag{3.30}$$

where the function $\mu(t)$ is independent of $\hbar$ and continuously differentiable with respect to $t$.

Similar to the proof of the estimates (3.17) and (3.18), it can be shown that the operators

$$\{\Delta\hat{z}\}^{\alpha}, \qquad \{-i\hbar\partial_t - \dot{S}(t,\hbar) + \langle \vec{P}(t,\hbar), \dot{\vec{X}}(t,\hbar)\rangle + \langle \dot{Z}(t,\hbar), J\Delta\hat{z}\rangle\} \tag{3.31}$$

do not disrupt the structure of the expansions (3.28), (3.29), and

$$\begin{gathered} \{\Delta\hat{z}\}^{\alpha} : \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),l) \longrightarrow \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),l+|\alpha|), \\ \{-i\hbar\partial_t - \dot{S}(t,\hbar) + \langle \vec{P}(t,\hbar), \dot{\vec{X}}(t,\hbar)\rangle + \langle \dot{Z}(t,\hbar), J\Delta\hat{z}\rangle\} : \\ \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),l) \longrightarrow \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),l+2). \end{gathered} \tag{3.32}$$

**Remark 3.8.** From Corollary 3.5, it follows that the solution $\Psi(\vec{x},t,\hbar)$ of (2.1), belonging to the class $\mathscr{P}_{\hbar}^{t}$, is semiclassically concentrated.

The limiting character of the conditions (3.20) and (3.21), and the asymptotic character of the estimates (3.7), (3.13), (3.17), and (3.18), valid for the class of trajectory-concentrated functions, make it possible to construct semiclassically concentrated solutions to the Hartree type equation, *not exactly*, but *approximately*. In this case, the $L_2$ norm of the error has an order of $\hbar^{\alpha}$, $\alpha > 1$, for $\hbar \to 0$ on any finite time interval $[0,T]$. Denote such an approximate solution as $\Psi_{\text{as}} = \Psi_{\text{as}}(\vec{x},t,\hbar)$. This solution satisfies the following problem:

$$\left[-i\hbar\frac{\partial}{\partial t} + \hat{\mathscr{H}}(t) + \varkappa\hat{V}(t,\Psi_{\text{as}})\right]\Psi_{\text{as}} = O(\hbar^{\alpha}), \quad \Psi_{\text{as}} \in \mathscr{P}_{\hbar}^{t}(Z(t,\hbar),S(t,\hbar),\hbar), \ t \in [0,T], \tag{3.33}$$

where $O(\hbar^{\alpha})$ denotes the function $g^{(\alpha)}(\vec{x},t,\hbar)$, the "*residual*" of (2.1). For the residual, the following estimate is valid:

$$\max_{0\le t\le T}\|g^{(\alpha)}(\vec{x},t,\hbar)\| = O(\hbar^{\alpha}), \quad \hbar \longrightarrow 0. \tag{3.34}$$

Below we refer to the function $\Psi_{\text{as}}(\vec{x},t,\hbar)$, satisfying the problem (3.33) and (3.34), as a *semiclassically concentrated solution* $(\operatorname{mod}\hbar^{\alpha},\ \hbar \to 0)$ of the Hartree type equation (2.1).

The main goal of this work is to construct semiclassically concentrated solutions to the Hartree type equation (2.1) with any degree of accuracy in small parameter $\sqrt{\hbar}$, $\hbar \to 0$, that is, functions $\Psi_{\text{as}}(\vec{x},t,\hbar) = \Psi^{(N)}(\vec{x},t,\hbar)$ satisfying the problem (3.33) and (3.34) in $\bmod(\hbar^{(N+1)/2})$, where $N \geq 2$ is any natural number.

Thus, the semiclassically concentrated solutions $\Psi^{(N)}(\vec{x},t,\hbar)$ of the Hartree type equation approximately describe the evolution of the initial state $\Psi_0(\vec{x},\hbar)$ if the latter has been taken from a class of trajectory-concentrated functions $\mathcal{P}_\hbar^0$. The operators $\hat{\mathcal{H}}(t)$ and $\hat{V}(t,\Psi)$, entering in the Hartree type equation (2.1), leave the class $\mathcal{P}_\hbar^t$ invariant on a finite time interval $0 \leq t \leq T$ since their symbols satisfy Supposition 2.1. Therefore, in constructing semiclassically concentrated solutions to the Cauchy problem, the initial conditions can be

$$\Psi(\vec{x},t,\hbar)\big|_{t=0} = \Psi_0(\vec{x},\hbar), \quad \Psi_0 \in \mathcal{P}_\hbar^0(z_0,S_0). \tag{3.35}$$

The functions from the class $\mathcal{P}_\hbar^0$ have the following form:

$$\begin{aligned}&\Psi_0(\vec{x},\hbar)\\&\quad= \exp\left\{\frac{i}{\hbar}\left[S(0,\hbar) + \langle \vec{P}_0(\hbar),(\vec{x}-\vec{X}_0(\hbar))\rangle\right]\right\}\varphi_0\left(\frac{\vec{x}-\vec{X}_0(\hbar)}{\sqrt{\hbar}},\hbar\right), \quad \varphi_0(\vec{\xi},\hbar) \in \mathbb{S}(\mathbb{R}^n_\xi),\end{aligned} \tag{3.36}$$

where $Z_0(\hbar) = (\vec{P}_0(\hbar),\vec{X}_0(\hbar))$ is an arbitrary point of the phase space $\mathbb{R}^{2n}_{px}$, and the constant $S_0(\hbar)$ can be put equal to zero, without loss of generality.

Let us bring two important examples of the initial conditions of type (3.36).

(1) The first case is

$$\varphi_0(\vec{\xi}) = e^{-\langle\vec{\xi},A\vec{\xi}\rangle/2}, \tag{3.37}$$

where the real $n\times n$ matrix $A$ is positive definite and symmetric. Then relationship (3.36) defines the *Gaussian packet*.

(2) The second case is

$$\varphi_0(\vec{\xi}) = e^{i\langle\vec{\xi},Q\vec{\xi}\rangle/2}H_\nu(\operatorname{Im}Q\vec{\xi}), \tag{3.38}$$

where the complex $n\times n$ matrix $Q$ is symmetric and has a positive definite imaginary part $\operatorname{Im}Q$, and $H_\nu(\vec{\eta})$, $\vec{\eta} \in \mathbb{R}^n$, are multidimensional Hermite polynomials of multi-index $\nu = (\nu_1,\ldots,\nu_n)$ [6]. In this case, relation (3.36) defines the *Fock states of a multidimensional oscillator*.

The solution of the Cauchy problem (2.1), (3.35) leads in turn to a set of Hamilton-Ehrenfest equations which we will study in the following section.

**4. The set of Hamilton-Ehrenfest equations.** In view of Supposition 2.1 for the symbols $\mathcal{H}(z,t)$ and $V(z,w,t)$, the operator $\mathcal{H}(\hat{z},t)$ in (2.2) is self-conjugate to the scalar product $\langle\Psi|\Phi\rangle$ in the space $L_2(\mathbb{R}^n_x)$ and the operator $V(\hat{z},\hat{w},t)$ (2.3) is self-conjugate to the scalar product $L_2(\mathbb{R}^{2n}_{xy})$:

$$\langle\Psi(t)|\Phi(t)\rangle_{\mathbb{R}^{2n}} = \int_{\mathbb{R}^{2n}} d\vec{x}\,d\vec{y}\,\Psi^*(\vec{x},\vec{y},t,\hbar)\Phi(\vec{x},\vec{y},t,\hbar). \tag{4.1}$$

Therefore, we have for the exact solutions of (2.1)

$$||\Psi(t)||^2 = ||\Psi(0)||^2, \tag{4.2}$$

and for the mean values of the operator $\hat{A}(t) = A(\hat{z},t)$, calculated for these solutions, the equality

$$\begin{aligned}\frac{d}{dt}\langle\hat{A}(t)\rangle = \left\langle\frac{\partial\hat{A}(t)}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[\hat{\mathcal{H}},\hat{A}(t)]_-\rangle \\ + \frac{i\varkappa}{\hbar}\left\langle\int d\vec{y}\Psi^*(\vec{y},t,\hbar)[V(\hat{z},\hat{w},t),\hat{A}(t)]_-\Psi(\vec{y},t,\hbar)\right\rangle,\end{aligned} \tag{4.3}$$

where $[\hat{A},\hat{B}]_- = \hat{A}\hat{B}-\hat{B}\hat{A}$ is the commutator of the operators $\hat{A}$ and $\hat{B}$, is valid. We refer to (4.3) as the *Ehrenfest equation for the operator* $\hat{A}$ *and function* $\Psi(\vec{x},t,\hbar)$. This term was chosen in view of the fact that in the linear case ($\varkappa = 0$), (2.1) goes into a quantum mechanical Schrödinger equation, and relation (4.3) into an Ehrenfest equation [17].

We have the following notations:

$$\hat{z} = (\hat{\vec{p}},\hat{\vec{x}}), \qquad Z(t,\hbar) = (\vec{P}(t,\hbar),\vec{X}(t,\hbar)), \qquad \Delta\hat{z} = \hat{z}-Z(t,\hbar). \tag{4.4}$$

Using the rules of composition for Weyl symbols [26], we find, for the symbol of the operator $\hat{C} = \hat{A}\hat{B}$,

$$C(z) = A\left(\overset{2}{z} + \frac{i\hbar}{2}J\overset{1}{\frac{\partial}{\partial z}}\right)B(z) = B\left(\overset{2}{z} - \frac{i\hbar}{2}J\overset{1}{\frac{\partial}{\partial z}}\right)A(z). \tag{4.5}$$

Here, the index over an operator symbol specifies the turn of its action. We suppose that, for the Hartree type equation (2.1), exact solutions (or solutions differing from exact ones by a quantity $O(\hbar^\infty)$) exist in the class of trajectory-concentrated functions. We write Ehrenfest equations (4.3) for the mean values of the operators $\hat{z}_j$ and $\{\Delta\hat{z}\}^\alpha$ calculated from such (trajectory-coherent) solutions of (2.1). After cumbersome, but not complicated calculations similar to those performed for the linear case with $\varkappa = 0$ (see [5], for details), we then obtain, restricting ourselves to the moments of order $N$, the following set of ordinary differential equations:

$$\begin{aligned}\dot{z} &= \sum_{|\mu|=0}^{N}\frac{1}{\mu!}J\left(\mathcal{H}_{z\mu}(z,t)\Delta_\mu + \tilde{\varkappa}\sum_{|\nu|=0}^{N}\frac{1}{\nu!}V_{z\mu\nu}(z,t)\Delta_\mu\Delta_\nu\right),\\ \dot{\Delta}_\alpha &= \sum_{|\mu+\gamma|=0}^{N}(-i\hbar)^{|\gamma|-1}\frac{[(-1)^{|\gamma_p|}-(-1)^{|\gamma_x|}]\alpha!\beta!\theta(\alpha-\gamma)\theta(\beta-\gamma)}{\gamma!(\alpha-\gamma)!(\beta-\gamma)!\mu!}\\ &\quad\times\left(\mathcal{H}_\mu(z,t) + \tilde{\varkappa}\sum_{|\nu|=0}^{N}\frac{1}{\nu!}V_{\mu\nu}(z,t)\Delta_\nu\right)\Delta_{\alpha-\gamma+J\beta-J\gamma} - \sum_{k=1}^{2n}\dot{z}_k\alpha_k\Delta_{\alpha(k)},\end{aligned} \tag{4.6}$$

with initial conditions

$$z|_{t=0} = z_0 = \langle\Psi_0|\hat{z}|\Psi_0\rangle, \qquad \Delta_\alpha|_{t=0} = \left\langle\Psi_0\middle|\{\hat{z}-z_0\}^\alpha\middle|\Psi_0\right\rangle, \quad \alpha\in\mathbb{Z}_+^{2n},\ |\alpha|\le N. \tag{4.7}$$

Here, $\tilde{\varkappa} = \varkappa\|\Psi_0(\vec{x},\hbar)\|^2$ and $\Psi_0(\vec{x},\hbar)$ is the initial function from (3.35),

$$\begin{gathered}
\mathcal{H}_\mu(z,t) = \frac{\partial^{|\mu|}\mathcal{H}(z,t)}{\partial z^\mu}, \qquad V_{\mu\nu}(z,t) = \frac{\partial^{|\mu+\nu|}V(z,w,t)}{\partial z^\mu \partial w^\nu}\Big|_{\omega=z}, \\
\mathcal{H}_{z\mu}(t,\hbar) = \partial_z\mathcal{H}_\mu(t,\hbar), \qquad \alpha = (\alpha_p,\alpha_x), \qquad J\alpha = (\alpha_x,\alpha_p), \\
\theta(\alpha-\beta) = \prod_{k=1}^{2n}\theta(\alpha_k-\beta_k), \quad \alpha(k) = (\alpha_1-\delta_{1,k},\ldots,\alpha_{2n}-\delta_{2n,k}).
\end{gathered} \tag{4.8}$$

By analogy with the linear theory ($\varkappa = 0$) [5], we refer to (4.6) as *Hamilton-Ehrenfest equations* of order $N$. In view of the estimates (3.7) for the class $\mathcal{P}^t_\hbar$, these equations are equivalent up to $O(\hbar^{(N+1)/2})$ to the nonlinear Hartree type equation (2.1).

For the case of $N = 2$, the Hamilton-Ehrenfest equations take the form

$$\begin{gathered}
\dot{z} = J\partial_z\left(1 + \frac{1}{2}\langle\partial_z,\Delta_2\partial_z\rangle + \frac{1}{2}\langle\partial_\omega,\Delta_2\partial_\omega\rangle\right)(\mathcal{H}(z,t) + \tilde{\varkappa}V(z,\omega,t))\big|_{\omega=z}, \\
\dot{\Delta}_2 = JM\Delta_2 - \Delta_2 MJ,
\end{gathered} \tag{4.9}$$

where

$$M = [\mathcal{H}_{zz}(z,t) + \tilde{\varkappa}V_{zz}(z,\omega,t)]\big|_{\omega=z}, \qquad \Delta_2 = \|\Delta_{ij}\|_{2n\times 2n}. \tag{4.10}$$

Equations (4.9) can be written in the equivalent form if we put in the second equation

$$\Delta_2(t) = A(t)\Delta_2(0)A^+(t), \tag{4.11}$$

and then it becomes

$$\dot{A} = JMA, \quad A(0) = \mathbb{I}. \tag{4.12}$$

**5. Linearization of the Hartree type equation.** Now, we construct a semiclassically concentrated (for $\hbar \to 0$) solution of (2.1), satisfying the initial condition (3.35).

