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PEANO COMPACTIFICATIONS AND PROPERTY S METRIC SPACES

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ABSTRACT. Let (X,d) denote a locally connected, connected separable metric space. We say the X is S-metrizable provided there is a topologically equivalent metric ρ on X such that (X,ρ) has Property S, i.e. for any $\varepsilon > 0$, X is the union of finitely many connected sets of ρ-diameter less than ε. It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X, then the usual metric completion $(\tilde{X},\tilde{\rho})$ of (X,ρ) is a compact, locally connected, connected metric space, i.e. $(\tilde{X},\tilde{\rho})$ is a Peano compactification of (X,ρ). There are easily constructed examples of locally connected connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space (X,d) which has a Peano compactification. In this paper we conjecture that: If (P,ρ) a Peano compactification of $(X,\rho|X)$, X must be S-metrizable. Several (new) necessary and sufficient for a space to be S-metrizable are given, together with an example of non-S-metrizable space which fails to have a Peano compactification.

KEY WORDS AND PHRASES. *Property S metrics, Peano spaces, compactifications.*
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1. INTRODUCTION.

Throughout this note let (X,d) denote a locally connected, connected separable metric space. We say that X is S-metrizable provided there is a topologically equivalent metric ρ on X such that (X,ρ) has Property S, i.e. for any $\varepsilon > 0$, X is the union of finitely many connected sets of ρ-diameter less than ε. It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X, then the usual metric completion $(\tilde{X},\tilde{\rho})$ of (X,ρ) is a compact, locally connected, connected metric space, i.e. $(\tilde{X},\tilde{\rho})$ is a Peano compactification of (X,ρ) [8,p.154].

Property S metric spaces (X,ρ) have been studied extensively in [1,2,3,4,8]. There are easily constructed examples of locally connected, connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space (X,d) which has a Peano compactification. We therefore ask:

QUESTION 1. If (P,ρ) is a Peano compactification of $(X,\rho|X)$, must X be S-metrizable?

2. DEFINITIONS AND BASIC RESULTS   A space Z is an extension of a space Y if Y is a dense subspace of Z. If Z is an extension of Y, we say that Y is locally connected in Z if Z has a basis consisting of regions (that is, open connected sets) whose intersections with Y are regions in Y. Z is a perfect extension of Y if Z is an extension of Y and whenever a closed subset H of Y separates two sets A, $B \subset Y$ in Y, the set $cl_Z H$ (the closure of H in Z) separates A, B in Z. [6]

For completeness we include the following:

THEOREM 2.1 [6]. Let Z be an extension of X. Then X is locally connected in Z if and only if Z is a perfect locally connected extension of X.

THEOREM 2.2 [6]. Let (X,d) be a metric space. Then X is S-metrizable if and only if X has a metrizable compactification Z in which it is locally connected.

THEOREM 2.3 [6]. A topological space is S-metrizable if and only if it has a perfect locally connected metrizable compactification.

THEOREM 2.4 [6]. Let X be a space having a perfect S-metrizable extension. Then X is S-metrizable.

THEOREM 2.5 [5]. Let X be a separable, locally connected, connected rim compact metric space. Then X is S-metrizable.

THEOREM 2.6 [6]. Every countable product of S-metrizable connected spaces $X_1, X_2, \ldots,$ is S-metrizable.

3. RELATED RESULTS AND QUESTIONS.

THEOREM 3.1. Let (P,d) be a Peano space and let X be a dense, locally connected, connected subset of P. Then there exists a G_δ-subset Y of P containing X such that X is locally connected in Y (as an extension of X).

PROOF. Let n be a positive integer and define $Z_n = \{y \in P$: if U is an open connected subset of P containing y and $\delta(U) < 2^{-n}$, then $U \cap X$ is not connected$\}$. (Here $\delta(U)$ denotes the d-diameter of U). We first assert that Z_n is closed. For suppose $y_1, y_2, \ldots,$ is a sequence in Z_n which converges to $y \in (P \setminus Z_n)$. Since $y \notin Z_n$, there exists an open connected subset U of P containing y and $\delta(U) < 2^{-n}$ and $U \cap Z_n \neq \phi$ and this is a contradiction. Hence Z_n is closed.

We next assert $Z_n \cap X = \phi$. For let $x \in X$ and let V be an open connected subset of X such that $\delta(\mathrm{cl}V) < 2^{-n}$. Then $U = \mathrm{int}\,\mathrm{cl}V$ is open in P and contains x and $\delta(U) < 2^{-n}$. Furthermore, $U \cap X$ is connected since $V \subseteq U \cap X \subseteq \mathrm{cl}\,V$ and V is connected. Thus $x \notin Z_n$ and $Z_n \cap X = \phi$.

Clearly $Z_1 \subset Z_2 \subset Z_3 \ldots$ is a monotonically increasing sequence and if for each $i \geq 1$, $Y_i = P \setminus Z_i$, $Y = \bigcap_{i=1}^{\infty} Y_i$ is a connected G_δ-subset of P which contains X.

We now assert that X is locally connected in Y, as an extension of X. For let $\epsilon > 0$ and let $y \in Y$. Then there exists a positive integer n so that $\epsilon > 2^{-n}$, and since $y \notin Z_n$, there exists an open connected subset U of P with $\delta(U) < 2^{-n}$ and such that $U \cap X$ is connected. This implies that $W = \mathrm{int}_Y \mathrm{cl}_Y U$ is an open connected subset of Y. Thus Y has a basis consisting of regions whose intersection with X is connected. This completes the proof.

