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# SEMIGROUP STRUCTURE UNDERLYING EVOLUTIONS

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ABSTRACT. A member of a class of evolution systems is defined by averaging a one-parameter family of invertible transformations $G$ with a semigroup $T$. The resulting evolution system, $U(t,s) = G(t)T(t-s)G(s)^{-1}$, preserves continuity and strong continuity, and in case $G$ is a linear family, may have an identifiable generator and resolvent both of which are constructed from $T$. Occurrences of the class of evolutions are given to show possible applications.

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## 1. INTRODUCTION.

Given a subset $C$ of a Banach space $H$, a strongly continuous evolution system of continuous transformations on $C$ is a family of functions $U = \{U(t,s): t \geq s \geq 0\}$ so that if $t \geq s \geq r \geq 0$, then

i. $U(t,s)$ is a continuous function from $C$ into $C$,

ii. $U(s,s) = I$, the identity on $C$,

iii. $U(t,s)U(s,r) = U(t,r)$, and (D1

iv. if $x \in C$, then $\{(p, U(p,s)x) : p \geq s\}$ is continuous.

Such systems arise as a result of having unique solutions to the evolution equation $y'(t) = A(t)(y(t))$, $y(s) = x$, where x is a point of C and if $t \geq 0$, A(t) is a function from a subset of C into H.

The study of evolution systems has in several papers been made through the study of semigroups (see for instance Kato[5]). A strongly continuous semigroup of continuous transformations on C is a family $T = \{T(t): t \geq 0\}$ of continuous functions from C into C so that if t and s are in $[0, \infty)$, then

i. $T(t+s) = T(t)(T(s))$,

ii. $T(0) = I$, and

iii. if x is an element of C, $\{(p, T(p)(x)) : p \geq 0\}$ is continuous. (D2

By the relationship $U(t,s) = T(t-s)$, each semigroup is a specialized evolution system. One approach to the study of evolution systems has been to study semigroups and try to generalize to evolution systems properties the semigroups display. A second approach (see Ball [1], or Neveu [8]) has been to embed U into a semigroup T on $[0, \infty)$xC by the action $T(t)(s,x) = (t+s, U(t+s,s)x)$ and study T. Goldstein ([4]), however, has shown that even if U has linear, non-expansive maps and analytic trajectories, T may not even have Lipschitz maps and hence may fail to satisfy the hypothesis which is typically used in theorems about semigroups.

This paper gives an alternative approach to studying evolution systems through semigroups. Here the evolution systems is not studied by analogy but rather by identifying a class of evolution systems for which semigroups are actual structural components. Results concern the structure of such evolution systems as they relate to the existing theory for evolution systems. Examples are given that illustrate under what conditions these evolution systems occur.

## 2. DEFINITIONS and THEOREMS.

In addition to the notation established in the introduction, the following notation will be used. L(H,H) will denote the normed vector space of continuous linear transformations on H with, for f in L(H,H), $|f| = \text{glb}\,\{m: \text{if } p \in H,\ \|f(p)\| \leq m \cdot \|p\|\}$.

$R(H,H)$ will denote the subset of $L(H,H)$ whose elements have inverses that are in $L(H,H)$. The identity function on the numbers will be denoted by j. The domain (range) of a function f will be denoted $D_f(R_f)$.

DEFINITION: Suppose that G is a family of functions so that if $t \geq 0$, $G(t) : C \to C$. The statement that G is <u>strongly</u> <u>continuous</u> at the point x of C means that $g_x = \{(t,G(t)(x)) : t \geq 0\}$ is continuous. $g_x$ will be called the <u>trajectory</u> of G from x.

DEFINITION: Suppose that U is a strongly continuous evolution system of continuous tranformations on C. The statement that A is the <u>infinitesimal</u> <u>generator</u> for U means that $A = \{A(t) : t \geq 0\}$ is so that if $t \geq 0$, $A(t) = \{(x,y) : x \in C$ and $y = \lim_{h \to 0} h^{-1}(U(t+h,t)(x) - x)\}$.

Theorem 1 establishes a way of combining a semigroup with a one parameter family so that the result exhibits evolution system structure.
Conditions are given under which the form of the infinitesimal generator can be guaranteed.