Designate by

$$y^{(N)}(t,\hbar) = (Z_{j_1},\Delta^{(2)}_{j_2j_3},\Delta^{(3)}_{j_4j_5j_6},\ldots) = (Z(t,\hbar),\Delta_\alpha(t,\hbar)), \quad |\alpha| \le N, \tag{5.1}$$

the solution of the Hamilton-Ehrenfest equations of order $N$, (4.6), with the initial data $y^{(N)}(0,\hbar)$, (4.7), determined by the initial function $\Psi_0(\vec{x},\hbar)$, (3.35), that is, the mean values $Z(0,\hbar)$ and $\Delta_\alpha(0,\hbar)$ are calculated from the function $\Psi_0(\vec{x},\hbar)$. Expand the "kernel" of the operator $\hat{V}(t,\Psi)$ in a Taylor power series of the operators $\Delta\hat{w} = \hat{w} - Z(t,\hbar)$,

$$V(\hat{z},\hat{w},t) = \sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}V(\hat{z},w,t)}{\partial w^\alpha}\Big|_{w=Z(t,\hbar)}\{\Delta\hat{w}\}^\alpha. \tag{5.2}$$

Substituting this series into (2.1), we obtain for the functions $\Psi \in \mathcal{P}_\hbar^t$

$$\left[ -i\hbar\partial_t + \mathcal{H}(\hat{z},t) + \tilde{\varkappa} \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} V(\hat{z},w,t)}{\partial w^\alpha} \Big|_{w=Z(t,\hbar)} \Delta_\alpha(t,\hbar) \right] \Psi = O(\hbar^{(N+1)/2}), \qquad (5.3)$$
$$\Psi|_{t=0} = \Psi_0,$$

where

$$Z(t,\hbar) = \frac{1}{\|\Psi(t,\hbar)\|^2} \langle \Psi(t,\hbar) | \hat{z} | \Psi(t,\hbar) \rangle,$$
$$\Delta_\alpha(t,\hbar) = \frac{1}{\|\Psi(t,\hbar)\|^2} \langle \Psi(t,\hbar) \,|\, \{\Delta\hat{z}\}^\alpha \,|\, \Psi(t,\hbar) \rangle. \qquad (5.4)$$

In view of the asymptotic estimates (3.7), the functions $z(t,\hbar)$ and $\Delta_\alpha(t,\hbar)$ can be determined with any degree of accuracy from the Hamilton-Ehrenfest equations (4.6) as

$$z(t,\hbar) = z(t,\hbar,N) + O(\hbar^{(N+1)/2});$$
$$\Delta_\alpha(t,\hbar) = \Delta_\alpha(t,\hbar,N) + O(\hbar^{(N+1)/2}), \quad |\alpha| \le N, \qquad (5.5)$$

where $z(t,\hbar,N)$ and $\Delta_\alpha(t,\hbar,N)$ are solutions of the Hamilton-Ehrenfest equations of order $N$, which are completely determined by the initial condition of the Cauchy problem for the Hartree type equation, $\Psi_0(\vec{x},t,\hbar)$, and do not use the explicit form of the solution $\Psi(\vec{x},t,\hbar)$ in (5.3). Thus, the change of the mean values of the operators for the solutions of the Hamilton-Ehrenfest equations of order $N$, (5.5), *linearizes* the Hartree type equation (5.3) up to $O(\hbar^{(N+1)/2})$. So, to find an asymptotic solution to the Hartree type equation (2.1), we should consider the *linear* Schrödinger type equation

$$\hat{L}^{(N)}(t,\Psi_0)\Phi = O(\hbar^{(N+1)/2}), \quad \Phi\big|_{t=0} = \Phi_0; \qquad (5.6)$$

$$\hat{L}^{(N)}(t,\Psi_0) = -i\hbar\partial_t + \mathcal{H}(\hat{z},t) + \tilde{\varkappa} \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} V(\hat{z},w,t)}{\partial w^\alpha} \Big|_{w=Z(t,\hbar,N)} \Delta_\alpha(t,\hbar,N). \qquad (5.7)$$

**Definition 5.1.** We call an equation of type (5.6) with a given $\Psi_0$ a Hartree equation in the trajectory-coherent approximation or a linear associated Schrödinger equation of order $N$ for the Hartree type equation (2.1).

The following statement is valid.

**Statement 5.2.** If the function $\Phi^{(N)}(\vec{x},t,\hbar,\Psi_0) \in \mathcal{P}_\hbar^t$ is an asymptotic (up to $O(\hbar^{(N+1)/2})$, $\hbar \to 0$) solution of (5.6), satisfying the initial condition $\Phi|_{t=0} = \Psi_0$, the function

$$\Psi^{(N)}(\vec{x},t,\hbar) = \Phi^{(N)}(\vec{x},t,\hbar,\Psi_0) \qquad (5.8)$$

is an asymptotic (up to $O(\hbar^{(N+1)/2})$, $\hbar \to 0$) solution of the Hartree type equation (2.1).

Now, we expand the operators

$$\mathcal{H}(\hat{z},t), \qquad \frac{\partial^{|\alpha|}V(\hat{z},w,t)}{\partial w^\alpha}\Big|_{w=Z(t,\hbar,N)} \tag{5.9}$$

in a Taylor power series of the operator $\Delta\hat{z}$ and present the operator $-i\hbar\partial_t$ in the form

$$\begin{aligned}-i\hbar\partial_t = {}&\{-\langle\vec{P}(t,\hbar,N),\dot{\vec{X}}(t,\hbar,N)\rangle+\dot{S}(t,\hbar)\}-\langle\dot{Z}(t,\hbar,N),J\Delta\hat{z}\rangle\\ &+\{-i\hbar\partial_t-\dot{S}(t,\hbar)+\langle\vec{P}(t,\hbar,N),\dot{\vec{X}}(t,\hbar,N)\rangle+\langle\dot{Z}(t,\hbar,N),J\Delta\hat{z}\rangle\}.\end{aligned} \tag{5.10}$$

Here, the group of terms in braces containing $-i\hbar\partial_t$, in view of (3.32), has an order of $\hat{O}(\hbar)$. The other terms can be estimated, in view of (3.18), by the parameter $\hbar$. Substitute the obtained expansions into (5.6). Take (to within $O(\hbar^{N/2})$) the real function $S(t,\hbar)$ entering in the definition of the class $\mathcal{P}^t_\hbar(Z(t,\hbar),S(t,\hbar))$ in the form

$$\begin{aligned}S(t,\hbar) &= S^{(N)}(t,\hbar)\\ &= \int_0^t\Big\{\langle\vec{P}(t,\hbar,N)\dot{\vec{X}}(t,\hbar,N)\rangle-\mathcal{H}(Z(t,\hbar,N),t)\\ &\qquad -\tilde{\varkappa}\sum_{|\alpha|=0}^{N}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}V(Z(t,\hbar,N),w,t)}{\partial w^\alpha}\Big|_{w=Z(t,\hbar,N)}\Delta_\alpha(t,\hbar,N)\Big\}dt.\end{aligned} \tag{5.11}$$

As a result, (5.6) will not contain operators of multiplication by functions depending only on $t$ and $\hbar$.

In view of the estimates (3.17) and (3.18), valid for the class $\mathcal{P}^t_\hbar(Z(t,\hbar),S(t,\hbar))$, we obtain for (5.3)

$$\{-i\hbar\partial_t+\hat{\mathfrak{H}}_0(t,\Psi_0)+\hbar\hat{\mathfrak{H}}^{(N)}(t,\Psi_0)\}\Phi = O(\hbar^{(N+1)/2}), \tag{5.12}$$

where

$$\hat{\mathfrak{H}}^{(N)}(t,\Psi_0) = \sum_{k=1}^{N}\hbar^{k/2}\hat{\mathfrak{H}}_k(t,\Psi_0), \tag{5.13}$$

$$\begin{aligned}\hat{\mathfrak{H}}_0(t,\Psi_0) &= -\dot{S}(t,\hbar)+\langle\vec{P}(t,\hbar),\dot{\vec{X}}(t,\hbar)\rangle\\ &\quad+\langle\dot{Z}(t,\hbar),J\Delta\hat{z}\rangle+\frac{1}{2}\langle\Delta\hat{z},\mathfrak{H}_{zz}(t,\Psi_0)\Delta\hat{z}\rangle,\end{aligned} \tag{5.14}$$

$$\mathfrak{H}_{zz}(t,\Psi_0) = [\mathcal{H}_{zz}(z,t)+\tilde{\varkappa}V_{zz}(z,w,t)]\big|_{z=w=Z(t,\hbar,N)},$$

$$\begin{aligned}\hbar^{(k+2)/2}\hat{\mathfrak{H}}_k(t,\Psi_0) &= -\hbar^{(k+1)/2}\langle\dot{Z}_{(k+1)}(t),J\Delta\hat{z}\rangle+\sum_{|\alpha|=k+2}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}\mathcal{H}(z,t)}{\partial z^\alpha}\Big|_{z=Z(t,\hbar,N)}\{\Delta\hat{z}\}^\alpha\\ &\quad+\tilde{\varkappa}\sum_{|\alpha+\beta|=k+2}\frac{1}{\alpha!\beta!}\frac{\partial^{|\alpha+\beta|}V(z,w,t)}{\partial z^\beta\partial w^\alpha}\Big|_{z=w=Z(t,\hbar,N)}\{\Delta\hat{z}\}^\beta\Delta_\alpha(t,\hbar,N).\end{aligned} \tag{5.15}$$

Here, $k=\overline{1,N}$ and the functions $Z_{(k)}(t)$ are the coefficients of the expansion of the projection $Z(t,\hbar)$ of the solution $y^{(N)}(t,\hbar)$ of the Hamilton-Ehrenfest equations on the phase space $\mathbb{R}^{2n}$ in a power series of $\sqrt{\hbar}$ in terms of the regular perturbation theory,

$$Z(t,\hbar) = Z(t,\hbar,N) = Z(t,0) + \sum_{k=2}^{N} \hbar^{k/2} Z_{(k)}(t). \tag{5.16}$$

From the Hamilton-Ehrenfest equations, in view of the fact that the first-order moments are zero ($\Delta_\alpha(t,\hbar,N) = 0$ for $|\alpha| = 1$), it follows that the coefficient $\dot{Z}_{(1)}(t)$ is equal to zero.

**Remark 5.3.** The solutions of the set of Hamilton-Ehrenfest equations depend on the index $N$ that denotes the highest order of the centered moments $\Delta_\alpha$, $\alpha \in \mathbb{Z}_+^{2n}$. We will omit the index $N$ if this does not give rise to ambiguity.

The operators $\hat{\mathfrak{H}}_0(t)$, (5.14), and $\hat{\mathfrak{H}}_k(t)$, (5.15), depend on the mean $Z(t,\hbar)$ and moments $\Delta_\alpha(t,\hbar)$, that is, on the solution $y^{(N)}(t,\hbar)$ of the Hamilton-Ehrenfest equations (4.6). The solutions of (5.12) in turn depend implicitly on $y^{(N)}(t,\hbar)$,

$$\Phi(\vec{x},t,\hbar) = \Phi(\vec{x},t,\hbar,y^{(N)}(t,\hbar)). \tag{5.17}$$

Below the function arguments $y^{(N)}(t,\hbar)$ or $\Psi_0$ can be omitted if this does not give rise to ambiguity. For example, we may put $\mathfrak{H}_0(t) = \hat{\mathfrak{H}}_0(t,\Psi_0)$.

In accordance with the expansions (3.29) and (3.28), the solution of (5.12) can be represented in the form

$$\Phi(\vec{x},t,\hbar,\Psi_0) = \sum_{k=0}^{N} \hbar^{k/2}\Phi^{(k)}(\vec{x},t,\hbar,\Psi_0) + O(\hbar^{(N+1)/2}), \tag{5.18}$$

where

$$\Phi^{(k)}(\vec{x},t,\hbar,\Psi_0) \in \mathscr{C}_\hbar^t(Z(t,\hbar),S(t,\hbar)). \tag{5.19}$$

In view of (3.32), for the operators $\{-i\hbar\partial_t + \hat{\mathfrak{H}}_0(t,\Psi_0)\}$ in (5.14) and $\hbar^{(k+2)/2}\hat{\mathfrak{H}}_k(t,\Psi_0)$, $k = \overline{1,N}$, in (5.15), the following is valid:

$$\begin{aligned}
&\hbar^{(k+2)/2}\hat{\mathfrak{H}}_k(t,\Psi_0) : \mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar),l) \longrightarrow \mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar),l+k+2),\\
&\{-i\hbar\partial_t + \hat{\mathfrak{H}}_0(t,\Psi_0)\} : \mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar),l) \longrightarrow \mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar),l+2).
\end{aligned} \tag{5.20}$$

Substitute (5.18) into (5.12) and equate the terms having the same order in $\hbar^{1/2}$, $\hbar \to 0$ in the sense of (5.20). As a result, we obtain a set of recurrent *associated linear equations* of order $k$ to determine the functions $\Phi^{(k)}(\vec{x},t,\hbar,\Psi_0)$,

$$\{-i\hbar\partial_t + \hat{\mathfrak{H}}_0(t,\Psi_0)\}\Phi^{(0)} = 0, \quad \text{for } \hbar^1 \tag{5.21}$$

$$\{-i\hbar\partial_t + \hat{\mathfrak{H}}_0(t,\Psi_0)\}\Phi^{(1)} + \hbar\hat{\mathfrak{H}}_1(t,\Psi_0)\Phi^{(0)} = 0, \quad \text{for } \hbar^{3/2} \tag{5.22}$$

$$\{-i\hbar\partial_t + \hat{\mathfrak{H}}_0(t,\Psi_0)\}\Phi^{(2)} + \hbar\hat{\mathfrak{H}}_1(t,\Psi_0)\Phi^{(1)} + \hbar^{3/2}\hat{\mathfrak{H}}_2(t,\Psi_0)\Phi^{(0)} = 0, \quad \text{for } \hbar^2 \tag{5.23}$$

$$\vdots$$

It is natural to call (5.21) for the principal term of the asymptotic solution as the Hartree type equation in the trajectory-coherent approximation in $\bmod \hbar^{3/2}$. This equation is a Schrödinger equation with the Hamiltonian quadric with respect to the operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$.

**6. The trajectory-coherent solutions of the Hartree type equation.** The solution of the Schrödinger equation with a quadric Hamiltonian is well known [9, 34]. For our purposes, it is convenient to take semiclassical trajectory-coherent states (TCSs) [5] as a basis of solutions to (5.21). We will refer to the solution of the nonlinear Hartree type equation, which coincides with the TCS at the time zero, as a *trajectory-coherent solution of the Hartree type equation*. Now, we pass to constructing solutions like this.

We write the symmetry operators $\hat{a}(t,\Psi_0)$ of (5.21), linear with respect to the operators $\Delta\hat{z}$, in the form

$$\hat{a}(t,\Psi_0) = N_a\langle b(t,\Psi_0),\Delta\hat{z}\rangle, \tag{6.1}$$

where $N_a$ is a constant and $b(t)$ is a $2n$-space vector. From the equation

$$-i\hbar\frac{\partial\hat{a}(t)}{\partial t} + [\hat{\mathfrak{H}}_0(t,\Psi_0),\hat{a}(t)]_- = 0, \tag{6.2}$$

which determines the operators $\hat{a}(t)$, in view of the explicit form of the operator $\hat{\mathfrak{H}}_0(t,\Psi_0)$ in (5.14), we obtain

$$\begin{aligned} -i\hbar\langle\dot{b}(t),\Delta\hat{z}\rangle &+ i\hbar\langle b(t),\dot{Z}(t,\hbar)\rangle \\ &+\Big[\Big\{-\dot{S}(t,\hbar)+\langle\vec{P}(t,\hbar),\dot{\vec{X}}(t,\hbar)\rangle+\langle\dot{Z}(t,\hbar),J\Delta\hat{z}\rangle \\ &\quad+\frac{1}{2}\langle\Delta z,\mathfrak{H}_{zz}(t,\Psi_0)\Delta\hat{z}\rangle\Big\},\langle b(t),\Delta\hat{z}\rangle\Big] = 0. \end{aligned} \tag{6.3}$$

Taking into account the commutative relations

$$[\Delta\hat{z}_j,\Delta\hat{z}_k] = i\hbar J_{jk}, \quad j,k=\overline{1,2n}, \tag{6.4}$$

which follow from (2.5), we find that

$$-i\hbar\langle\dot{b}(t),\Delta\hat{z}\rangle + i\hbar\langle\Delta\hat{z},\mathfrak{H}_{zz}(t)Jb(t)\rangle = 0. \tag{6.5}$$

Hence, we have

$$\dot{b} = \mathfrak{H}_{zz}(t,\Psi_0)Jb. \tag{6.6}$$

Denote $b(t) = -Ja(t)$. Then, for the determination of the $2n$-space vector $a(t)$ from (6.6), we obtain

$$\dot{a} = J\mathfrak{H}_{zz}(t,\Psi_0)a. \tag{6.7}$$

We call the set of (6.7), by analogy with the linear case [37], a *set of equations in variations*.

Thus, the operator

$$\hat{a}(t) = \hat{a}(t,\Psi_0) = N_a\langle b(t),\Delta\hat{z}\rangle = N_a\langle a(t),J\Delta\hat{z}\rangle \tag{6.8}$$

is a symmetry operator for (5.21) if the vector $a(t) = a(t,\Psi_0)$ is a solution of the equations in variations (6.7).

For each given solution $Z(t,\hbar)$ of the Hamilton-Ehrenfest equations (4.6), we can find $2n$ linearly independent solutions $a_k(t)\in\mathbb{C}^{2n}$ to the equations in variations (6.7). Since each $2n$-space vector $a_k(t)$ is associated with an operator $\hat{a}_k(t,\Psi_0)$, we obtain $2n$ operators, $n$ of which commutate with one another and form a complete set of symmetry operators for (5.21).

Now, we turn to constructing the basis of solutions to (5.21) with the help of the operators $\hat{a}_k(t,\Psi_0)$. Equation (5.21) is a (linear) Schrödinger equation with a quadric Hamiltonian and admits solutions in the form of Gaussian wave packets

$$\begin{aligned}\Phi(\vec{x},t,\Psi_0) = N_\hbar \exp\Big\{\frac{i}{\hbar}\Big[&S(t,\hbar)+i\phi_0(t)+i\hbar\phi_1(t)\\&+\langle\vec{P}(t,\hbar),\Delta\vec{x}\rangle+\frac{1}{2}\langle\Delta\vec{x},Q(t)\Delta\vec{x}\rangle\Big]\Big\},\end{aligned} \tag{6.9}$$

where the real phase $S(t,\hbar)$ is defined in (5.11), $N_\hbar$ is a normalized constant, while the real functions $\phi_0(t)$ and $\phi_1(t)$ and the complex $n\times n$ matrix $Q(t)$ are to be determined.

