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RINGS WITH INVOLUTION WHOSE SYMMETRIC ELEMENTS ARE CENTRAL

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ABSTRACT. In a ring R with involution whose symmetric elements S are central, the skew-symmetric elements K form a Lie algebra over the commutative ring S. The classification of such rings which are 2-torsion free is equivalent to the classification of Lie algebras K over S equipped with a bilinear form f that is symmetric, invariant and satisfies $[[x,y],z] = f(y,z)x - f(z,x)y$. If S is a field of char $\neq 2$, $f \neq 0$ and $\dim K > 1$ then K is a semisimple Lie algebra if and only if f is nondegenerate. Moreover, the derived algebra K' is either the pure quaternions over S or a direct sum of mutually orthogonal abelian Lie ideals of $\dim \leq 2$.

KEY WORDS AND PHRASES. *Ring with involution, symmetric and skew-symmetric elements, Lie algebra, symmetric and invariant bilinear form, Cartan's Criterion of semisimplicity of Lie algebras, pure quaternions, mutually orthogonal abelian Lie ideals.*

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1. INTRODUCTION and MAIN RESULTS.

Let R be a ring with an involution $*$, i.e., a map $R \to R$ such that for all $a,b \in R$

$$(a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^* \quad \text{and} \quad a^{**} = a.$$

The sets of symmetric and skew-symmetric elements of R are respectively

$$S = \{a \in R | a^* = a\}, \quad K = \{a \in R | a^* = -a\}.$$

As usual, $[x,y] = xy - yx$ denotes the commutator of $x,y \in R$ and the symbol Z denotes the center of R.

If the symmetric elements of R are central, i.e., $S \subset Z$, then for abbreviation, R is called a CS-ring.

For all $x \in R$, $2x = x + x^* + x - x^*$ with $x + x^* \in S$, $x - x^* \in K$ and thus $2R \subset S + K$. If R is 2-torsion free then $S \cap K = 0$ and hence $\frac{1}{2} \in R$ implies that R is a group direct sum $S \oplus K$. If, additionally, R is a CS-ring then for $a \in S$, $x \in K$, $ax = xa = -(ax)^* \in K$ and therefore K is a Lie algebra over the commutative ring S with respect to commutation.

We have the following converse:

THEOREM 1. If S is a commutative ring, K is a 2-torsion free Lie algebra over S and $f:K \times K \to S$ is an S-bilinear map such that

(1) $f(x,y) = f(y,x)$ (f is symmetric)

(2) $f(x,[y,z]) = f([x,y],z)$ (f is invariant)

(3) $[[x,y],z] = f(y,z)x - f(z,x)y$

then the group direct sum $R = S \oplus K$ can be made into a CS-ring by defining the multiplication and the involution, for all $a,b \in K$, as follows:

(4) $(a + x)(b + y) = ab + f(x,y) + ay + bx + [x,y]$

(5) $(a + x)^* = a - x$.

PROOF. Let $a,b,c \in S$ and $x,y,z \in K$. Multiplication in R is associative because

$$((a + x)(b + y))(c + z) - (a + x)((b + y)(c + z))$$
$$= f(x,y)z+f([x,y],z)+[[x,y],z]-f(y,z)x-f(x,[y,z])-[x,[y,z]]$$
$$= f(x,y)z - f(y,z)x - [[z,x],y] \quad \text{by (2) and Jacobi identity}$$
$$= 0 \quad \text{by (3).}$$

From (1), (4) and (5),

$$((a + x)(b + y))^* = (b + y)^*(a + x)^*.$$

Hence, $*$ is an involution in R.

Since K is 2-torsion free, S and K are precisely the symmetric and skew-symmetric elements of R and therefore R is a CS-ring. ∎

A CS-ring will, of course, satisfy identities (1) - (3) if we put $f(x,y) = 2(xy + yx)$ for all $x,y \in K$.