THEOREM 1. Suppose that T is a strongly continuous semigroup of continuous transformations on C with infinitesimal generator B and G is a strongly continuous family of functions on C so that if $t \geq 0$, $G(t)$ is Lipschitz and has a Lipschitz inverse on C; and if a and b are numbers so that $a \leq b$, $\{k : a \leq p \leq b$ and k is the Lipschitz norm of $G(p)\}$ is a bounded set. For $t \geq s \geq 0$, define $U(t,s) = G(t)T(t-s)G(s)^{-1}$.

Then U is a strongly continuous evolution system of continuous transformations on C. Suppose, in addition, that $C=H$, $x \in H$, $G(s) \in R(H,H)$ for each s, and t is a number so that $G(t)^{-1}x$ is in the domain of B and the trajectory of G from $G(t)^{-1}x$ is differentiable at t. Then, if A denotes the infinitesimal generator for U, x is in the domain of $A(t)$ and $A(t)x = G(t)BG(t)^{-1}(x)+g_y'(t)$ where $y = G(t)^{-1}x$.

PROOF: To see that U is an evolution system of continuous transformations on C, note that (i) $U(t,s) = G(t)T(t-s)G(s)^{-1}$ is the composition of continuous functions each mapping C into C; (ii) $U(s,s) = G(s)T(s-s)G(s)^{-1} = I$; and (iii) $U(t,s)U(s,r) = G(t)T(t-s)G(s)^{-1}(G(s)T(s-r)G(r)^{-1}) = G(t)T(t-r)G(r)^{-1} = U(t,r)$.

For strong continuity, suppose that $s \geq 0$, $x \in C$, and $\{t_n\}_{n=1}^{\infty}$ converges to t in $[s,\infty)$. $\| U(t_n,s)x - U(t,s)x\| = \|G(t_n)T(t_n-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)\| \leq \|G(t_n)T(t_n-s)G(s)^{-1}(x) - G(t_n)T(t-s)G(s)^{-1}(x)\| + \|G(t_n)T(t-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)\| \leq |G(t_n)|_{Lip} \cdot \|T(t_n-s)G(s)^{-1}(x)-T(t-s)G(s)^{-1}(x)\| + \|G(t_n)T(t-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)\|$. T is strongly continuous at $G(s)^{-1}(x)$, $\{|G(t_n)|\}_{n=1}^{\infty}$ is a bounded number sequence, and G is strongly continuous at $T(t-s)G(s)^{-1}(x)$. Thus each term of the final inequality can be made arbitrarily small and each trajectory of U is continuous.

With G satisfying the additional assumptions, suppose x is an element of H so that $y = G(t)^{-1}(x)$ is in the domain of B and consider $\lim_{h\to 0} h^{-1}(U(t+h,t)(x) - x) = \lim_{h\to 0} h^{-1}(G(t+h)T(h)G(t)^{-1}(x) - x) = \lim_{h\to 0} h^{-1}(G(t+h)T(h)G(t)^{-1}(x) - G(t+h)G(t)^{-1}(x)) + h^{-1}(G(t+h)G(t)^{-1}(x) - G(t)G(t)^{-1}(x)) = \lim_{h\to 0} G(t+h)(h^{-1}(T(h)-I)(G(t)^{-1}(x))) + \lim_{h\to 0} h^{-1}(g_y(t+h) - g_y(t)) = G(t)BG(t)^{-1}(x) + g_y'(t)$. Hence x is in the domain of $A(t)$ and $A(t)(x) = G(t)BG(t)^{-1}(x) + g_y'(t)$.

Theorem 2 provides a partial converse to Theorem 1. The thrust of the theorem is that a non-linear problem may, through the structure from Theorem 1, be exchanged for a linear problem.

THEOREM 2: Suppose that $\{A(t)\}_{t \geq 0}$ is so that there is a unique solution to $v'(t) = A(t)(v(t))$, $v(s) = x$ for each point $x$ in the domain of $A(s)$. Suppose also that there is a function $G$ with domain $[0,\infty)$ and range in $R(H,H)$ and a strongly continuous semigroup $T$ of continuous transformations on $H$ with infinitesimal generator $B$ so that

i. if $t > 0$ and $x \in H$, then $T(t)(x) \in D_B$;

ii. if $t \in [0,\infty)$, then $G(t)(D_B) = D_{A(t)}$;

iii. if $x \in D_B$ and $y_x : [0,\infty) \to H$ is defined by $y_x(t) = G(t)(x)$, then $y_x'(t) = A(t)G(t)(x) - G(t)B(x)$; and

iv. if $t \geq s \geq 0$, $\{|G(r)| : s \leq r \leq t\}$ is a bounded set.