**Remark 6.1.** Asymptotic solutions in the form of Gaussian packets (6.9) for equations with an integral nonlinearity of more general form than (2.1) were constructed in [52]. In this case, the Hamilton-Ehrenfest equations depend substantially on the initial condition for the original nonlinear equation.

Substitution of (6.9) into (5.21) yields

$$\begin{aligned}\Phi\Big\{&\dot{S}(t,\hbar)+i\dot{\phi}_0(t)+i\hbar\dot{\phi}_1(t)+\langle\dot{\vec{P}}(t,\hbar),\Delta\vec{x}\rangle-\langle\vec{P}(t,\hbar),\dot{\vec{X}}(t,\hbar)\rangle+\frac{1}{2}\langle\Delta\vec{x},\dot{Q}(t)\Delta\vec{x}\rangle\\&-\langle\Delta\vec{x},Q(t)\dot{\vec{X}}(t,\hbar)\rangle-\dot{S}(t,\hbar)+\langle\vec{P}(t,\hbar),\dot{\vec{X}}(t,\hbar)\rangle+\langle\dot{\vec{X}}(t,\hbar),Q(t)\Delta\vec{x}\rangle-\langle\dot{\vec{P}}(t,\hbar),\Delta\vec{x}\rangle\\&+\frac{1}{2}\Big\{\langle\Delta\vec{x},\mathfrak{H}_{xx}(t,\Psi_0)\Delta\vec{x}\rangle+\langle\Delta\vec{x},\mathfrak{H}_{px}(t,\Psi_0)Q(t)\Delta\vec{x}\rangle\\&\qquad+\langle[-i\hbar\nabla+Q(t)\Delta\vec{x}],\mathfrak{H}_{px}(t,\Psi_0)\Delta\vec{x}\rangle\\&\qquad+\langle[-i\hbar\nabla+Q(t)\Delta\vec{x}],\mathfrak{H}_{pp}(t,\Psi_0)[-i\hbar\nabla+Q(t)\Delta\vec{x}]\rangle\Big\}\Big\}=0.\end{aligned} \tag{6.10}$$

Equating the coefficients of the terms with the same powers of the parameter $\hbar$ and the operator $\Delta\vec{x}$, we obtain

$$\begin{aligned}&i\dot{\phi}_0(t)=0\quad\text{for }(\Delta\vec{x})^0\hbar^0;\\&i\dot{\phi}_1(t)+\frac{-i}{2}\operatorname{Sp}[\mathfrak{H}_{px}(t,\Psi_0)+\mathfrak{H}_{pp}(t,\Psi_0)Q(t)]=0\quad\text{for }(\Delta\vec{x})^0\hbar^1;\\&\langle\Delta\vec{x},0\rangle=0\quad\text{for }(\Delta\vec{x})^1\hbar^0;\\&\langle\Delta\vec{x},[\dot{Q}(t)+\mathfrak{H}_{xx}(t,\Psi_0)+\mathfrak{H}_{xp}(t,\Psi_0)Q(t)+Q(t)\mathfrak{H}_{px}(t,\Psi_0)\\&\qquad+Q(t)\mathfrak{H}_{pp}(t,\Psi_0)Q(t)]\Delta\vec{x}\rangle=0\quad\text{for }(\Delta\vec{x})^2\hbar^0.\end{aligned} \tag{6.11}$$

As a result, we have

$$\phi_0(t) = 0,$$
$$\phi_1(t) = \frac{1}{2}\int_0^t \mathrm{Sp}\left[\mathfrak{H}_{px}(t) + \mathfrak{H}_{pp}(t)Q(t)\right]dt. \tag{6.12}$$

The matrix $Q(t)$ is determined from the Riccati type equation

$$\dot{Q}(t) + \mathfrak{H}_{xx}(t) + Q(t)\mathfrak{H}_{px}(t) + \mathfrak{H}_{xp}(t)Q(t) + Q(t)\mathfrak{H}_{pp}(t)Q(t) = 0. \tag{6.13}$$

Thus, the construction of a solution to (5.21) in the form of the Gaussian packet (6.9) is reduced to solving the set of ordinary differential equations (6.13).

Now, we construct the Fock basis of solutions to a (linear) Hartree type equation in the trajectory-coherent approximation (5.21). This is the first step in constructing the solution to recurrent equations (5.21), (5.22), and (5.23).

Consider the properties of the symmetry operators $\hat{a}_k(t)$ in (6.8) of the zero-order associated Schrödinger equation (5.21), which are necessary to construct the Fock basis.

**Statement 6.2.** Let $a_1(t)$ and $a_2(t)$ be two solutions of the equations in variations and let $\hat{a}_1(t)$ and $\hat{a}_2(t)$ be the respective symmetry operators of (5.21), defined in (6.8). Then the equality

$$[\hat{a}_1(t), \hat{a}_2(t)] = i\hbar N_1 N_2 \{a_1(t), a_2(t)\} = i\hbar N_1 N_2 \{a_1(0), a_2(0)\} \tag{6.14}$$

is valid.

Actually, upon direct checking, we are convinced that

$$\begin{aligned}[\hat{a}_1(t), \hat{a}_2(t)] &= N_1 N_2 [\langle a_1(t), J\Delta\hat{z}\rangle, \langle a_2(t), J\Delta\hat{z}\rangle] \\ &= i\hbar N_1 N_2 \langle Ja_1(t), JJa_2(t)\rangle \\ &= i\hbar N_1 N_2 \langle a_1(t), JJJa_2(t)\rangle \\ &= i\hbar N_1 N_2 \langle a_1(t), Ja_2(t)\rangle \\ &= i\hbar N_1 N_2 \{a_1(t), a_2(t)\}.\end{aligned} \tag{6.15}$$

Here, we have used the rules of commutation for the operators $\Delta\hat{z}$. The skew scalar product holds and, hence, the statement is proved.

**Remark 6.3.** If the initial conditions for the equations in variations are taken such that

$$\{a_j(0), a_k(0)\} = \{a_j^*(0), a_k^*(0)\} = 0, \quad \{a_j(0), a_k^*(0)\} = id_k\delta_{kj}, \quad d_k > 0,\ k, j = \overline{1, n}, \tag{6.16}$$

and $N_k = 1/\sqrt{\hbar d_k}$, then the following canonical commutation relations for the boson operators of *creation* $(\hat{a}_k^+(t))$ and *annihilation* $(\hat{a}_k(t))$ are valid:

$$[\hat{a}_k(t), \hat{a}_j(t)] = [\hat{a}_k^+(t), \hat{a}_j^+(t)], \qquad [\hat{a}_k(t), \hat{a}_j^+(t)] = \delta_{kj}. \tag{6.17}$$

The simplest example of initial data satisfying the conditions (6.16) is

$$
\begin{aligned}
&a_1(0) = (b_1, 0, \dots, 0, 1, 0, \dots);\\
&a_2(0) = (0, b_2, \dots, 0, 0, 1, \dots);\\
&\vdots
\end{aligned}
\tag{6.18}
$$

Here, $d_k = 2\operatorname{Im} b_k > 0$, $k = \overline{1, n}$.

**Theorem 6.4.** *The function*

$$
\begin{aligned}
|0, t\rangle &= |0, t, \Psi_0\rangle\\
&= \frac{N_\hbar}{\det C(t)} \exp\left\{\frac{i}{\hbar}\left[S(t, \hbar) + \langle \vec{P}(t, \hbar), \Delta\vec{x}\rangle + \frac{1}{2}\langle \Delta\vec{x}, Q(t)\Delta\vec{x}\rangle\right]\right\},
\end{aligned}
\tag{6.19}
$$

*where* $N_\hbar = [(\pi\hbar)^{-n} \det D_0]^{1/4}$ *is a* vacuum *state for the operators* $\hat{a}_j(t)$*, such that*

$$
\hat{a}_j(t)|0, t\rangle = 0, \quad j = \overline{1, n}. \tag{6.20}
$$

**Proof.** Actually, substituting (6.8) and (6.19) into (6.20), we get

$$
|0, t\rangle[\langle \vec{Z}_j(t), Q(t)\Delta\vec{x}\rangle - \langle \vec{W}_j(t), \Delta\vec{x}\rangle] = 0, \tag{6.21}
$$

since

$$
Q(t)\vec{Z}_j(t) = B(t)C^{-1}(t)\vec{Z}_j(t) = \vec{W}_j(t). \tag{6.22}
$$

Recall that from the fact that the matrix $D_0$ is positive definite and diagonal, it follows that $\det C(t) \neq 0$, and so the matrix $\operatorname{Im} Q(t)$ is positive definite as well (see the appendix).

Define the denumerable set of states $|\nu, t\rangle$ as the result of the action of the creation operators upon the vacuum state $|0, t\rangle$,

$$
|\nu, t\rangle = |\nu, t, \Psi_0\rangle = \frac{1}{\sqrt{\nu!}}(\hat{a}^+(t, \Psi_0))^\nu |0, t, \Psi_0\rangle = \prod_{k=1}^{n} \frac{1}{\sqrt{\nu_k!}}(\hat{a}_k^+(t, \Psi_0))^{\nu_k} |0, t, \Psi_0\rangle. \tag{6.23}
$$

By analogy with the linear theory ($\varkappa = 0$), we call the functions $|\nu, t\rangle$ in (6.23) *semiclassical trajectory-coherent states* and consider their simplest properties. □

**Statement 6.5.** The relations

$$
\begin{aligned}
\hat{a}_k|\nu, t\rangle &= \sqrt{\nu_k}\,|\tilde{\nu}_k^{(-)}, t\rangle,\\
\hat{a}_k^+|\nu, t\rangle &= \sqrt{\nu_k + 1}\,|\tilde{\nu}_k^{(+)}, t\rangle,\\
\tilde{\nu}_k^{(\pm)} &= (\nu_1 \pm \delta_{1,k}, \nu_2 \pm \delta_{2,k}, \dots, \nu_n \pm \delta_{n,k})
\end{aligned}
\tag{6.24}
$$

are valid.

Actually, we have

$$
[\hat{a}_j, (\hat{a}_k^+)^{\nu_k}] = \nu_k(\hat{a}_k^+)^{\nu_k - 1}\delta_{j,k}. \tag{6.25}
$$

It follows that

$$
\begin{aligned}
\hat{a}_j|\nu,t\rangle &= \prod_{k=1}^{n}\frac{1}{\sqrt{\nu_k!}}[\hat{a}_j,(\hat{a}_k^+)^{\nu_k}]|0,t\rangle = \prod_{k=1}^{n}\frac{\nu_j}{\sqrt{\nu_k!}}(\hat{a}_k^+)^{\nu_k-\delta_{k,j}}|0,t\rangle \\
&= \sqrt{\nu_j}\prod_{k=1}^{n}\frac{1}{\sqrt{(\nu_k-\delta_{k,j})!}}(\hat{a}_k^+)^{\nu_k-\delta_{k,j}}|0,t\rangle \\
&= \sqrt{\nu_j}\,|\tilde{\nu}_k^{(-)},t\rangle; \\
\hat{a}_j^+|\nu,t\rangle &= \prod_{k=1}^{n}\frac{1}{\sqrt{\nu_k!}}(\hat{a}_k^+)^{\nu_k+\delta_{k,j}}|0,t\rangle \\
&= \prod_{k=1}^{n}\frac{\sqrt{\nu_j+1}}{\sqrt{(\nu_k+\delta_{k,j})!}}(\hat{a}_k^+)^{\nu_k+\delta_{k,j}}|0,t\rangle \\
&= \sqrt{\nu_j+1}\,|\tilde{\nu}_k^{(+)},t\rangle,
\end{aligned}
\tag{6.26}
$$

and thus the statement is proved.

**Statement 6.6.** The states $|\nu,t,\Psi_0\rangle$ with $t\in\mathbb{R}$ and $\Psi_0\in\mathcal{P}_\hbar^0$ form a set of orthonormal functions

$$\langle\Psi_0,t,\nu'|\nu,t,\Psi_0\rangle = \delta_{\nu,\nu'},\quad \nu,\nu'\in\mathbb{Z}_+^n. \tag{6.27}$$

Consider the expression

$$\langle\Psi_0,t,\nu'|\nu,t,\Psi_0\rangle = \frac{1}{\sqrt{\nu'!\nu!}}\langle\Psi_0,t,0|\hat{\vec{a}}^{\nu'}(t,\Psi_0)[\hat{\vec{a}}^+]^{\nu}(t,\Psi_0)|0,t,\Psi_0\rangle. \tag{6.28}$$

Commuting the operators of creation and annihilation in view of commutation relations (6.17) and using relation (6.20), we obtain

$$\langle t,\nu'|\nu,t\rangle = \langle t,0|0,t\rangle\delta_{\nu,\nu'}. \tag{6.29}$$

Then we calculate

$$\langle t,0|0,t\rangle = \frac{N_\hbar^2}{|\det C(t)|}\int\exp\Big[-\frac{2}{\hbar}\operatorname{Im}S(\vec{x},t)\Big]d\vec{x}. \tag{6.30}$$

In view of (A.18) and the explicit form of the complex phase in (6.19), we have

$$\operatorname{Im}S(\vec{x},t) = \frac{1}{2}\langle\Delta\vec{x},\operatorname{Im}Q(t)\Delta\vec{x}\rangle. \tag{6.31}$$

The matrix $\operatorname{Im}Q(t)$ is real and positive definite; hence, the matrix $\sqrt{\operatorname{Im}Q(t)}$ does exist, such that

$$\det\sqrt{\operatorname{Im}Q(t)} = \frac{\sqrt{\det D_0}}{|\det C(t)|}. \tag{6.32}$$

We perform in the integral of (6.30) the change

$$\vec{\xi} = \frac{1}{\sqrt{\hbar}} \sqrt{\operatorname{Im} Q(t)} \Delta \vec{x}, \tag{6.33}$$

then we obtain

$$\langle t, 0|0, t\rangle = \frac{N_\hbar^2}{\sqrt{\det D_0}} \hbar^{n/2} \int e^{-\vec{\xi}^2} d\vec{\xi} = \frac{(\pi\hbar)^{n/2} N_\hbar^2}{\sqrt{\det D_0}} = 1, \tag{6.34}$$

since $\det D_0 = \prod_{k=1}^{n} \operatorname{Im} b_k$. Thus, the functions $|\nu, t, \Psi_0\rangle$ (6.23) form the Fock basis of solutions to (5.21).

**Theorem 6.7.** *Let the symbols of the operators $\hat{\mathcal{H}}(t)$ and $\hat{V}(t, \Psi)$ satisfy the conditions of Supposition 2.1. Then, for any $\nu \in \mathbb{Z}_+^n$, the function*

$$\Psi_\nu(\vec{x}, t, \hbar) = |\nu, t\rangle, \tag{6.35}$$

*where the functions $|\nu, t\rangle$ are defined by formula (6.23), is an asymptotic (up to $O(\hbar^{3/2})$, $\hbar \to 0$) solution of the Hartree type equation (2.1) with the initial conditions*

$$\Psi_\nu(\vec{x}, t, \hbar)|_{t=0} = |\nu, t\rangle|_{t=0}. \tag{6.36}$$

**7. The principal term of the semiclassical asymptotic.** The solution of the Cauchy problem (2.1), (6.36) is a special case of the semiclassically concentrated solutions of (2.1). However, in the case of arbitrary initial conditions (3.35) belonging to the class $\mathcal{P}_\hbar^t$, the functions $|\nu, t\rangle$ are not asymptotic solutions of the Hartree type equation (2.1). This is a fundamental difference between the complex germ method for the Hartree type equation (2.1), being developed here, and a similar method developed for linear equations [5, 37]. The coefficients of the Hartree type equation in the trajectory-coherent approximation (5.21) depend on the initial condition (3.35) since they are determined by the solutions of the set of Hamilton-Ehrenfest equations. It follows that among the whole set of solutions to (5.21) only one (satisfying the condition $\Psi(\vec{x}, t, \hbar)|_{t=0} = \Psi_0(\vec{x}, \hbar)$) will be an asymptotic (up to $O(\hbar^{3/2})$) solution of (2.1). However, the Fock basis (TCS's) $|\nu, t\rangle$ makes it possible to construct in explicit form an asymptotic solution to (2.1), with any degree of accuracy in $\hbar^{1/2}$, $\hbar \to 0$, which would satisfy the initial condition (3.35).