Note that if K has an S-basis and f is the dot product then (2) is the triple dot product and (3) is the "triple cross product" with opposite sign, that is, $[[x,y],z] = z \times (x \times y)$. We must, however, recall that the cross product of vectors is valid only for dimension ≤ 3 and it can also be ([3,p.61] or [5]) that CS-rings satisfy the standard polynomial of degree 4.

An example of a CS-ring is a ring of quaternions $\mathcal{Q}$ over a 2-torsion free commutative ring S, where $\mathcal{Q}$ admits an S-basis $1,i,j,ij$ such that given $a,b \in S$

$$i^2 = a,\ j^2 = b,\ ij = -ji$$

and

$$1^* = 1,\ i^* = -i,\ j^* = -j.$$

The skew-symmetric part K of $\mathcal{Q}$ is a Lie algebra (with respect to commutation) of pure quaternions.

Henceforth, we shall tacitly <u>assume</u> that K is a Lie algebra over a field F of char $\neq 2$ and that K is equipped with an F-bilinear form f satisfying identities (1) - (3) such that $R = F \oplus K$ is a CS-ring with the multiplication and the involution defined according to (4) and (5).

As usual, the derived algebra and the radical of K are respectively $K' = [K,K]$, $K^{\perp} = \{x \in K | f(x,K) = 0\}$. A Lie ideal I of K is a subspace of K with $[I,K] \subset I$. It can be verified that a Lie ideal of K contained in $K^{\perp}$ is a proper ideal of R.

If $x,y \in K$ then $\langle x,y \rangle$ shall denote the F-subspace of K generated by x and y.

PROPOSITION 1. If $\dim K \neq 3$ then K' is abelian.

PROOF. We may assume $\dim K > 3$ since the prop. is trivially true for dimension < 3. From (3), we have $[[x,y],z] \in \langle x,y \rangle$ for all x,y,z in K. Thus, if x,y,z,w are linearly independent vectors of K then

$$[[x,y], [z,w]] \in \langle x,y \rangle \cap \langle z,w \rangle = 0.$$

If $w \in \langle x,y,z \rangle$ where x,y,z are linearly independent then we choose a vector v in K such that $v \notin \langle x,y,z \rangle$ and thus $w + v \notin \langle x,y,z \rangle$. Consequently, $0 = [[x,y],[z,w + v]] = [[x,y],[z,w]]$.

Continuing this argument, we obtain $[[x,y],[z,w]] = 0$ for arbitrary x,y,z,w in K and thus $[K',K'] = 0$. ∎

PROPOSITION 2. If $f \neq 0$ then $K^{\perp}$ is an ideal of R contained in K' and $\dim K/K' = 0$ or 1.

PROOF. If $z \in K^{\perp}$ then by (2), $f(x,[y,z]) = f([x,y],z) = 0$ for all $x,y \in K$ and thus $[K,z] \subset K^{\perp}$. Hence, $K^{\perp}$ is a Lie ideal of K and an ideal of R. Since $f \neq 0$ there is a nonzero vector y in K with $f(y,y) \neq 0$. If $z \in K^{\perp}$ then by (3), $[[z,y],y] = f(y,y)z$ and thus $z = f(y,y)^{-1}[[z,y],y] \in K'$. Hence, $K^{\perp} \subset K'$.

If $\dim K/K' > 1$ then let x,y be vectors in K which are linearly independent modulo K'. By (3), $[[x,y],K] \in \langle x,y \rangle \cap K' = 0$ which forces $x \in K^{\perp}$. Since $K^{\perp} \subset K'$ we have $x \in K'$, a contradiction. Hence, $\dim K/K' = 0$ or 1.

Putting $K' = 0$, we have

COROLLARY 1. If K is abelian and $\dim K > 1$ then $f = 0$ and $xy = 0$ for all $x,y \in K$.

COROLLARY 2. If $K' \neq 0$ then $\dim K/K' > 1 \Leftrightarrow f = 0 \Leftrightarrow [K',K] = 0$.

PROOF. The second equivalence follows from (3). If $f \neq 0$ then by prop. 2, $\dim K/K' \leq 1$.