Then if $U$ is the evolution system generated by $A$ and $t \geq s$, $U(t,s) = G(t)T(t-s)G(s)^{-1}$.

PROOF: Suppose that $t \geq s$ and that $x \in D_{A(s)}$. By assumption, there is a unique function $v$ so that $v'(t) = A(t)(v(t))$ and $v(s) = x$. Also, $U(t,s)(x)$ is defined to be $v(t)$. Consider $f_x : [0,\infty) \to H$ defined by $f_x(t) = G(t)T(t-s)G(s)^{-1}(x)$. $f_x(s) = x$.
$\|h^{-1}(f_x(t+h) - f_x(t)) - A(t)(f_x(t))\| = \|h^{-1}(G(t+h)T(t+h-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)) - A(t)G(t)T(t-s)G(s)^{-1}(x))\|$.
Notice that $A(t)(f_x(t))$ makes sense since by assumption $x$ is in $D_{A(s)}$ and if $t > s$, $R_{T(t-s)}$ is a subset of $D_B$ and $G(t)(D_B) = D_{A(t)}$. From assumption iii. on the strong differentiability of $G$, and denoting $T(t-s)G(s)^{-1}x$ by $y$, the above norm can be written $\|h^{-1}(G(t+h)T(t+h-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)) - (g_y'(t) - G(t)BT(t-s)G(s)^{-1}(x)\| \leq \|G(t+h)(h^{-1}(T(h)(T(t-s)G(s)^{-1}(x)) - T(t-s)G(s)^{-1}(x)) - G(t+h)(B(T(t-s)G(s)^{-1}(x))\| + \|h^{-1}(G(t+h)(T(t-s)G(s)^{-1}(x)) - G(t+h)(B(T(t-s)G(s)^{-1}(x))) - g_y'(t)\| + \|G(t+h)(BT(t-s)G(s)^{-1}(x)) - G(t)BT(t-s)G(s)^{-1}(x))\|$.
The first term converges to 0 by the differentiability of T at

$T(t-s)G(s)^{-1}(x)$ and the boundedness of $\{|G(r)| : r \in [t, t+m]\}$ the second converges to 0 by the strong differentiability of G at $T(t-s)G(s)^{-1}(x)$; and the third converges to 0 by the strong continuity of G at $BT(t-s)G(s)^{-1}(x)$. Thus $f_x'(t) = A(t)(f_x(t))$, $f_x(s)=x$ and by the uniqueness of solutions $U(t,s)(x)=v(t)=f_x(t) = G(t)T(t-s)G(s)^{-1}(x)$.

In earlier work on nonlinear evolution systems (see for instance Crandall [3]) the assumption of accretiveness on the arguments of A provides existence of resolvents for A from which product formulas for the evolution system can be constructed. Theorem 3 shows that if the linear differentiability of the hypothesis of Theorem 2 is strengthened, then the resolvent structure for the underlying semigroup is carried forward to the evolution system.

THEOREM 3: Suppose that $G: [0, \infty) \to R(H,H)$ is differentiable in $L(H,H)$ and that T is a strongly continuous semigroup of continuous transformations on H with generator B so that if $t>0$, $(I-tB)^{-1}$ is a Lipschitz transformation on H, and that there is a number M so that the Lipschitz norms for $\{(I - tB)^{-1}: 0 < t \leq M\}$ is a bounded set. Define $U(t,s)$ to be $G(t)T(t-s)G(s)^{-1}$ and let A denote the infinitesimal generator for U. Then if $t > 0$, there is $k > 0$ so that if $0 < m < k$, $(I - mA(t))^{-1}$ is a Lipschitz transformation with domain H and Lipschitz norm no greater than
$(|G(t)|\ |(I - mB)^{-1}|\ |G(t)^{-1}|\ ) / (1 - m|G(t)|\ |(I - mB)^{-1}|\ |G(t)^{-1}|\ \ |G'(t)|\ )$.