We illustrate in more detail the relation of the solutions of an associated linear Schrödinger equation to the solution of a Hartree type equation. To do this, we construct the Green function of the Cauchy problem for the zero-order associated Schrödinger equation. Although the Green function $G^{(0)}(\vec{x}, \vec{y}, t, s)$ for quadric quantum systems is well known [15, 16, 34, 44], we give for completeness its explicit form, as convenient to us. This function will allow us to demonstrate explicitly the nontrivial dependence of the evolution operator of the associated linear equation on the initial conditions for the original Hartree type equation.

By definition,

$$\begin{gathered}[-i\hbar\partial_t+\hat{\mathfrak{H}}_0(t,\Psi_0)]G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)=0,\\ \lim_{t\to s}G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)=\delta(\vec{x}-\vec{y}),\end{gathered}\tag{7.1}$$

where the operator $\hat{\mathfrak{H}}_0$ is defined in (5.14). We make use of the simplifying assumption that

$$\det\mathfrak{H}_{pp}(s)\neq 0,\qquad \det\left\|\frac{\partial p_k(t,z_0)}{\partial p_{0j}}\right\|\neq 0.\tag{7.2}$$

If condition (7.2) is not valid, the solution of the problem can be found following the work [15, 16]. For the problem under consideration, exact solutions of the Schrödinger equation (5.21) are known, these are the functions $|\nu,t,\Psi_0\rangle$ in (6.23) that form a complete set of functions. Thus we have

$$G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)=\sum_{|\nu|=0}^{\infty}\Phi_\nu(\vec{x},t,\hbar)\Phi_\nu^*(\vec{y},s,\hbar),\tag{7.3}$$

where

$$\Phi_\nu(\vec{x},t,\hbar)=|\nu,t,\Psi_0\rangle.\tag{7.4}$$

Details of similar calculations can be found, for instance, in [34]. However, for our purposes, the following approach seems to be convenient.

We carry out an $\hbar^{-1}$ Fourier transform in (7.1). For the Fourier transformation of the Green function

$$\tilde{G}^{(0)}(\vec{p},\vec{y},t,s,\Psi_0)=\int_{\mathbb{R}^n}\frac{d\vec{x}}{(2\pi i\hbar)^{n/2}}G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)\exp\left\{-\frac{i}{\hbar}\langle\vec{x},\vec{p}\rangle\right\},\tag{7.5}$$

we obtain

$$\begin{gathered}[-i\hbar\partial_t+\tilde{\mathfrak{H}}_0(\hat{\vec{p}},\hat{\vec{x}},t,\Psi_0)]\tilde{G}^{(0)}(\vec{p},\vec{y},t,s,\Psi_0)=0,\\ \lim_{t\to s}\tilde{G}(\vec{p},\vec{y},t,s,\Psi_0)=\frac{1}{(2\pi i\hbar)^{n/2}}\exp\left\{-\frac{i}{\hbar}\langle\vec{p},\vec{y}\rangle\right\}.\end{gathered}\tag{7.6}$$

Here, $\hat{\vec{p}}=\vec{p}$, $\hat{\vec{x}}=i\hbar(\partial/\partial\vec{p})$, and the symbols of the operators of (7.1) and (7.6) coincide,

$$\tilde{\mathfrak{H}}_0(\vec{p},\vec{x},t)=\mathfrak{H}_0(\vec{p},\vec{x},t).\tag{7.7}$$

Equation (7.6) coincides up to notations with (5.21) and, hence, admits solutions of type (6.9),

$$\begin{aligned}&\tilde{G}^{(0)}(\vec{p},\vec{y},t,s,\Psi_0)\\ &\quad=\exp\left\{-\frac{i}{\hbar}\left[S_0(t,s,\vec{y})+\langle\vec{G}(t,s,\vec{y}),\Delta\vec{p}\rangle+\frac{1}{2}\langle\Delta\vec{p},\tilde{Q}(t,s,\vec{y})\Delta\vec{p}\rangle\right]\right\},\end{aligned}\tag{7.8}$$

where $\Delta\vec{p} = \vec{p} - \vec{P}(t,\hbar)$. Here, the functions $S_0(t,s,\vec{y}) = S_0(t)$, $\vec{G}(t,s,\vec{y}) = \vec{G}(t)$, and $\tilde{Q}(t,s,\vec{y}) = \tilde{Q}(t)$ are to be determined and, according to (7.6), satisfy the initial conditions

$$\lim_{t\to s}\tilde{Q}(t,s,\vec{y}) = 0, \qquad \lim_{t\to s}\vec{G}(t,s,\vec{y}) = \vec{y}, \qquad \lim_{t\to s}S_0(t,s,\vec{y}) = \langle\vec{p}_0,\vec{y}\rangle. \tag{7.9}$$

Substituting (7.8) into (7.6), we write

$$\begin{aligned}
\tilde{G}^{(0)}(\vec{p},\vec{y},t,s,\Psi_0)\Big\{&-\dot{S}_0(t) - \langle\dot{\vec{G}}(t),\Delta\vec{p}(t)\rangle + \langle\vec{G}(t),\dot{\vec{P}}(t,\hbar)\rangle - \frac{1}{2}\langle\Delta\vec{p},\dot{\tilde{Q}}(t)\Delta\vec{p}\rangle \\
&+ \langle\dot{\vec{P}}(t,\hbar),\tilde{Q}(t)\Delta\vec{p}\rangle - \dot{S}(t,\hbar) - \langle\dot{\vec{P}}(t,\hbar),(\vec{G}(t)+\tilde{Q}(t)\Delta\vec{p}-\vec{P}(t,\hbar))\rangle \\
&+ \langle\dot{\vec{X}}(t,\hbar),\Delta\vec{p}\rangle \\
&+ \frac{1}{2}\Big[\langle(\vec{G}(t)+\tilde{Q}(t)\Delta\vec{p}-\vec{P}(t,\hbar)),\mathfrak{H}_{xx}(t)(\vec{G}(t)+\tilde{Q}(t)\Delta\vec{p}-\vec{P}(t,\hbar))\rangle \\
&\quad + \langle(\tilde{G}(t)+\tilde{Q}(t)\Delta\vec{p}-\vec{P}(t,\hbar)),\mathfrak{H}_{xp}(t)\Delta\vec{p}\rangle + \langle\Delta\vec{p},\mathfrak{H}_{pp}(t)\Delta\vec{p}\rangle \\
&\quad + \langle\Delta\vec{p},\mathfrak{H}_{px}(t)(\tilde{G}(t)+\tilde{Q}(t)\Delta\vec{p}-\vec{P}(t,\hbar))\rangle\Big] \\
&+ \frac{i\hbar}{2}\,\mathrm{Sp}\left[\mathfrak{H}_{xx}(t)\tilde{Q}(t)+\mathfrak{H}_{xp}(t)\right]\Big\} = 0.
\end{aligned} \tag{7.10}$$

Equating the terms with the same powers of $\Delta\vec{p}$, we obtain the following set of equations:

$$-\dot{\tilde{Q}} = \mathfrak{H}_{px}(t)\tilde{Q} + \tilde{Q}\mathfrak{H}_{xp}(t) + \tilde{Q}\mathfrak{H}_{xx}(t)\tilde{Q} + \mathfrak{H}_{pp}(t) = 0, \tag{7.11}$$

$$-(\dot{\vec{G}} - \dot{\vec{P}}(t,\hbar)) + \tilde{Q}(t)\mathfrak{H}_{xx}(t)(\vec{G}-\vec{P}(t,\hbar)) + \mathfrak{H}_{px}(t)(\vec{G}-\vec{P}(t,\hbar)) = 0, \tag{7.12}$$

$$\begin{aligned}
-\dot{S}_0 + \langle\vec{X}(t,\hbar),\dot{\vec{P}}(t,\hbar)\rangle - \dot{S}(t,\hbar) + \frac{i\hbar}{2}\,\mathrm{Sp}\left[\mathfrak{H}_{xx}(t)\tilde{Q}(t)+\mathfrak{H}_{xp}(t)\right]& \\
+\frac{1}{2}\langle(\vec{G}(t)-\vec{P}(t,\hbar)),\mathfrak{H}_{xx}(t)(\vec{G}(t)-\vec{P}(t,\hbar))\rangle = 0,&
\end{aligned} \tag{7.13}$$

with the initial conditions (7.9).

Let $\tilde{B}(t)$ and $\tilde{C}(t)$ be solutions of equations in variations (A.4) with the initial conditions

$$\tilde{B}(t)|_{t=s} = B_0(s), \qquad \tilde{C}(t)|_{t=s} = 0, \qquad B_0^t(s) = B_0(s), \tag{7.14}$$

and let the matrix $\mathrm{Im}\,B_0(s)$ be positive definite.

In view of (7.2), the solution of the Cauchy problem (A.4) and (7.14) will then have the form

$$\tilde{B}(t) = \lambda_4^t(\Delta t)B_0(s), \quad \tilde{C}(t) = -\lambda_3^t(\Delta t)B_0(s), \quad \Delta t = t-s, \tag{7.15}$$

where the matrices $\lambda_3^t(t)$ and $\lambda_4^t(t)$ are defined in (A.30). The matrix

$$\tilde{Q}(t) = \tilde{C}(t)\tilde{B}^{-1}(t) = -\lambda_3^t(\Delta t)(\lambda_4^{-1}(\Delta t))^t \tag{7.16}$$

will then satisfy (7.11) with the initial conditions (7.9).

Provided that (7.2) is valid, from (A.25) and (A.21), it follows that

$$\vec{G}(t) = (\tilde{B}^{-1}(t))^t B_0^t(s)(\vec{y}-\vec{x}_0) + \vec{X}(t,\hbar) = \lambda_4^{-1}(\Delta t)(\vec{y}-\vec{x}_0) + \vec{X}(t,\hbar). \tag{7.17}$$

In a similar manner, we obtain, for $S_0$,

$$\begin{aligned} S_0(t,s,\hbar) = S(t,\hbar) - S(s,\hbar) + \frac{i\hbar}{2}\int_s^t d\tau\,\mathrm{Sp}\,[\mathfrak{H}_{xp}(\tau) + \mathfrak{H}_{xx}(\tau)\tilde{Q}(\tau)] \\ + \frac{1}{2}\int_s^t d\tau \langle (\vec{G}(\tau) - \vec{X}(\tau,\hbar)), \mathfrak{H}_{xx}(\tau)(\vec{G}(\tau) - \vec{X}(\tau,\hbar))\rangle + \langle \vec{p}_0, \vec{y}\rangle. \end{aligned} \tag{7.18}$$

In view of (A.25) and Liouville's lemma (Lemma A.14), we obtain

$$\begin{aligned} \frac{1}{2}\int_s^t d\tau\,\mathrm{Sp}\,[\mathfrak{H}_{xp}(\tau) + \mathfrak{H}_{xx}(\tau)\tilde{Q}(\tau)] &= \frac{1}{2}\ln\det\tilde{B}^{-1}(\tau)|_s^t \\ &= \frac{1}{2}\ln\left(\frac{\det B_0(s)}{\det\tilde{B}(t)}\right) \\ &= -\frac{1}{2}\ln\det\lambda_4(\Delta t). \end{aligned} \tag{7.19}$$

To calculate the integral in (7.19), we use relation (A.29) and, in view of (7.18), we get

$$\begin{aligned} \frac{1}{2}\int_s^t d\tau \langle (\vec{G}(\tau) - \vec{X}(\tau,\hbar)), \mathfrak{H}_{xx}(\tau)(\vec{G}(\tau) - \vec{X}(\tau,\hbar))\rangle \\ = \frac{1}{2}\langle (\vec{y}-\vec{x}_0), \lambda_2(\Delta t)\lambda_4^{-1}(\Delta t)(\vec{y}-\vec{x}_0)\rangle, \end{aligned} \tag{7.20}$$

where the matrix $\lambda_2^t(t)$ is defined in (A.30). Hence,

$$\begin{aligned} S_0(t,s,\hbar) = S(t,\hbar) - S(s,\hbar) - \frac{i\hbar}{2}\ln(\det\lambda_4(\Delta t)) \\ + \frac{1}{2}\langle (\vec{y}-\vec{x}_0), \lambda_2(\Delta t)\lambda_4^{-1}(\Delta t)(\vec{y}-\vec{x}_0)\rangle + \langle \vec{p}_0, \vec{y}\rangle. \end{aligned} \tag{7.21}$$

Substituting (7.21), (7.18), and (7.16) into (7.8), we obtain the well-known expression (see, e.g., [15])

$$\begin{aligned} \tilde{G}^{(0)}(\vec{p},\vec{y},t,s,\Psi_0) = \frac{1}{(2\pi i\hbar)^{n/2}}\frac{1}{\sqrt{\det\lambda_4(\Delta t)}} \\ \times\exp\Big\{ -\frac{i}{\hbar}(S(t,\hbar) - S(s,\hbar)) \\ - \frac{i}{2\hbar}\langle (\vec{y}-\vec{x}_0), \lambda_2(\Delta t)\lambda_4^{-1}(\Delta t)(\vec{y}-\vec{x}_0)\rangle \\ - \frac{i}{\hbar}\langle \vec{p}_0, \vec{y}\rangle - \frac{i}{\hbar}\langle \Delta\vec{p}, \vec{X}(t,\hbar)\rangle - \frac{i}{\hbar}\langle \Delta\vec{p}, \lambda_4^{-1}(\Delta t)(\vec{y}-\vec{x}_0)\rangle \\ + \frac{i}{2\hbar}\langle \Delta\vec{p}, \lambda_4^{-1}(\Delta t)\lambda_3(\Delta t)\Delta\vec{p}\rangle\Big\}. \end{aligned} \tag{7.22}$$

Now, we substitute (7.22) into (7.5) and make use of the relation

$$\int_{\mathbb{R}^n} d\vec{x}\exp\left[-\frac{1}{2}\langle \vec{x}, \Gamma\vec{x}\rangle + \langle \vec{b}, \vec{x}\rangle\right] = \sqrt{(2\pi)^n\det\Gamma^{-1}}\exp\left\{\frac{\langle \vec{b}, \Gamma^{-1}\vec{b}\rangle}{2}\right\}, \tag{7.23}$$

in which we put $\Gamma = -(i/\hbar)\lambda_4^{-1}(\Delta t)\lambda_3(\Delta t)$, $\vec{b} = -(i/\hbar)[\vec{X}(t,\hbar) - \vec{x} + \lambda_4^{-1}(\Delta t)(\vec{y} - \vec{x}_0)]$. Then we obtain

$$\begin{aligned} G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0) = {} & \frac{1}{\sqrt{\det\left(-i2\pi\hbar\lambda_3(\Delta t)\right)}} \\ & \times \exp\left\{\frac{i}{\hbar}\left[S(t,\hbar) - S(s,\hbar) + \langle \vec{P}(t,\hbar), \Delta\vec{x}\rangle - \langle \vec{p}_0, (\vec{y} - \vec{x}_0)\rangle\right.\right. \\ & \qquad - \frac{1}{2}\langle (\vec{y} - \vec{x}_0), \lambda_1(\Delta t)\lambda_3^{-1}(\Delta t)(\vec{y} - \vec{x}_0)\rangle \\ & \qquad - \frac{1}{2}\langle \Delta\vec{x}, \lambda_3^{-1}(\Delta t)(\vec{y} - \vec{x}_0)\rangle \\ & \qquad \left.\left. - \frac{1}{2}\langle \Delta\vec{x}, \lambda_3^{-1}(\Delta t)\lambda_4(\Delta t)\Delta\vec{x}\rangle\right]\right\}. \end{aligned} \tag{7.24}$$

Here, we used the relations

$$\lambda_1^t(t)\lambda_4(t) - \lambda_3^t(t)\lambda_2(t) = \mathbb{I}_{n\times n}, \qquad \lambda_3(t)\lambda_4^t(t) - \lambda_4(t)\lambda_3^t(t) = 0, \tag{7.25}$$

that follow immediately from (A.10), (A.37), and from the definition of matriciant (A.30).