COROLLARY 3.1.1. Every dense, locally connected, connected G_δ-subset of a Peano continuum is S-metrizable if and only if dense, locally connected, connected subset of a Peano continuum is S-metrizable.

PROOF. This follows from (2.1), (2.4) and (3.1).

Since every nested intersection of countably many sets can be represented as an inverse limit space and since every Y_i above is S-metrizable, by (2.5), we ask:

QUESTION 2. If $\{Y_i, f_{i,j}, \mathbb{N}\}$ is an inverse limit sequence of S-metrizable spaces and continuous maps (bicontinuous injections), must Y_∞ = inv lim $\{Y_i, f_{ij}, \mathbb{N}\}$ be S-metrizable?

Of course an affirmative answer to Question 2 would yield an affirmative answer to Question 1.

THEOREM 3.2. Let (X,d) be a locally connected, connected separable metric space, let βX denote the Stone-Čech compactification of X. Then X is S-metrizable if and only if there exists a Peano compactification P of X such that βf, the continuous extension of the identity injection $f: X \to P$ to βX, is monotone.

PROOF. Recall that a map between compact Hausdorff spaces is monotone if every point inverse is connected. Suppose that (X,d) is S-metrizable, say ρ is an S-metric for X. By (2.3), there exists a Peano compactification P of X and X is locally connected in P. Let $\beta f: \beta X \to P$ be the continuous extension of the identity map $f: X \to P$ to βX. We need to show that for $y \in P$, $\beta f^{-1}(y)$ is connected. But since P is a metric space and X is locally connected in P, there exists a neighborhood basis for y in P, $\{U_i\}_{i=1}^{\infty}$ such that for $i \in \mathbb{N}$, $\mathrm{cl}\, U_{i+1} \subseteq U_i$ and

$U_i \cap X$ is connected. Then, if $\beta f^{-1}(U_i) = W_i$, $\beta f^{-1}(U_i \cap X) = f^{-1}(U_i \cap X)$ is connected and $W_i \cap X = \beta f^{-1}(U_i \cap X)$. Thus by (1.4) of [7], W_i is connected. It then follows that $\beta f^{-1}(y) = \bigcap_{i=1}^{\infty} \mathrm{cl}\, W_i$ is connected and that completes the proof of the necessity.

Now suppose (P,ρ) is a Peano compactification of X and $\beta f : \beta X \to P$ is a monotone map. Let $y \in P$ and let V be an open connected subset of P containing y. Since βf is monotone, $\beta f^{-1}(V) = W$ is a connected open subset of βX. Again, by (1.4) of [7], $W \cap X$ is connected. This implies that $\beta f(W \cap X) = f(W \cap X) = V \cap X$ is connected and so X is locally connected in P. By (2.3), S is S-metrizable.

4. AN EXAMPLE. This is an example which fails to be S-metrizable, however it also fails to have a Peano compactification.

Let L_i be the line in $\mathbb{R}^2$ defined by $L_i = \{(x,y) : y = x/i, 0 \le x \le 1\}$ and let $X = \bigcup_{i=1}^{\infty} L_i$ with the relative topology inherited from $\mathbb{R}^2$. We first assert that X is not S-metrizable. For in any (Hausdorff) compactification Z of X, $U_i = L_i \setminus \{0,0)\}$ is an open subset of Z and since $A = \{(0,0)\}$ is compact, A and $B = \bigcup_{i=1}^{\infty} \{(1, i^{-1})\}$ are subsets of X whose closures are disjoint in Z. Thus if Z is a metric space with metric r and the distance from A to $\mathrm{cl}_Z B$ is ε, then $\varepsilon > 0$. It then follows that no finite collection of connected sets with r-diameter less than $\varepsilon/2$ fails to cover Z. Thus r is not a Property S metric for Z and X is not S-metrizable.

We will now show that X fails to have a locally connected metric compactification. Suppose (Z,r) is a locally connected metric compactification of X. Let U and V be open subsets of Z containing (0,0) such that $\mathrm{cl}\, U \subseteq V \subseteq (Z \setminus \mathrm{cl}\, B)$ (B is defined above). Then each L_i intersects bd U and bd V and contains a subarc S_i such that $S_i \subseteq (\mathrm{cl}\, V \setminus U)$ and S_i meets each of bd V and bd U in a single point, say $S_i \cap \mathrm{bd}\, V = \{a_i\}$ and $S_i \cap \mathrm{bd}\, U = \{b_i\}$. Without loss of generality we may suppose that $\{a_i\}_{i=1}^{\infty}$ converges to a point $a \in \mathrm{bd}\, V$ and $\{b_i\}_{i=1}^{\infty}$ converges to a point $b \in$ bd $b \in \mathrm{bd}\, U$. Then $L = \lim\sup \{S_i : i \in \mathbb{N}\}$ is a connected set subset of $\mathrm{cl}\, V \setminus U$ meeting bd U and bd V [8, p. 14]. Then since every point of $L \setminus (\mathrm{bd}\, U \cup \mathrm{bd}\, V)$ is a limit point of $\bigcup_{i=1}^{\infty} S_i$ and each S_i is a component of $\mathrm{cl}\, V \setminus U$, Z fails to be locally connected at any point of $L \setminus (\mathrm{bd}\, U \cup \mathrm{bd}\, V)$. Thus X fails to have a Peano compactification.

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