Conversely, if $f = 0$ then let x,y be vectors in K with $[x,y] \neq 0$ and thus $x,y \notin K'$. If suffices to show that the images $\underset{\sim}{x}, \underset{\sim}{y}$ in K/K' are linearly independent over F. Indeed, if $\underset{\sim}{y} = a\underset{\sim}{x}$ for some $a \in F$ then $y - ax \in K'$ and thus $0 = [x,y - ax] = [x,y]$, a contradiction. Hence, $\dim K/K' > 1$. ∎

For the Lie algebras that we are considering there is a simple proof of Cartan's criterion for semisimplicity.

THEOREM 2. If $f \neq 0$ and $\dim K > 1$ then K has no nonzero abelian Lie ideals if and only if f is nondegenerate.

PROOF. If $K^{\perp} \neq 0$ then by prop. 2, $K^{\perp}$ is a nonzero Lie ideal contained in K'. By (3), $[K',K^{\perp}] = 0$ and hence $K^{\perp}$ is abelian.

Conversely, if K has a nonzero abelian Lie ideal I then let y,z be nonzero vectors in I and x be any vector of K such that x and y are linearly independent. By (3), $f(y,z)x - f(z,x)y = [[x,y],z] \in [I,I] = 0$ and thus $f(y,z) = f(z,x) = 0$. Hence, $f(z,K) = 0$ and $K^{\perp} \neq 0$. ∎

THEOREM 3. If $f \neq 0$ then K' is either a Lie algebra of pure quaternions over F or a direct sum of mutually orthogonal abelian Lie ideals of K with $\dim \leq 2$.

PROOF. We may assume $K' \neq 0$ for otherwise, prop. 2 would imply that K is of dim 1. We have only to consider the two cases, $\dim K/K' = 0, 1$.

Suppose $K' = K$. Since $[K',K'] = [K,K] \neq 0$, $\dim K = 3$ by prop. 1. If $K^{\perp} \neq 0$ then let $K = K^{\perp} \oplus V$ where $\dim V \leq 2$ and by (3), $0 = [K',K^{\perp}] = [K,K^{\perp}]$ which implies that $K' = [V,V]$ is of $\dim \leq 1$, contradictory to $K' = K$.

Hence, $K^{\perp} = 0$. Since the bilinear form f is symmetric and nondegenerate, K has an orthogonal basis x, y, z. As $K' = K$, the commutators $[x,y]$, $[y,z]$, $[z,x]$ also form a basis of K. By (2), $f(x,[x,y]) = f(y,[x,y]) = 0$ and hence $[x,y]$ is orthogonal to x and y. Consequently, $[x,y] = z$, $[y,z] = ax$, $[z,x] = by$ where $a,b \in F$. We can now easily derive from (3) that $f(x,x) = -b$, $f(y,y) = -a$ and $f(z,z) = -ab$. Hence, $K' = K$ is a Lie algebra of pure quaternions over F.

Suppose $\dim K/K' = 1$. We have $K^{\perp} \subset K'$ by prop. 2. To show $K' \subset K^{\perp}$, let $x \in K'$ and choose $0 \neq y \notin K'$. By (3), $f(y,z)x - f(z,x)y = [[x,y],z] \in K'$ for all $z \in K$ and thus $f(K,x) = 0$. Hence, $x \in K^{\perp}$ and $K^{\perp} = K'$. Moreover, $0 = [K', K^{\perp}] = [K', K']$. Since $f \neq 0$, there exists a nonzero vector $e \in K/K'$ with $f(e,e) \neq 0$. Let $d(x) = [x,e]$ for all $x \in K'$. By (3), $d^2(x) = [[x,e],e] = f(e,e)x$ and hence $d^2 = f(e,e)I$ where I is the identity map of K'. Since every nonzero vector x and K' is in the d-invariant subspace $L_x = \langle x, d(x) \rangle$, it follows from [2,p.87] that K' is completely reducible as a module for d. Clearly, each L_x is an abelian Lie ideal of K with $\dim \leq 2$ and $f(L_x, L_y) = 0$ for $x \neq y$. ∎

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