Consider the limit of expression (7.24) for $\Delta t = t - s \to 0$. We obtain

$$\begin{gathered} \lambda_1(\Delta t) = \mathbb{I}_{n\times n} + O(\Delta t), \qquad \lambda_3^t(\Delta t) = -\mathfrak{H}_{pp}(s)\Delta t + O\left((\Delta t)^2\right), \\ \lambda_3^{-1}(\Delta t) = -\frac{1}{\Delta t}\mathfrak{H}_{pp}^{-1}(s) + O\left((\Delta t)^0\right), \qquad \lambda_4(\Delta t) = \mathbb{I}_{n\times n} + O(\Delta t), \\ \lambda_2(\Delta t) = O(\Delta t). \end{gathered} \tag{7.26}$$

It follows that, for short times (see, e.g., [40])

$$\begin{aligned} & \lim_{\Delta t\to 0} G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0) \\ & \quad = \frac{1}{\sqrt{\det\left(-i2\pi\hbar\Delta t\mathfrak{H}_{pp}(s)\right)}} \exp\left\{\frac{i}{2\hbar\Delta t}\langle (\vec{x} - \vec{y}), \mathfrak{H}_{pp}^{-1}(s)(\vec{x} - \vec{y})\rangle + O(\Delta t^0)\right\}. \end{aligned} \tag{7.27}$$

Thus we have proved the following theorem.

**Theorem 7.1.** *Let the symbols of the operators $\hat{\mathcal{H}}(t)$ and $\hat{V}(t,\Psi)$ satisfy the conditions of Supposition 2.1. Then the function*

$$\Psi^{(0)}(\vec{x},t,\hbar) = \hat{U}^{(0)}(t,0,\Psi_0)\Psi_0, \tag{7.28}$$

*where $\hat{U}^{(0)}(t,0,\Psi_0)$ is the evolution operator of the zero-order associated Schrödinger equation (5.21) with the kernel $G^{(0)}(\vec{x},\vec{y},t,0,\Psi_0)$, is an asymptotic (up to $O(\hbar^{3/2})$, $\hbar\to 0$) solution of the Hartree type equation (2.1) and satisfies the initial condition*

$$\Psi^{(0)}(\vec{x},t,\hbar)|_{t=0} = \Psi_0. \tag{7.29}$$

**Remark 7.2.** The principal term of the semiclassical asymptotic $\Psi^{(0)}(\vec{x},t,\hbar)$ will not change (to within $O(\hbar^{3/2})$, $\hbar \to 0$) if the phase function $S^{(N)}(t,\hbar)$ in the operator $\hat{\mathfrak{H}}_0(t,\Psi_0)$ is substituted by its value $S^2(t,\hbar)$ for $N = 2$ and we restrict ourselves to the first terms in $\hbar \to 0$ in the phase trajectory $Z^{(2)}(t,\hbar)$, and in the other expressions $Z^{(N)}(t,\hbar)$ is changed by $Z^0(t,\hbar)$.

**8. Semiclassically concentrated solutions.** Now, we construct asymptotic solutions to the Hartree type equation (2.1) with an arbitrary accuracy in powers of $\sqrt{\hbar}$. To do this, we find asymptotic solutions of the associated linear Schrödinger equation (5.6) with an arbitrary accuracy in powers of $\sqrt{\hbar}$. We present an arbitrary initial condition $\Phi_0(\vec{x},\hbar) \in \mathcal{P}_\hbar^0$ as

$$\Phi_0(\vec{x},\hbar) = \sum_{k=0}^{N} \hbar^{k/2}\Phi_0^{(k)}(\vec{x},\hbar), \tag{8.1}$$

where

$$\Phi_0^{(k)}(\vec{x},\hbar) \in \mathscr{C}_\hbar^t(z_0,S_0). \tag{8.2}$$

Then, for the recurrent associated linear equations (5.21), (5.22), and (5.23), we arrive at a Cauchy problem with initial data

$$\Phi^{(k)}\big|_{t=0} = \Phi_0^{(k)}(\vec{x},\hbar), \quad k = \overline{0,N}. \tag{8.3}$$

The solution of these recurrent equations can readily be constructed as its expansion over the complete set of orthonormal Fock functions $|\nu,t\rangle$ in (6.20). As a result, we obtain

$$\Phi^{(0)}(\vec{x},t,\hbar) = \sum_{|\nu|=0}^{\infty} |\nu,t,\Psi_0\rangle\langle\Psi_0,0,\nu|\Phi_0^{(0)}(\vec{x},\hbar)\rangle, \tag{8.4}$$

$$\begin{aligned}\Phi^{(1)}(\vec{x},t,\hbar) &= \sum_{|\nu|=0}^{\infty} |\nu,t,\Psi_0\rangle\langle\Psi_0,0,\nu|\Phi_0^{(1)}(\vec{x},\hbar)\rangle \\ &\quad -\frac{i}{\hbar}\sum_{|\nu|=0}^{\infty} |\nu,t,\Psi_0\rangle\int_0^t d\tau\langle\Psi_0,\tau,\nu|\hat{\mathfrak{H}}_1(t,\Psi_0)\Phi^{(0)}(\vec{x},\tau,\hbar)\rangle,\end{aligned} \tag{8.5}$$

$$\begin{aligned}\Phi^{(2)}(\vec{x},t,\hbar) &= \sum_{|\nu|=0}^{\infty} |\nu,t,\Psi_0\rangle\langle\Psi_0,0,\nu|\Phi_0^{(2)}(\vec{x},\hbar)\rangle \\ &\quad -\frac{i}{\hbar}\sum_{|\nu|=0}^{\infty} |\nu,t,\Psi_0\rangle\int_0^t d\tau\langle\Psi_0,\tau,\nu|\hat{\mathfrak{H}}_1(t,\Psi_0)\Phi^{(1)}(\vec{x},\tau,\hbar)\rangle \\ &\quad -\frac{i}{\hbar}\sum_{|\nu|=0}^{\infty} |\nu,t,\Psi_0\rangle\int_0^t d\tau\langle\Psi_0,\tau,\nu|\hat{\mathfrak{H}}_2(t,\Psi_0)\Phi^{(0)}(\vec{x},\tau,\hbar)\rangle,\end{aligned} \tag{8.6}$$

$$\vdots$$

Denote by $\hat{\mathcal{F}}^{(N)}(t,\Psi_0)$ the operator defined by the relation

$$\hat{\mathcal{F}}^{(N)}(t,\Psi_0)\Phi(\vec{x},t) = \int_0^t d\tau\hat{U}_0(t,\tau,\Psi_0)\hat{\mathfrak{H}}^{(N)}(\tau,\Psi_0)\Phi(\vec{x},\tau), \tag{8.7}$$

where $\hat{U}_0(t,\tau)$ is the evolution operator of the associated Schrödinger equation (5.21) and

$$\hat{\mathfrak{H}}^{(N)}(t,\Psi_0) = \sum_{k=1}^{N} \hbar^{k/2}\hat{\mathfrak{H}}_k(t,\Psi_0). \tag{8.8}$$

Thus we have proved the following theorem.

**Theorem 8.1.** *Let the symbols of the operators $\hat{\mathscr{H}}(t)$ and $\hat{V}(t,\Psi)$ satisfy the conditions of Supposition 2.1. Then the function*

$$\Psi^{(N)}(\vec{x},t,\hbar) = \sum_{k=0}^{N} \frac{1}{k!}\left\{-\frac{i}{\hbar}\hat{\mathscr{F}}^{(N)}(t,\Psi_0)\right\}^k \hat{U}_0(t,0,\Psi_0)\Psi_0(\vec{x},\hbar), \tag{8.9}$$

*where $N \geq 2$, is an asymptotic, up to $O(\hbar^{(N+1)/2})$, solution of (2.1) and satisfies the initial condition (3.20).*

**9. The Green function and the nonlinear superposition principle.** We show that in the class of trajectory-concentrated functions for the Hartree type equation (2.1) we can construct, with any given accuracy in $\hbar^{1/2}$, the kernel of the evolution operator or the Green function of the Cauchy problem for (2.1). The explicit form of the semiclassical asymptotics $\Psi^{(N)}(\vec{x},t,\hbar)$ in (8.9) makes it possible to obtain an expression for the Green function $G^{(N)}(\vec{x},\vec{y},t,s,\Psi_0)$ valid on finite time intervals $t \in [0,T]$. Actually, according to (8.9), for any function $\varphi(\vec{x},\hbar) \in \mathscr{P}_\hbar^0$, the solution of the Cauchy problem with the initial condition

$$\Phi(\vec{x},t,\hbar)|_{t=0} = \varphi(\vec{x},\hbar), \tag{9.1}$$

for the associated linear Schrödinger equation (5.12), has the form

$$\begin{aligned}\Phi^{(N)}(\vec{x},t,\hbar) &= \hat{R}^{(N)}(t,\Psi_0)\int_{\mathbb{R}^n} d\vec{y}\,G^{(0)}(\vec{x},\vec{y},t,0,\Psi_0)\varphi(\vec{y},\hbar) + O(\hbar^{(N+1)/2})\\ &= \int_{\mathbb{R}^n} d\vec{y}\,G^{(N)}(\vec{x},\vec{y},t,0,\Psi_0)\varphi(\vec{y},\hbar) + O(\hbar^{(N+1)/2}),\end{aligned} \tag{9.2}$$

where

$$\hat{R}^{(N)}(t,\Psi_0) = \sum_{k=0}^{N} \frac{1}{k!}\left\{-\frac{i}{\hbar}\hat{\mathscr{F}}^{(N)}(t,\Psi_0)\right\}^k, \tag{9.3}$$

and the function $G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)$ is defined in (7.24).

It follows that

$$G^{(N)}(\vec{x},\vec{y},t,0,\Psi_0) = \hat{R}^{(N)}(t,\Psi_0)G^{(0)}(\vec{x},\vec{y},t,0,\Psi_0). \tag{9.4}$$

Since $\hat{R}^{(N)}(0,\Psi_0) = 1$, we have, for an arbitrary $s < t$,

$$G^{(N)}(\vec{x},\vec{y},t,s,\Psi_0) = \hat{R}^{(N)}(t,\Psi_0)G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)(\hat{R}^{(N)}(s,\Psi_0))^+, \tag{9.5}$$

which is the Green function of the Cauchy problem (9.1) with $s \neq 0$. Obviously, for the functions $G^{(N)}(\vec{x},\vec{y},t,s,\Psi_0)$, the following composition rule is valid:

$$\int d\vec{u}G^{(N)}(\vec{x},\vec{u},t,\tau,\Psi_0)G^{(N)}(\vec{u},\vec{y},\tau,s,\Psi_0) = G^{(N)}(\vec{x},\vec{y},t,s,\Psi_0) + O(\hbar^{(N+1)/2}). \quad (9.6)$$

Denoting by $\hat{U}^{(N)}(t,0,\Psi_0)$ the approximate evolution operator of the linear equation (5.12), then

$$\hat{U}^{(N)}(t,0,\Psi_0)\varphi(\vec{x},\hbar) = \int d\vec{y}\,G^{(N)}(\vec{x},\vec{y},t,0,\Psi_0)\varphi(\vec{y},\hbar), \quad (9.7)$$

we obtain it from (9.5) in the form of the $T$-ordered Dyson expansion,

$$\hat{U}^{(N)}(t,0,\Psi_0) = \sum_{k=0}^{N}\left(-\frac{i}{\hbar}\right)^k \int_{\Delta_k^>} d^k\tau\hat{\mathfrak{H}}_1(\tau_1,t,\Psi_0)\cdots\hat{\mathfrak{H}}_1(\tau_k,t,\Psi_0)\hat{U}_0(t,0,\Psi_0). \quad (9.8)$$

Here, we have used the following notations [40]: the domain of integration is an open hypertriangle,

$$\Delta_k^> \equiv \{\tau \in [0,t]^k \mid t > \tau_1 > \tau_2 > \cdots > \tau_N > s\}, \quad (9.9)$$

the operator $\hat{\mathfrak{H}}_1(\tau,t,\Psi_0)$ is a perturbation operator in the representation of the interaction

$$\hat{\mathfrak{H}}_1(\tau,t) = \hat{U}_0(t,\tau,\Psi_0)\hat{\mathfrak{H}}^{(N)}(\tau,\Psi_0)\hat{U}_0^+(\tau,t,\Psi_0), \quad (9.10)$$

where $\hat{\mathfrak{H}}^{(N)}(t,\Psi_0)$ has been defined in (5.13), and $\hat{U}_0(t,s,\Psi_0)$ is the evolution operator of the associated linear Schrödinger equation with the kernel $G^{(0)}(\vec{x},\vec{y},t,s,\Psi_0)$ in (7.24).

In view of Statement 5.2, the action of operator (9.8) on the function $\varphi = \Psi_0(\vec{x},\hbar)$ determines the asymptotic solution of the Cauchy problem (2.1), (3.35) for the Hartree type equation (2.1)

$$\Psi^{(N)}(\vec{x},t,\hbar) = \hat{U}^{(N)}(t,0,\Psi_0)\Psi_0(\vec{x},\hbar), \quad \Psi_0(\vec{x},\hbar) \in \mathcal{P}_\hbar^0. \quad (9.11)$$

It follows that the operator (9.8) is an approximate evolution operator for the Hartree type equation (2.1) in the class of trajectory-concentrated functions.

For the constructed asymptotic solutions, from expression (9.11) immediately follows the following theorem.

**Theorem 9.1** (nonlinear superposition principle). *Let $\Psi_1(\vec{x},t,\hbar,y_1^{(N)}(t,\hbar))$ be an asymptotic, up to $O(\hbar^{(N+1)/2})$, solution of the Cauchy problem for the Hartree type equation (2.1) with the initial condition $\Psi_{01}(\vec{x},\hbar)$, and let the function $\Psi_2(\vec{x},t,\hbar, y_2^{(N)}(t,\hbar))$ be a solution of the same problem with the initial condition $\Psi_{02}(\vec{x},\hbar)$. Then the solution of the Cauchy problem with the initial condition*

$$\Psi_{03}(\vec{x},\hbar) = c_1\Psi_{01}(\vec{x},\hbar) + c_2\Psi_{02}(\vec{x},\hbar), \quad c_1,c_2 = \text{const}, \quad (9.12)$$

*has the form*

$$\begin{aligned}\Psi_3(\vec{x},t,\hbar,y_3^{(N)}(t,\hbar)) &= \hat{U}^{(N)}(t,0,\Psi_{03})\Psi_{03}(\vec{x})\\ &= c_1\hat{U}^{(N)}(t,0,\Psi_{03})\Psi_{01}(\vec{x})+c_2\hat{U}^{(N)}(t,0,\Psi_{03})\Psi_{02}(\vec{x})\\ &= c_1\Psi_1(\vec{x},t,\hbar,y_3^{(N)}(t,\hbar))+c_2\Psi_2(\vec{x},t,\hbar,y_3^{(N)}(t,\hbar)).\end{aligned} \tag{9.13}$$

*Here, $y_k^{(N)}(t,\hbar)$ denotes the solution of the Hamilton-Ehrenfest equations of order $N$, $N \geq 2$, (4.6), with an initial condition which is determined from the functions $\Psi_{0k}(\vec{x},\hbar)$, $k=\overline{1,3}$, respectively.*

**10. The one-dimensional Hartree type equation with a Gaussian potential.** We illustrate the above scheme for asymptotic solutions construction by the example of the Hartree type equation with a Gaussian self-action potential

$$\left\{-i\hbar\partial_t+\frac{(\hat{p})^2}{2m}+\tilde{\varkappa}V_0\int_{-\infty}^{+\infty}dy\exp\left[-\frac{(x-y)^2}{2\gamma^2}\right]\frac{|\Psi(y,t)|^2}{\|\Psi\|^2}\right\}\Psi=0. \tag{10.1}$$

Here, $\gamma$ and $V_0$ are parameters of the potential, $\tilde{\varkappa}$ is a nonlinear parameter, and $\hat{p} = -i\hbar\partial/\partial x$, $x\in\mathbb{R}^1$.

We will solve the example in two steps. First, we will seek solutions concentrated on a phase trajectory $Z(t) = (P(t),X(t))$ that does not depend on $\hbar$. In this case, a countable set of semiclassical trajectory-concentrated solutions can be constructed. Each of these solutions is a squeezed coherent state that is well known in quantum mechanics (see, e.g., [34]) and is an asymptotic solution of (10.1) with the accuracy of $O(\hbar^{3/2})$, $\hbar\to 0$. Provided that, a linear combination of the constructed functions is not a solution (even not an approximate one), which, however, is natural since we deal with a nonlinear equation.

At the second step, we solve the problem of constructing an asymptotic solution for (10.1) under the additional condition: any linear combination (with a modification) of such solutions is an approximate solution of the equation, too.

It can be found that this problem is solvable if, *first*, the sought-for functions are concentrated on the phase trajectory $Z(t,\hbar) = (P(t,\hbar),X(t,\hbar))$, regularly depends on $\sqrt{\hbar}$, that is, they belong to the class $\mathcal{P}_\hbar^t(S(t,\hbar),Z(t,\hbar))$ in (3.1). *Second*, the functions $(P(t,\hbar)$ and $X(t,\hbar))$, that determine the phase trajectory $z = Z(t,\hbar)$, are approximate solutions of the Hamilton-Ehrenfest system.