PROOF:

Initially note that the differentiability assumed for Theorem 3 is in the norm topology of $L(H,H)$, that is, if $t \geq 0$, $G'(t)$ is in $L(H,H)$. Hence, from Theorem 1, $A(t)$ can be written as $T(t)BG(t)^{-1} + G'(t)G(t)^{-1}$. Pick x from H. For $I - mA(t)$

to be invertible with domain H there must be exactly one point y so that $(I - mA(t))(y) = x$. For $m > 0$, define a function $F_m$: $H \to H$ by $F_m(p) = G(t)(I - mB)^{-1}(mG(t)^{-1}G'(t)G(t)^{-1}(p) + G(t)^{-1}(x))$. Given $c>0$, let M denote an upper bound for $\{(I - nB)^{-1} : n \le c\}$.

$$\| F_m(p) - F_m(q) \| \le m|G(t)| \; |(I - mB)^{-1}| \; |G(t)^{-1}|^2|G'(t)| \; \| p - q \| .$$

Thus $m < \min\{(|G(t)| \; M \; |G(t)^{-1}|^2|G'(t)\ )^{-1}|, c\}$ implies that $F_m$ is a strict contraction and has a unique fixed point y.

Since $y = G(t)(I -mB)^{-1}(mG(t)^{-1}G'(t)G(t)^{-1}(y)+G(t)^{-1}(x))$, it follows that

$(I - mB)G(t)^{-1}(y) = mG(t)^{-1}G'(t)G(t)^{-1}(y)+G(t)^{-1}(x)$,

$G(t)^{-1}(y) - m(BG(t)^{-1}(y) + G(t)^{-1}G'(t)G(t)^{-1}(y)) = G(t)^{-1}(x)$, and

$y - m((G(t)BG(t)^{-1} + G'(t)G(t)^{-1})(y)) = x$ , that is, $(I - mA(t))(y) = x$.

Reversing the calculations shows that any such point must be a fixed point of $F_m$. Thus $y = (I -mA(t))^{-1}(x)$.

For $y_1 = (I - mA(t))^{-1}(x_1)$ and $y_2 = (I - mA(t))^{-1}(x_2)$,

$$\|(I - mA(t))^{-1}(x_1) - (I - mA(t))^{-1}(x_2)\| = \| G(t)(I - mB)^{-1}(mG(t)^{-1}G'(t)G(t)^{-1}(y_1) + G(t)^{-1}(x_1)) - G(t)(I - mB)^{-1}(mG(t)^{-1}G'(t)G(t)^{-1}(y_2) + G(t)^{-1}(x_2))\| \le |G(t)| \; |(I - mB)^{-1}| (m|G(t)^{-1}|^2|G'(t)| \; \|y_1 - y_2\| + |G(t)^{-1}| \; \| x_1 - x_2\| ).$$

Thus $\| y_1 - y_2\| \le ((|G(t)| \; |(I - mB)^{-1}| |G(t)^{-1}|)/(1 - m|G(t)| \; |(I - mB)^{-1}| |G(t)^{-1}|^2 |G'(t)| )) \; \| x_1 - x_2\|$ .

## 3. OCCURRENCES.

There appear to be several possibilities for the application of the evolution structure identified in the preceding theorems.

Example 1. In [7], Neuberger studied semigroups of Lipschitz transformations so that if T is one of them, then there is $c \ge 0$ so that if x and y are elements of H, and $t \ge 0$, then $\| T(t)(x) - T(t)(y)\| \le e^{ct} \| x - y \|$ . By taking $G(t) = e^{-ct} I$, Neuberger's special evolution system which produced product formulas for the semigroup can be seen to be an instance of the theory.

Example 2. Suppose that T is a semigroup of continuous transformations on H with infinitesimal generator B and G is a group of continuous linear transformations on H with generator C. Then there is an evolution system U on H with generator A so that $B + C \subset A(0)$.

PROOF: Define U by $U(t,s) = G(t)^{-1}T(t-s)G(s)^{-1}$. If $x \in D_{B+C}$ the hypothesis to Theorem 1 is satisfied and $A(0)(x) = G(0)BG(0)^{-1}(x) + g_{G(0)^{-1}(x)}{}'(0)$. Hence $A(0)(x) = B(x) + \lim_{h\to 0} h^{-1}(G(h)(x) - x) = B(x) + C(x)$.
Thus, even though such sums may fail to guarantee the generation of semigroups (see Chernoff[2]), they must always support the generation of globally defined evolutions.