In addition, we can clarify a nonlinearity of the generalised superposition principle for the Hartree type equation. Namely, the constructed solutions depend on the two parameters $\Theta_1$ and $\Theta_2$ that are determined by the initial condition $\Psi_0\in\mathcal{P}_\hbar^t$. Then, a linear combination of approximate solutions

$$G_1\Psi_1(x,t,\hbar,\Theta_1^{(1)},\Theta_2^{(1)})+G_2\Psi_2(x,t,\hbar,\Theta_1^{(2)},\Theta_2^{(2)}) \tag{10.2}$$

is an asymptotic solution too, if the parameters $\Theta_k^{(l)}$ $k,l=1,2$ are substituted by the parameters $\Theta_k^{(3)}$, respectively, that are calculated by the linear combination at zero time moment,

$$G_1\Psi_2(x,0,\hbar)+G_2\Psi_2(x,0,\hbar). \tag{10.3}$$

We seek a solution of (10.1) in the form of the following statement:

$$\Psi(x,t,\hbar)=\varphi\left(\frac{\Delta x}{\sqrt{\hbar}},t,\sqrt{\hbar}\right)\exp\left[\frac{i}{\hbar}\left(S(t,\hbar)+P(t)\Delta x\right)\right]. \tag{10.4}$$

Here, $\varphi(\xi,t,\sqrt{\hbar})\in\mathbb{S}$ is a function in Schwartz space with respect to the variable $\xi=\Delta x/\sqrt{\hbar}$, and depends regularly on $\sqrt{\hbar}$, $\hbar\to 0$, and $\Delta x=x-X(t)$. The real functions $S(t,\hbar)$, $Z(t)=(P(t),X(t))$ are to be determined.

We expand the exponential in (10.1) in a Taylor series of $\Delta x=x-X(t)$, $\Delta y=y-X(t)$ and restrict ourselves to the terms of the order of not above four in $\Delta x$ and $\Delta y$. In view of estimates (3.19), (10.1) will then take the form

$$\begin{aligned}\Big\{&-i\hbar\partial_t+\frac{P^2(t)}{2m}+\frac{P(t)\Delta\hat{p}}{m}+\frac{\Delta\hat{p}^2}{2m}\\&+\tilde{\varkappa}V_0\Big[1-\frac{1}{2\gamma^2}\left(\Delta x^2-2\Delta x\,\alpha_\Psi^{(1)}(t,\hbar)+\alpha_\Psi^{(2)}(t,\hbar)\right)\\&\qquad+\frac{1}{8\gamma^4}\left(\Delta x^4-4\Delta x^3\alpha_\Psi^{(1)}(t,\hbar)+6\Delta x^2\alpha_\Psi^{(2)}(t,\hbar)\right.\\&\qquad\qquad\left.-4\Delta x\,\alpha_\Psi^{(3)}(t,\hbar)+\alpha_\Psi^{(4)}(t,\hbar)\right)\Big]\Big\}\Psi=O(\hbar^{5/2}),\end{aligned} \tag{10.5}$$

where $\Delta\hat{p}=\hat{p}-P(t)$, and

$$\alpha_\Psi^{(k)}(t,\hbar)=\frac{1}{\|\Psi\|^2}\int_{-\infty}^{+\infty}(\Delta y)^k|\Psi(y,t)|^2dy,\quad k=\overline{0,\infty}, \tag{10.6}$$

are the $k$-order moments centered about $X(t)$. Equation (10.5) can be simplified if we make the change

$$\begin{aligned}\Psi(x,t,\hbar)=\exp\Big\{\frac{i}{\hbar}\int_0^t\Big[&P(t)\dot{X}(t)-\frac{P^2(t)}{2m}-\tilde{\varkappa}V_0\\&-\frac{\tilde{\varkappa}}{2\gamma^2}V_0\sigma_{xx}(t,\hbar)+\frac{1}{8\gamma^4}\alpha_\Psi^{(4)}(t,\hbar)\Big]dt\Big\}\Phi(x,t,\hbar),\end{aligned} \tag{10.7}$$

where

$$\sigma_{xx}(t,\hbar)=\alpha_\Psi^{(2)}(t,\hbar)=\frac{1}{\|\Psi\|^2}\int_{-\infty}^{+\infty}\Delta y^2|\Psi(y,t)|^2dy \tag{10.8}$$

is the variance. The function $\Phi(x,t,\hbar)$ belongs to the class $\mathcal{P}_\hbar^t(\tilde{S}(t,\hbar),Z(t))$, where

$$\tilde{S}(t,\hbar)=S(t,\hbar)-\int_0^t\left[P(t)\dot{X}(t)-\frac{P^2(t)}{2m}-\tilde{\varkappa}V_0-\frac{\tilde{\varkappa}}{2\gamma^2}V_0\sigma_{xx}(t,\hbar)+\frac{1}{8\gamma^4}\alpha_\Psi^{(4)}(t,\hbar)\right]dt, \tag{10.9}$$

and satisfies the equation

$$\begin{aligned}&\Big\{-i\hbar\partial_t+P(t)\dot{X}(t)+\frac{P(t)\Delta\hat{p}}{m}+\frac{\Delta\hat{p}^2}{2m}\\&\quad+\tilde{\varkappa}V_0\Big[-\frac{1}{2\gamma^2}(\Delta x^2-2\Delta x\alpha_\Phi^{(1)}(t,\hbar))\\&\qquad+\frac{1}{8\gamma^4}(\Delta x^4-4\Delta x^3\alpha_\Phi^{(1)}(t,\hbar)+6\Delta x^2\alpha_\Phi^{(2)}(t,\hbar)-4\Delta x\alpha_\Phi^{(3)}(t,\hbar))\Big]\Big\}\Phi\\&=O(\hbar^{5/2}).\end{aligned}\tag{10.10}$$

Here, we have made use of

$$\alpha_\Psi^{(k)}(t,\hbar)=\alpha_\Phi^{(k)}(t,\hbar),\quad k=\overline{1,N}.\tag{10.11}$$

We will seek the approximate $(\bmod\,\hbar^{5/2})$ solution $\Phi(x,t,\hbar)$ to (10.10) in the form

$$\Phi(x,t,\hbar)=\Phi^{(0)}(x,t)+\sqrt{\hbar}\Phi^{(1)}(x,t)+\hbar\Phi^{(2)}(x,t)+\cdots,\tag{10.12}$$

where $\Phi^{(k)}(x,t)\in\mathscr{C}_\hbar^t(S(t,\hbar),Z(t))$. Denote by $\hat{L}_0$ and $\hat{L}_1$ the operators

$$\begin{aligned}\hat{L}_0&=-i\hbar\partial_t+P(t)\dot{X}(t)+\frac{1}{m}P(t)\Delta\hat{p}+\frac{1}{2m}\Delta\hat{p}^2-\frac{\tilde{\varkappa}V_0}{2\gamma^2}\Delta x^2;\\\hat{L}_1&=\frac{\tilde{\varkappa}V_0}{8\hbar\gamma^4}[\Delta x^4-4\Delta x^3\alpha_\Phi^{(1)}(t,\hbar)+6\Delta x^2\alpha_\Phi^{(2)}(t,\hbar)-4\Delta x\alpha_\Phi^{(3)}(t,\hbar)].\end{aligned}\tag{10.13}$$

Earlier we have shown that $\hat{L}_0=\hat{O}(\hbar)$ and $\hat{L}_1=\hat{O}(\hbar)$. In (10.10), we equate the terms having the same estimate in $\sqrt{\hbar}$ in the sense of (3.19). Then,

$$\Big[\hat{L}_0+\frac{\tilde{\varkappa}}{\gamma^2}\Delta x\alpha_{\Phi^{(0)}}^{(1)}\Big]\Phi^{(0)}=0,\quad\text{for }\hbar^1;\tag{10.14}$$

$$\Big[\hat{L}_0+\frac{\tilde{\varkappa}}{\gamma^2}\Delta x\alpha_{\Phi^{(0)}}^{(1)}\Big]\Phi^{(1)}=-2\frac{\tilde{\varkappa}V_0}{\gamma^2}\Delta x\,\mathrm{Re}\,\langle\Phi^{(0)}|\Delta x|\Phi^{(1)}\rangle\Phi^{(0)},\quad\text{for }\hbar^{3/2};\tag{10.15}$$

$$\begin{aligned}\Big[\hat{L}_0+\frac{\tilde{\varkappa}}{\gamma^2}\Delta x\alpha_{\Phi^{(0)}}^{(1)}\Big]\Phi^{(2)}&=-\frac{\tilde{\varkappa}V_0}{\gamma^2}\Delta x\Big\{[2\,\mathrm{Re}\,\langle\Phi^{(0)}|\Delta x|\Phi^{(2)}\rangle+\alpha_{\Phi^{(1)}}^{(1)}]\Phi^{(0)}\\&\quad+2\,\mathrm{Re}\,\langle\Phi^{(0)}|\Delta x|\Phi^{(1)}\rangle\Phi^{(1)}\Big\}-\hat{L}_1\Phi^{(0)},\quad\text{for }\hbar^2.\end{aligned}\tag{10.16}$$

The function

$$\Phi_0^{(0)}(x,t)=N_\hbar\Big(\frac{1}{C(t)}\Big)^{1/2}\exp\Big\{\frac{i}{\hbar}\Big(P(t)\Delta x+\frac{m}{2}\frac{\dot{C}(t)}{C(t)}\Delta x^2\Big)\Big\}\tag{10.17}$$

is a solution of (10.14). Here, we have made use of the fact that $X(t)$ and $P(t)$ are solutions of the ordinary differential equations

$$\dot{P}=0,\qquad\dot{X}=\frac{P}{m},\tag{10.18}$$

and $C(t)$ denotes the complex function satisfying the equations

$$\dot{B} = \frac{\tilde{\varkappa} V_0}{\gamma^2} C, \qquad \dot{C} = \frac{B}{m}. \tag{10.19}$$

Equations (10.18) are Hamiltonian equations with the Hamiltonian $\mathcal{H}(p,x,t) = p^2/(2m)$ and their solution is

$$P(t) = p_0, \qquad X(t) = \frac{p_0}{m} t + x_0. \tag{10.20}$$

Similarly, (10.19) are Hamiltonian equations for a harmonic oscillator with frequency

$$\Omega = \sqrt{\frac{\tilde{\varkappa}|V_0|}{m\gamma^2}}, \tag{10.21}$$

and their solution is

$$\begin{aligned} C(t) &= c_1 \exp\left(-\sqrt{\frac{\tilde{\varkappa}|V_0|}{m\gamma^2}}\, t\right) + c_2 \exp\left(\sqrt{\frac{\tilde{\varkappa}|V_0|}{m\gamma^2}}\, t\right) \quad C_1, C_2 = \text{const}, \\ B(t) &= m\dot{C}(t). \end{aligned} \tag{10.22}$$

For the initial conditions (6.18)

$$C(0) = 1, \qquad B(0) = b, \quad \operatorname{Im} b > 0, \tag{10.23}$$

two cases are possible,

$$C(t) = \begin{cases} \operatorname{ch}(\Omega t) + \dfrac{b}{\Omega} \operatorname{sh}(\Omega t), & \tilde{\varkappa} V_0 > 0, \\ \cos(\Omega t) + \dfrac{b}{\Omega} \sin(\Omega t), & \tilde{\varkappa} V_0 < 0. \end{cases} \tag{10.24}$$

In the linear case ($\tilde{\varkappa} = 0$), the frequency $\Omega = 0$ and (10.19) become equations in variations for (10.18). In view of (10.19), $N_\hbar = (m \operatorname{Im} b/\pi\hbar)^{1/4}$ and we find the variance of the coordinate $x$ in explicit form,

$$\sigma_{xx}(t,\hbar) = \sqrt{\frac{m \operatorname{Im} b}{\pi\hbar}} \cdot \int_{-\infty}^{+\infty} \frac{\Delta x^2}{|C(t)|} \exp\left[-\frac{m}{\hbar}\Delta x^2 \frac{\operatorname{Im} b}{|C(t)|^2}\right] dx = \frac{|C(t)|^2 \hbar}{2m \cdot \operatorname{Im} b}. \tag{10.25}$$

Then,

$$\Psi_0^{(0)}(x,t,\hbar) = \exp\left\{\frac{i}{\hbar}\left[\left(\frac{p_0^2}{2m} - \tilde{\varkappa} V_0\right) t + \frac{\hbar \tilde{\varkappa} V_0}{2m \cdot \operatorname{Im} b} \int_0^t |C(t)|^2 dt\right]\right\} \Phi_0^{(0)}(x,t,\hbar). \tag{10.26}$$

It can readily be noticed that $\alpha^{(2)}_{\Psi(0)}(t,\hbar) = \alpha^{(2)}_{\Phi(0)}(t,\hbar)$. Hence, from (10.24) and (10.25), it can be inferred that for $\tilde{\varkappa} V_0 < 0$ the variance $\alpha^{(2)}_{\Phi(0)}(t,\hbar)$ is limited in $t$, that is, $|\sigma_{xx}(t)| \leq M$, $M = \text{const}$, while for $\tilde{\varkappa} V_0 > 0$ it increases exponentially. In the limit of $\gamma \to 0$ and with $V_0 = (2\pi\gamma)^{-1/2}$, (10.10) becomes a nonlinear Schrödinger equation, while in the case where $\tilde{\varkappa} V_0 < 0$ ($\tilde{\varkappa} V_0 > 0$), it corresponds to the condition of existence (nonexistence) of solitons. Note that, if $\alpha^{(1)}_{\Phi^{(0)}}(t,\hbar) = 0$, the equation for the function $\Phi^{(0)}$ takes the form

$$\hat{L}_0\Phi^{(0)} = 0, \tag{10.27}$$

becoming a Schrödinger equation with a quadric Hamiltonian. We will find the solution to (10.27) satisfying an additional condition $\alpha^{(1)}_{\Phi^{(0)}}(t,\hbar) = 0$. To do this, we denote

$$\hat{a}(t) = N_a(C(t)\Delta\hat{p} - B(t)\Delta x). \tag{10.28}$$

If $C(t)$ and $B(t)$ are solutions of (10.19), the operator $\hat{a}(t)$ commutates with the operator $\hat{L}_0$. So the function

$$\Phi^{(0)}_k = \frac{1}{k!}(\hat{a}^+(t))^k\Phi^{(0)}_0 \tag{10.29}$$

will also be a solution of the Schrödinger equation (10.10). Commuting the operators $\hat{a}^+(t)$ with the function $\Phi^{(0)}_0(x,t,\hbar)$, we obtain the Fock basis of solutions for linear equation (10.27)

$$\begin{aligned}\Phi^{(0)}_k(x,t) &= \frac{1}{k!}N^k_a\Phi^{(0)}_0(x,t)(-i)^k[C^*(t)]^k\left[\hbar\frac{\partial}{\partial x} - \frac{2m\,\mathrm{Im}\,b}{|C(t)|^2}\Delta x\right]^k 1\\ &= \frac{1}{k!}N^k_a\Phi^{(0)}_0(x,t)(-i)^k[C^*(t)]^k\left(\frac{\sqrt{\hbar m\,\mathrm{Im}\,b}}{|C(t)|}\right)^k H_k\left(\Delta x\frac{\sqrt{m\,\mathrm{Im}\,b}}{|C(t)|\sqrt{\hbar}}\right),\end{aligned} \tag{10.30}$$

where $H_n(\xi)$ are Hermite polynomials. Determining $N_a$ from the condition $[\hat{a}(t), \hat{a}^+(t)] = 1$ and representing the solution of the equations in variations as

$$C(t) = |C(t)|\exp\{i\arg[C(t)]\}, \tag{10.31}$$

we get

$$\Phi^{(0)}_k(x,t) = \frac{1}{k!}(-i)^k\exp\{-ik\arg[C(t)]\}\left(\frac{1}{\sqrt{2}}\right)^k H_k\left(\Delta x\frac{\sqrt{m\,\mathrm{Im}\,b}}{|C(t)|\sqrt{\hbar}}\right)\Phi^{(0)}_0(x,t). \tag{10.32}$$

Using the properties of Hermite polynomials, we can obtain that the mean $\alpha^{(1)}_{\Phi^{(0)}_k}(t,\hbar) = 0$, $k = \overline{0,\infty}$. Then,

$$\Psi^{(0)}_k(x,t,\hbar) = \exp\left\{\frac{i}{\hbar}\left[\left(\frac{p_0^2}{2m} - \tilde{\varkappa}V_0\right)t - \frac{\tilde{\varkappa}V_0}{2\gamma^2}\alpha^{(2)}_{\Phi^{(0)}_k}(t,\hbar)\right]\right\}\Phi^{(0)}_k(x,t,\hbar). \tag{10.33}$$

Similarly, we find

$$\alpha^{(2)}_{\Phi^{(0)}_\nu}(t,\hbar) = \frac{1}{2^\nu\nu!\sqrt{\pi}}\int_{-\infty}^{\infty}\Delta x^2|\Phi^{(0)}_\nu(x,t)|^2dx = \frac{\hbar|C(t)|^2(2\nu+1)}{2m\,\mathrm{Im}\,b}, \tag{10.34}$$

and for the functions $\Psi_k^{(0)}(t,\hbar)$

$$\Psi_k^{(0)}(x,t,\hbar)=\exp\left\{\frac{i}{\hbar}\left[\left(\frac{p_0^2}{2m}-\tilde{\varkappa}V_0\right)t-\frac{\tilde{\varkappa}V_0}{2\gamma^2}\frac{\hbar(2k+1)}{2m\,\mathrm{Im}\,b}\int_0^t|C(\tau)|^2d\tau\right]\right\}\Phi_k^{(0)}(x,t,\hbar). \tag{10.35}$$

The functions (10.35) are approximate, up to $O(\hbar^{3/2})$, solutions of the Hartree type equation (10.1). However, since, for the linear combination

$$\Phi(x,t)=c_1\Phi_k^{(0)}(x,t)+c_2\Phi_l^{(0)}(x,t), \tag{10.36}$$

the condition $\alpha_{\Phi}^{(1)}(t,\hbar)=0$ is not fulfilled, $\Phi(x,t)$ is not a solution of (10.14) and, hence, the linear superposition principle is invalid for the functions (10.35) even in the class of asymptotic solutions $\mathcal{P}_\hbar^t(S(t,\hbar),P(t),X(t))$ up to $O(\hbar^{3/2})$. Thus, the presence of the term $\alpha_{\Phi(0)}^{(1)}(t,\hbar)$ in (10.14) violates the linear superposition principle (10.36).

We seek the solution to (10.1) in the class $\mathcal{P}_\hbar^t(S(t,\hbar),Z(t,\hbar))$, that is, we localize the solution asymptotically in the neighborhood of the trajectory $z=Z(t,\hbar)$ depending explicitly on parameter $\hbar$. With that, the estimates (3.19) remain valid. We take the dependence of $Z(t,\hbar)$ on the parameter $\hbar\to 0$ such that the equation for the function $\Phi^{(0)}(x,t,\hbar)$ is linear. For doing this, we subject the functions $X(t,\hbar)$ and $P(t,\hbar)$ to the equations

$$\dot{P}=\frac{\tilde{\varkappa}V_0}{\gamma^2}\left(\alpha_{\Phi(0)}^{(1)}(t,\hbar)+\frac{1}{2\gamma^2}\alpha_{\Phi(0)}^{(3)}(t,\hbar)\right),\qquad \dot{X}=\frac{P}{m}, \tag{10.37}$$

and the functions $C(t)$ and $B(t)$ to the equations

$$\dot{B}=\frac{\tilde{\varkappa}V_0}{\gamma^2}C+\frac{3}{4\gamma^2}\alpha_{\Phi(0)}^{(2)}(t,\hbar)C,\qquad \dot{C}=\frac{B}{m}. \tag{10.38}$$

The function $\Phi^{(0)}(x,t,\hbar)$ will then satisfy the equation

$$\hat{L}_0\Phi^{(0)}=0. \tag{10.39}$$

In contrast to (10.14), (10.18), (10.19), equations (10.37), (10.38), and (10.39) are dependent. Note that, within the accuracy under consideration, the principal term of the asymptotic will not change if (10.37) and (10.38) are solved accurate to $O(\hbar^{3/2})$ and $O(\hbar)$, respectively. Then (10.37) become

$$\dot{p}=\frac{\tilde{\varkappa}V_0}{\gamma^2}\alpha_{\Phi(0)}^{(1)}(t,\hbar),\qquad \dot{x}=\frac{p}{m}, \tag{10.40}$$

and (10.38) coincide with (10.19) and their solution has the form (10.24). Equation (10.39) is linear and its general solution can be represented as an expansion over a complete set of orthonormal functions $\Phi_k^{(0)}(x,t,\hbar)$,

$$\Phi^{(0)}(x,t,\hbar)=\sum_{k=0}^{\infty}c_k\Phi_k^{(0)}(x,t,\hbar). \tag{10.41}$$

Here, $\Phi_k^{(0)}(x,t)$ is determined by expression (10.32), where $X(t)$ and $P(t)$ ought to be replaced by $X(t,\hbar)$ and $P(t,\hbar)$, respectively. Substitute (10.41) into (10.24). In view of the properties of Hermite polynomials,

$$\begin{aligned}\int_{-\infty}^{+\infty}\xi H_n(\xi)H_l(\xi)e^{-\xi^2}\,d\xi &= \int_{-\infty}^{+\infty}\xi\Big[\frac{1}{2}H_{n+1}(\xi)+nH_{n-1}(\xi)\Big]H_l(\xi)e^{-\xi^2}\,d\xi\\ &= \frac{1}{2}\delta_{n+1,l}+n\delta_{n-1,l},\end{aligned}\tag{10.42}$$

we obtain

$$\begin{aligned}\alpha^{(1)}_{\Phi^{(0)}}(t,\hbar) &= \frac{\sqrt{\hbar}\,|C(t)|}{m\,\mathrm{Im}\,b}\sum_{n=0}^{\infty}\Big(\frac{1}{2}\delta_{n+1,l}+n\delta_{n-1,l}\Big)c_nc_l^*\\ &= \frac{\sqrt{\hbar}\,|C(t)|}{m\,\mathrm{Im}\,b}\sum_{n=0}^{\infty}\Big(\frac{1}{2}c_{n+1}^*+nc_{n-1}^*\Big)c_n.\end{aligned}\tag{10.43}$$

Equations (10.40) will then take the form

$$\dot{P}=\sqrt{\hbar}\Theta_1|C(t)|,\qquad \dot{X}=\frac{P}{m},\tag{10.44}$$

where

$$\Theta_1=\frac{\tilde{\varkappa}V_0}{m\gamma^2\,\mathrm{Im}\,b}\sum_{n=0}^{\infty}\Big(\frac{1}{2}c_{n+1}^*+nc_{n-1}^*\Big)c_n.\tag{10.45}$$

Integration of the above equations, (10.44) yields

$$\begin{aligned}&P(t,\hbar)=m\dot{X}(t,\hbar),\\ &X(t,\hbar)=\sqrt{\hbar}\frac{\Theta_1}{m}\int_0^t d\tau\int_0^\tau|C(s)|\,ds+\frac{p_0}{m}t+x_0.\end{aligned}\tag{10.46}$$

As a result, the principal term of the asymptotic can be represented in the form

$$\Psi^{(0)}(x,t,\hbar)=\exp\Big\{\frac{i}{\hbar}\Big[\int_0^t\Big(\frac{m}{2}\dot{X}^2(\tau,\hbar)-\tilde{\varkappa}V_0-\hbar\Theta_2|C(\tau)|^2\Big)d\tau\Big]\Big\}\Phi^{(0)}(x,t,\hbar),\tag{10.47}$$

where

$$\Theta_2=\frac{\tilde{\varkappa}V_0}{2m\,\mathrm{Im}\,b}\sum_{n=0}^{\infty}\Big[\frac{1}{4}c_{n+2}^*+\Big(n+\frac{1}{2}\Big)c_n^*+(n^2-n)c_{n-2}^*\Big]c_n.\tag{10.48}$$

It follows that the function (10.47) depends on $\Theta_1$ and $\Theta_2$ as on parameters:

$$\Psi^{(0)}(x,t,\hbar)=\Psi^{(0)}(x,t,\hbar,\Theta_1,\Theta_2).\tag{10.49}$$

Here, $\Theta_1$ and $\Theta_2$ are determined by the sets of (10.45) and (10.48), respectively.

Consider the Cauchy problem for (10.1)

$$\begin{gathered}\Psi_1\big|_{t=0}=\Psi_{10}(x),\qquad \Psi_2\big|_{t=0}=\Psi_{20}(x),\\ \Psi_3\big|_{t=0}=\Psi_{30}(x)=G_1\Psi_{10}(x)+G_2\Psi_{20}(x),\qquad \Psi_k(x)\in\mathcal{P}_\hbar^t,\end{gathered}\tag{10.50}$$

where $G_k=\text{const}$. Denote by $\Psi_k(x,t,\hbar,\Theta_1^{(k)},\Theta_2^{(k)})$ the principal term of the asymptotic solution of (10.1), satisfying the initial conditions (10.50). Then, from the explicit form of function (10.35) the following equation follows:

$$\Psi_3(x,t,\hbar,\Theta_1^{(3)},\Theta_2^{(3)})=G_1\Psi_1(x,t,\hbar,\Theta_1^{(3)},\Theta_2^{(3)})+G_2\Psi_2(x,t,\hbar,\Theta_1^{(3)},\Theta_2^{(3)}).\tag{10.51}$$

The relation (10.51) represents the nonlinear superposition principle for the asymptotic solutions of (10.1) in the class $\mathcal{P}_\hbar^t(S(t,\hbar),Z(t,\hbar))$.

## Appendix

**The set of equations in variations.** We already mentioned that to construct solutions to (5.21) in the class $\mathcal{P}_\hbar^t$, it is necessary to find solutions to the equations in variations (6.7) and to the Riccati type matrix equation (6.13). We show that the solutions of the Riccati type matrix equation can be completely expressed in terms of the solutions of the equations in variations $a(t)$.

We present the $2n$-space vector $a(t)$ in the form

$$a(t,\Psi_0)=(\vec{W}(t,\Psi_0),\vec{Z}(t,\Psi_0)),\tag{A.1}$$

where the $n$-space vector $\vec{W}(t)=\vec{W}(t,\Psi_0)$ is the "momentum" part and $\vec{Z}(t)=\vec{Z}(t,\Psi_0)$ is the "coordinate" part of the solution of the equations in variations. Thus we can write the latter as

$$\begin{aligned}\dot{\vec{W}}&=-\mathfrak{H}_{xp}(t,\Psi_0)\vec{W}-\mathfrak{H}_{xx}(t,\Psi_0)\vec{Z},\\ \dot{\vec{Z}}&=\mathfrak{H}_{pp}(t,\Psi_0)\vec{W}+\mathfrak{H}_{px}(t,\Psi_0)\vec{Z}.\end{aligned}\tag{A.2}$$

The set of (A.2) is called a *set of equations in variations in vector form*. Denote by $B(t)$ and $C(t)$ the $n\times n$ matrices composed of the "momentum" and "coordinate" parts of the solution of the equations in variations:

$$B(t)=(\vec{W}_1(t),\vec{W}_2(t),\dots,\vec{W}_n(t)),\qquad C(t)=(\vec{Z}_1(t),\vec{Z}_2(t),\dots,\vec{Z}_n(t)).\tag{A.3}$$

The matrices $B(t)$ and $C(t)$ satisfy the set of equations

$$\begin{aligned}\dot{B}&=-\mathfrak{H}_{xp}(t,\Psi_0)B-\mathfrak{H}_{xx}(t,\Psi_0)C,\\ \dot{C}&=\mathfrak{H}_{pp}(t,\Psi_0)B+\mathfrak{H}_{px}(t)C,\end{aligned}\tag{A.4}$$

which is called a set of equations in variations (6.7) in matrix form.

Consider some properties of the solutions of this set of equations, which determine the explicit form of the asymptotic solution of the Hartree type equation and its approximate evolution operator.

**Remark A.1.** The set of equations in variations (6.7) is a set of linear Hamiltonian equations with the Hamiltonian function

$$H(a,t) = \frac{1}{2}\langle a, \mathfrak{H}_{zz}(t)a\rangle, \quad a \in \mathbb{C}^{2n}. \tag{A.5}$$

The complex number $\{a_1, a_2\} = \langle a_1, Ja_2\rangle$ is called a skew-scalar product of the vectors $a_1$ and $a_2$, $a_k \in \mathbb{C}^{2n}$.

Obviously, the skew-scalar product is antisymmetric

$$\{a_1, a_2\} = -\{a_2, a_1\}. \tag{A.6}$$

**Statement A.2.** The skew-scalar product $\{a_1(t), a_2(t)\}$ of the solutions $a_1(t)$ and $a_2(t)$ of the equations in variations (6.7) is invariable in time, that is,

$$\{a_1(t), a_2(t)\} = \{a_1(0), a_2(0)\} = \text{const}, \tag{A.7}$$

$$\{a_1(t), a_2^*(t)\} = \{a_1(0), a_2^*(0)\} = \text{const}. \tag{A.8}$$

This statement can be checked immediately by differentiating the skew-scalar product $\{a_1(t), a_2(t)\}$ with respect to $t$,

$$\begin{aligned}\frac{d}{dt}\{a_1(t), a_2(t)\} &= \langle \dot{a}_1(t), Ja_2(t)\rangle + \langle a_1(t), J\dot{a}_2(t)\rangle \\ &= \langle J\mathfrak{H}_{zz}(t)a_1(t), Ja_2(t)\rangle + \langle a_1(t), JJ\mathfrak{H}_{zz}(t)a_2(t)\rangle \\ &= \langle a_1(t), \mathfrak{H}_{zz}(t)a_2(t)\rangle - \langle a_2(t), \mathfrak{H}_{zz}(t)a_2(t)\rangle = 0.\end{aligned} \tag{A.9}$$

Here, we have made use of the fact that $J^2 = -\mathbb{I}_{2n\times 2n}$ and $J^t = -J$. Relation (A.8) follows from (A.8) since $a_2^*(t)$ is also a solution of the equations in variations in view of the fact that these equations are real and linear.

For the set of equations in variations in matrix form, Statement A.2 will be as follows.

**Statement A.3.** The matrices

$$D_0 = \frac{1}{2i}\left[C^+(t)B(t) - B^+(t)C(t)\right], \tag{A.10}$$

$$\tilde{D}_0 = C^t(t)B(t) - B^t(t)C(t), \tag{A.11}$$

where the matrices $B(t)$ and $C(t)$ are arbitrary solutions of the set of equations in variations (A.4), are invariable in time, and so we have

$$D_0 = \frac{1}{(2i)\left[C^+(0)B(0) - B^+(0)C(0)\right]}, \qquad \tilde{D}_0 = C^t(0)B(0) - B^t(0)C(0). \tag{A.12}$$

The relation of the matrices $B(t)$ and $C(t)$ to the matrix $Q(t)$ and, in view of (6.9), to the function $\phi_1(t)$ yields the following statement.

**Statement A.4.** Let the $n\times n$ matrices $B(t)$ and $C(t)$ be solutions to equations in variations (6.7). Then, if $\det C(t)\neq 0$, $t\in[0,T]$, the matrix $Q(t)=B(t)C^{-1}(t)$ satisfies the Riccati matrix equation (6.13).

Actually, in view of

$$\dot{C}^{-1}(t)=-C^{-1}(t)\dot{C}(t)C^{-1}(t), \tag{A.13}$$

and since from $C^{-1}(t)C(t)=\mathbb{I}$ it follows that $\dot{C}^{-1}(t)C(t)+C^{-1}(t)\dot{C}(t)=0$, we have

$$\begin{aligned}\dot{Q}&=\dot{B}(t)C^{-1}(t)+B(t)\dot{C}^{-1}(t)=\dot{B}(t)C^{-1}(t)-Q(t)\dot{C}(t)C^{-1}(t)\\&=[-\mathfrak{H}_{xp}(t)B-\mathfrak{H}_{xx}(t)C]C^{-1}-Q[\mathfrak{H}_{pp}(t)B+\mathfrak{H}_{px}(t)C]C^{-1}\\&=-\mathfrak{H}_{xp}(t)Q-\mathfrak{H}_{xx}(t)-Q\mathfrak{H}_{pp}(t)Q-Q\mathfrak{H}_{px}(t).\end{aligned} \tag{A.14}$$

A similar property is also valid for the matrix $Q^{-1}(t)$

$$-\dot{Q}^{-1}+Q^{-1}\mathfrak{H}_{xx}(t)Q^{-1}+\mathfrak{H}_{px}(t)Q^{-1}+Q^{-1}\mathfrak{H}_{xp}(t)+\mathfrak{H}_{pp}(t)=0. \tag{A.15}$$

**Statement A.5.** If at the time zero the matrix $Q(t)$ is symmetric ($Q(0)=Q^t(0)$ at $t=0$), it is symmetric at any time $t\in[0,T]$ (i.e., $Q(t)=Q^t(t)$). Here, $A^t$ denotes the transpose to the matrix $A$.