Generation theories for evolution systems typically require that $\bigcap_{s \leq t} D_{A(s)}$ be dense in C and that $(I - mA(t))^{-1}$ be Lipschitz with domain C. These restrictions need not apply for the theory discussed in Theorem 1 to be viable.

Example 3. Let H be the Banach space of bounded uniformly continuous functions from $[0, \infty)$ into R with for $f \in H$, $\|f\| = \text{lub}\,\{|f(x)|\ x \geq 0\}$. Define A(s) by

$$(A(s)(f))(x) = \begin{cases} f'(x) \text{ if } x \leq s \\ 0 \text{ if } x > s \end{cases}$$

By taking T to be the semigroup generated by $\{(f,g) : f \in H, g \in H, \text{ and } g = f'\}$ and G defined by

$$(G(t)(f))(x) = \begin{cases} f(x) \text{ if } x \geq t \\ 2f(x) - f(t) \text{ if } x < t \end{cases},$$

U, defined by $U(t,s) = G(t)T(t-s)G(s)^{-1}$ has generator A. Notice that the two conditions on the domain of A(s) are provided separately by the differentiability of T and G respectively. Here U(t,s) is defined globally although $\bigcap_{0 \leq a \leq b} D_{A(a)}$ is nowhere dense in any neighborhood and $I - mA(t)$ does not have range H.

Even when existing theories cover a given evolution system, the structures studied here may provide alternative computation possibilities.

Example 4. Consider the ordinary differential equation

$$y'(t) = A/((Bt + C)y(t)) + -Dy(t)/(Bt + C),\ y(s) = x,\ D/B = \tfrac{1}{2}\ .$$

A typical solution might involve solving for $y^2$ in the associated linear equation. However, by letting G solve the linear term, a semigroup generator can be found by purely algebraic means from which $U(t,s) = G(t)T(t-s)G(s)^{-1}$ can be built to solve the original equation. Here the forms in which the two methods give the solution are computationally different.

Example 5. Suppose that A is a linear group generator and that C is a continuous function from H into the compact subset K of H. Consider, for $x \varepsilon H$, the problem

$$y'(t) = (A + C)(y(t)),\ y(0) = x$$

which can be solved by the method of convergence of approximate solutions as employed by Martin[6].

Using the approach suggested in the theorems, let G denote the group generated by A and look for a function f from [0, d] into H so that $y = \{(t, G(t)(f(t))) : 0 \leq t \leq d\}$ solves the problem. For such a function to be a solution it must be true that $f'(t) = G(t)^{-1}CG(t)(f(t))$. Letting $X = C([0,\infty),H)$, this in turn translates into a fixed point problem for the function F defined by, for $k \varepsilon X$,

$$(F(k))(t) = x + \int_0^t G(j)^{-1}CG(j)k$$

and $G(j)^{-1}CG(j)$ is a compact map on [0,d] xH. If f is a fixed point for F, then $\lim_{h\to 0} h^{-1}(G(t+h)(f(t+h)) - G(t)(f(t))) = \lim_{h\to 0} h^{-1}(G(h)G(t)(f(t)) - G(t)(f(t))) + G(t+h)(h^{-1}(f(t+h) - f(t)) = AG(t)(f(t)) + G(t)(f'(t)) = AG(t)(f(t)) + G(t)G(t)^{-1}CG(t)(f(t))$. Hence $y'(t) = (A + C)(y(t))$ and $y(0) = x$. Notice that the burden for solution is shifted from evaluating the convergence of approximate solutions to finding a fixed point associated with a compact map.

4. QUESTIONS.

In Example 4, when G is produced by solving the linear problem $u'(t) = -Du(t)/(Bt+c)$, finding the appropriate associated semigroup generator depends on finding a function f so that $A/((Bt + C)x) = C^{D/B}f(C^{-D/B}(Bt + C)x)$. In this case $f(p) = A/p$ works. Any systematic application for the evolution system in this paper must include an analysis of the conditions under which a family $\{B(t)\}_{t \leq 0}$ can be algebraically resolved to the form $\{G(t)fG(t)^{-1}\}_{t \leq 0}$.

Theorem 1 identifies a closed form for the evolution generator in case G is a linear family. Can other forms of the generator for $G(t)T(t-s)G(s)^{-1}$ be identified when G is not linear?

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