Actually, from (6.13) it follows that

$$\dot{Q}^t+\mathfrak{H}^t_{xx}(t)+\mathfrak{H}^t_{px}(t)Q^t+Q^t\mathfrak{H}^t_{xp}(t)+Q^t\mathfrak{H}^t_{pp}(t)Q^t=0, \tag{A.16}$$

since

$$\mathfrak{H}_{xx}=\mathfrak{H}^t_{xx},\qquad \mathfrak{H}_{pp}=\mathfrak{H}^t_{pp},\qquad \mathfrak{H}_{px}=\mathfrak{H}^t_{xp}. \tag{A.17}$$

Hence, the matrix $Q^t(t)$ satisfies (6.13) with the same initial conditions as the matrix $Q(t)$, since, as agreed, the matrix $Q(0)$ is symmetric. The validity of the statement follows from the uniqueness of the solution of the Cauchy problem.

**Statement A.6.** The imaginary parts of the matrices $Q(t)$ and $Q^{-1}(t)$ can be represented in the form

$$\operatorname{Im}Q(t)=(C^+(t))^{-1}D_0C^{-1}(t), \tag{A.18}$$

$$\operatorname{Im}Q^{-1}(t)=-(B^{-1}(t))^{+}D_0B^{-1}(t). \tag{A.19}$$

Here, the matrix $D_0$ is defined by relation (A.10).

Actually, by definition,

$$\begin{aligned}\operatorname{Im}Q(t)&=\frac{i}{2}[Q^+(t)-Q(t)]=\frac{i}{2}\{[B(t)C^{-1}(t)]^+-B(t)C^{-1}(t)\}\\&=\frac{i}{2}[C^+(t)]^{-1}[B^+(t)C(t)-C^+(t)B(t)]C^{-1}(t)\\&=[C^+(t)]^{-1}D_0C^{-1}(t),\end{aligned} \tag{A.20}$$

Similarly, relation (A.19) can be proved.

**Statement A.7.** Let the matrix $Q(t)$ be definite and symmetric and the components of the vector $\vec{y}_j^t(t)$, $j=\overline{1,n}$, be the row elements of the matrix $C^{-1}(t)$ in (A.45). Then the vectors $\vec{y}_j^t(t)$ satisfy the set of equations

$$\dot{\vec{y}} = -\mathcal{H}_{xp}(t)\vec{y} - Q(t)\mathcal{H}_{pp}(t)\vec{y}. \tag{A.21}$$

Actually, we have

$$\dot{C}^{-1} = -C^{-1}\dot{C}C^{-1}, \tag{A.22}$$

and hence,

$$\dot{C}^{-1} = -C^{-1}[\mathcal{H}_{pp}(t)Q(t)+\mathcal{H}_{px}(t)]. \tag{A.23}$$

Transposing relation (A.23) for the vectors $\vec{y}(t)$ (A.45), we obtain (A.21).

**Remark A.8.** If the matrix

$$Q^{-1}(t) = C(t)B^{-1}(t) \tag{A.24}$$

is definite and symmetric, the matrix $B^{-1}$ satisfies the equation

$$\dot{B}^{-1} = B^{-1}[\mathcal{H}_{xx}(t)Q^{-1}(t)+\mathcal{H}_{xp}(t)]. \tag{A.25}$$

The proof is similar to that of Statement A.7.

**Statement A.9.** If the matrix $D_0$ (A.10) is positive definite, the relation

$$2iB^{-1}(t)\mathcal{H}_{xx}(t)\left(B^{-1}(t)\right)^t = \frac{d}{dt}\left[D_0^{-1}B^+(t)\left(B^t(t)\right)^{-1}\right] \tag{A.26}$$

is valid.

Actually, from (A.19) it follows that

$$\begin{aligned}
&B^{-1}(t)\mathcal{H}_{xx}(t)\left(B^{-1}(t)\right)^t \\
&\quad= \frac{i}{2}D_0^{-1}B^+(t)\left[Q^{-1}(t)-\left(Q^*(t)\right)^{-1}\right]\mathcal{H}_{xx}(t)\left(B^{-1}(t)\right)^t \\
&\quad= \frac{i}{2}D_0^{-1}\Big[B^{-1}(t)\Big(\mathcal{H}_{xx}(t)C(t)B^{-1}(t)-\mathcal{H}_{xx}(t)C^+(t)\left(B^{-1}(t)\right)^*+\mathcal{H}_{px}(t)B(t)B^{-1}(t) \\
&\qquad\qquad -\mathcal{H}_{px}(t)B^*(t)\left(B^{-1}(t)\right)^*\Big)\left(2i\operatorname{Im}Q^{-1}(t)\right)B^*(t)D_0^{-1}\Big]^t \\
&\quad= \frac{i}{2}D_0^{-1}\left[B^{-1}(t)\dot{B}(t)B^{-1}(t)B^*(t)-B^{-1}(t)\dot{B}^*(t)\right]^t \\
&\quad= \frac{i}{2}D_0^{-1}\left[\frac{d}{dt}B^{-1}(t)B^*(t)\right]^t.
\end{aligned} \tag{A.27}$$

**Statement A.10.** If for the equations in variations (A.4) the Cauchy problem is formulated as

$$\tilde{B}(t)|_{t=s} = B_0, \quad \tilde{C}(t)|_{t=s} = 0, \quad B_0^t = B_0, \tag{A.28}$$

then the relation

$$\int_s^t \tilde{B}^{-1}(\tau)\mathcal{H}_{xx}(t)(\tilde{B}^{-1}(\tau))^t\,d\tau = (B_0^{-1})^t\lambda_2(\Delta t,\Psi_0)\lambda_4^{-1}(\Delta t,\Psi_0)(B_0^{-1})^t \tag{A.29}$$

is valid. Here, $\lambda_k(\Delta t,\Psi_0)$, $k=\overline{1,4}$, denote the $n\times n$ matrices being blocks of the matriciant of the set of equations in variations (6.7),

$$A(t,\Psi_0) = \begin{pmatrix} \lambda_4^t(t,\Psi_0) & \lambda_2^t(t,\Psi_0) \\ \lambda_3^t(t,\Psi_0) & \lambda_1^t(t,\Psi_0) \end{pmatrix}, \qquad A(0,\Psi_0) = \mathbb{I}_{2n\times 2n}. \tag{A.30}$$

Consider an auxiliary Cauchy problem formulated as

$$B(t,\epsilon)|_{t=s} = B_0, \qquad C(t,\epsilon)|_{t=s} = \epsilon\mathbb{I}, \quad \mathbb{I} = ||\delta_{k,j}||_{n\times n}. \tag{A.31}$$

Obviously, we have

$$\begin{gathered} \lim_{\epsilon\to 0} B(t,\epsilon) = \tilde{B}(t), \qquad \lim_{\epsilon\to 0} C(t,\epsilon) = \tilde{C}(t), \\ D_0(\epsilon) = \frac{\epsilon}{2i}(B_0 - B_0^*). \end{gathered} \tag{A.32}$$

We assume that the matrix $D_0(\epsilon)$ is symmetric and positive definite for $\epsilon\neq 0$. Hence, we may use relationship (A.26) and then obtain

$$\int_s^t B^{-1}(\tau,\epsilon)\mathcal{H}_{xx}(\tau)(B^{-1}(\tau,\epsilon))^t\,d\tau = -\frac{1}{\epsilon}(B_0-B_0^*)^{-1}B^+(\tau,\epsilon)(B^{-1}(\tau,\epsilon))^t\Big|_s^t. \tag{A.33}$$

In view of (A.30), we have

$$B(t,\epsilon) = \lambda_4^t(\Delta t)B_0 - \epsilon\lambda_2^t(\Delta t), \tag{A.34}$$

and, hence,

$$B^{-1}(t,\epsilon) = (1+\epsilon B_0^{-1}(\lambda_4^{-1}(\Delta t))^t\lambda_2^t(\Delta t))B_0^{-1}(\lambda_4^{-1}(\Delta t))^t + O(\epsilon^2). \tag{A.35}$$

Then we obtain

$$\begin{aligned} &\lim_{\epsilon\to 0}\frac{1}{\epsilon}B^+(\tau,\epsilon)(B^{-1}(t,\epsilon))^t\Big|_s^t \\ &\quad= \Big[\lim_{\epsilon\to 0}[B^{-1}(t,\epsilon)B^*(t,\epsilon) - B_0^{-1}B_0^*]\Big]^t \\ &\quad= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{B_0^{-1}B_0^* - \epsilon B_0^{-1}(\lambda_4^{-1}(\Delta t))^t\lambda_2^t(\Delta t)(1-B_0^{-1}B_0^*) - B_0^{-1}B_0^* + O(\epsilon^2)\}^t \\ &\quad= -(B_0-B_0^*)(B_0^{-1})^t\lambda_2(\Delta t)\lambda_4^{-1}(\Delta t)(B_0^{-1})^t. \end{aligned} \tag{A.36}$$

Substitution of the obtained expression into (A.33) yields (A.29).

**Statement A.11.** If the matrix $D_0$ in (A.10) is positive definite and, symmetric, and the matrix $\tilde{D}_0$ (A.11) is zero, the following relationships are valid:

$$C^*(t)D_0^{-1}B^t(t)-C(t)D_0^{-1}B^+(t)=B(t)D_0^{-1}C^+(t)-B^*(t)D_0^{-1}C^t(t)=2i\mathbb{I}_{n\times n}, \tag{A.37}$$

$$C^*(t)D_0^{-1}C^t(t)-C(t)D_0^{-1}C^+(t)=B(t)D_0^{-1}B^+(t)-B^*(t)D_0^{-1}B^t(t)=0_{n\times n}. \tag{A.38}$$

Consider an auxiliary matrix $T(t)$ of dimension $2n\times 2n$,

$$T(t)=\frac{1}{\sqrt{2}}\begin{pmatrix} D_0^{-1/2}C^t(t) & -D_0^{-1/2}B^t(t) \\ D_0^{-1/2}C^+(t) & -D_0^{-1/2}B^+(t)\end{pmatrix}, \tag{A.39}$$

and find its inverse matrix. Direct checking makes us convinced that

$$T^{-1}(t)=-\frac{i}{\sqrt{2}}\begin{pmatrix} -(B(t)D_0^{-1/2})^* & B(t)D_0^{-1/2} \\ -(C(t)D_0^{-1/2})^* & C(t)D_0^{-1/2}\end{pmatrix}. \tag{A.40}$$

Actually, we have

$$\begin{aligned} &T(t)T^{-1}(t)\\ &\quad=-\frac{i}{2}\begin{pmatrix} -[C^t(t)B^*(t)-B^t(t)C^*(t)](D_0^{-1})^* & [C^t(t)B(t)-B^t(t)C(t)]D_0^{-1} \\ -[C^+(t)B^*(t)-B^+(t)C^*(t)](D_0^{-1})^* & [C^+(t)B(t)-B^+(t)C(t)]D_0^{-1}\end{pmatrix}\\ &\quad=-\frac{i}{2}\begin{pmatrix} -D_0^{-1/2}(2iD_0)^*(D_0^{-1/2})^* & D_0^{-1/2}\tilde{D}_0D_0^{-1/2} \\ -D_0^{-1/2}\tilde{D}_0^*(D_0^{-1/2})^* & iD_0^{-1/2}D_0D_0^{-1/2}\end{pmatrix}=\mathbb{I}_{2n\times 2n}. \end{aligned} \tag{A.41}$$

From the uniqueness of the inverse matrix follows,

$$T(t)T^{-1}(t)=T^{-1}(t)T(t)=\mathbb{I}_{2n\times 2n}, \tag{A.42}$$

that is,

$$\begin{aligned} T(t)T^{-1}(t)&=-\frac{i}{2}\begin{pmatrix} -(BD_0^{-1})^*C^t+BD_0^{-1}C^+ & (BD_0^{-1})^*B^t+BD_0^{-1}B^+ \\ -(CD_0^{-1})^*C^t+CD_0^{-1}C^+ & (CD_0^{-1})^*B^t+CD_0^{-1}B^+\end{pmatrix}\\ &=\mathbb{I}_{2n\times 2n}. \end{aligned} \tag{A.43}$$

However, as assigned, we have $D_0^t=D_0$, and from definition (A.10) it follows that

$$D_0^+=-\frac{1}{2i}(C^+B-B^+C)^+=-\frac{1}{2i}(B^+C-C^+B)=D_0. \tag{A.44}$$

We then have $D_0^*=D_0$ and, hence, $(D_0^{-1})^*=D_0^{-1}$. Then from (A.43) we obtain (A.37) and (A.38).

The following properties of the solutions to the set of equations in variations are dramatically important for the construction of semiclassical asymptotics in the class of functions $\mathcal{P}_\hbar^t(Z(t,\hbar),S(t,\hbar))$.

**Lemma A.12.** *Let the matrix $D_0$ be diagonal and positive definite and* $\det C(t) \neq 0$. *The matrix* $\operatorname{Im} Q(t)$ *will then be positive definite as well.*

**Proof.** Let $D_0 = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0$, $j = \overline{1,n}$. Denote by $\vec{y}_j^t$ the rows of the matrix $C^{-1}(t)$,

$$C^{-1}(t) = \begin{pmatrix} \vec{y}_1^t(t) \\ \vec{y}_2^t(t) \\ \vdots \\ \vec{y}_n^t(t) \end{pmatrix}. \tag{A.45}$$

Then for an arbitrary complex vector $|\vec{p}| \neq 0$, we obtain

$$(\vec{p})^+ \operatorname{Im} Q(t) \vec{p} = \sum_{j=1}^{n} \langle \vec{p}, \vec{y}_j(t) \rangle^+ \alpha_j \langle \vec{y}_j(t), \vec{p} \rangle = \sum_{j=1}^{n} \alpha_j \left| \langle \vec{p}, \vec{y}_j(t) \rangle \right|^2 > 0. \tag{A.46}$$

Inequality (A.46) is true since $|\vec{y}_j(t)| \neq 0$ and $\alpha_j > 0$, $j = \overline{1,n}$. From this inequality, in view of the arbitrariness of the vector $\vec{p} \in \mathbb{C}^n$, $|\vec{p}| \neq 0$, it follows that the lemma is true. □

**Lemma A.13.** *The matrix $C(t)$ is nondegenerate,* $\det C(t) \neq 0$, *if the matrix* $D_0 = (2i)^{-1}(C^+(0)B(0) - B^+(0)C(0))$ *is positive definite.*

**Proof.** Assume that, for some $t_1$, $\det C(t_1) = 0$. Then a vector $\vec{k}$, $|\vec{k}| \neq 0$, exists, such that

$$C(t_1) \cdot \vec{k} = 0, \quad (\vec{k}^+ C^+(t_1) = 0). \tag{A.47}$$

Since relation (A.11) is valid for any $t$, then

$$\vec{k}^+ D_0 \vec{k} = \vec{k}^+ \left\{ \frac{i}{2} [B^+(t_1) C(t_1) - C^+(t_1) B(t_1)] \right\} \vec{k} = 0. \tag{A.48}$$

As agreed, the matrix $D_0$ is positive definite, and, hence, the above equality holds only for $|\vec{k}| = 0$. The obtained contradiction proves the lemma. □

**Lemma A.14** (Liouville's lemma). *If the matrix $Q(t)$ is continuous, the relation*

$$\exp\left\{ -\frac{1}{2} \int_0^t \operatorname{Sp}[\mathfrak{H}_{pp}(t) Q(t) + \mathfrak{H}_{px}(t)] dt \right\} = \sqrt{\frac{\det C(0)}{\det C(t)}} \tag{A.49}$$

*is valid.*

**Proof.** From (A.4) it follows that

$$\dot{C} = [\mathfrak{H}_{pp}(t) Q(t) + \mathfrak{H}_{px}(t)] C, \tag{A.50}$$

where the matrix $Q(t)$ is a solution of (6.13), and the matrices $\mathfrak{H}_{pp}(t)$ and $\mathfrak{H}_{px}(t)$ are continuous. Then for the matrix $C(t)$ the Jacobi identity,

$$\det C(t) = \left[\det C(0)\right] \exp \int_0^t \operatorname{Sp}\left[\mathfrak{H}_{pp}(t)Q(t) + \mathfrak{H}_{px}(t)\right] dt \tag{A.51}$$

is valid. Raising the left and right parts of the equality to the power $-1/2$ yields (A.49). □

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