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NONTRIVIAL ISOMETRIES ON $s_p(\alpha)$

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ABSTRACT. $s_p(\alpha)$ is a Banach space of sequences x with $\|x\| = \left(\sum_{i=0}^{\infty} |x_i|^p + \alpha \sum_{i=0}^{\infty} |x_{i+1} - x_i|^p \right)^{1/p}$. For $1 < p < \infty$, $p \neq 2$, $0 < \alpha < \infty$, $\alpha \neq 1$, there are no nontrivial surjective isometries in $s_p(\alpha)$. It has been conjectured that there are no nontrivial isometries. This note gives two distinct counterexamples to this conjecture and a partial affirmative answer for the case of isometries with finite codimension.

KEY WORDS AND PHRASES. *Isometry, sequential Banach space, Banach space.*

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1. INTRODUCTION.

Banach spaces provide a natural setting for a variety of pure and applied problems in functional analysis. The geometry of a Banach space is much more complex then that of the simpler Hilbert spaces. The geometry of a Banach space is closely related to the types of linear isometries that the Banach space admits [1].

There are examples of Banach spaces for which the only isometries from the space into itself are the trivial ones, $\pm I$ [2]. These examples are constructed by placing extreme points on the unit ball in an asymmetrical pattern. While these examples are of interest it is also helpful to have examples which closely resemble the Banach spaces commonly encountered.

As one step in this direction, Jamison and Fleming [3] studied a discrete analogue of the classical Sobolov spaces, which they denoted as $s_p(\alpha)$.

For $1 \le p < \infty$, $p \neq 2$, $\alpha \ge 0$, let $s_p(\alpha)$ denote the linear space of all real or complex sequences $x = \{x_k\}$ for which

$$\|x\| = \left(\sum_{i=0}^{\infty} |x_i|^p + \alpha \sum_{i=0}^{\infty} |x_{i+1} - x_i|^p \right)^{1/p} < \infty. \tag{1.1}$$

$s_p(0) = \ell_p$. If $\alpha > 0$, then $s_p(\alpha)$ is isomorphic but not isometric to ℓ_p. $s_p(\alpha)$ is isometric to a subspace of ℓ_p.

In [3], it is shown that for $1 < p < \infty$, $\alpha > 0$, $\alpha \neq 1$, the only surjective isometries in $s_p(\alpha)$ are scalar multiples of the identity. In [3], it is conjectured that all isometries in $s_p(\alpha)$ must be surjective and hence scalar multiples of the identity.

In this note we exhibit non-surjective isometries for all $\alpha > 0$. We also show that there are essentially two types of isometries in $s_p(\alpha)$ and that if $\alpha > 0$, then one kind cannot have finite codimension.

It is always assumed that $p \neq 2$. Unless stated otherwise, we allow $\alpha = 1$, $\alpha = 0$, or $p = 1$.

2. EXAMPLES AND TERMINOLOGY.

The simplest type of isometry on $s_p(\alpha)$ is one that preserves both sums in (1.1). Such an isometry is an isometry independent of α and is an isometry on ℓ_p. Thus, it has the structure developed in [4]. We shall call such an isometry a Lamperti isometry.

EXAMPLE 1. For $x \in s_p(\alpha)$, $1 \le p < \infty$, define T by

$$Tx = 3^{-1/p}\{x_o,x_o,x_1,x_o,x_1,x_2,x_1,x_2,x_3,x_2,x_3,x_4,x_3,x_4,\dots\}. \tag{2.1}$$

Then T is a Lamperti isometry.

Note, that if T is an isometry on $s_p(\alpha)$ for two distinct values of α, then T is a Lamperti isometry.

EXAMPLE 2. Suppose that $\alpha > 0$ and $m > 0$, $n > 0$ are integers. For $x \in s_p(\alpha)$, define T as

$$Tx = \{0,\beta x_o,\dots,\beta x_o,0,\gamma(x_1 - x_o),\dots,\gamma(x_1 - x_o),0,\beta x_1,\dots,\beta x_1,0,\dots\} \tag{2.2}$$

where each string of βx_i is repeated m times and each string of $\gamma(x_{i+1} - x_i)$ is repeated n times. Then

$$\|Tx\|^p = m\beta^p \sum |x_i|^p + n\gamma \sum |x_{i+1} - x_i|^p + 2\alpha\beta^p \sum |x_i|^p + 2\gamma^p\alpha \sum |x_{i+1} - x_i|^p$$

$$= (m + 2\alpha)\beta^p \sum |x_i|^p + (n + 2\alpha)\gamma^p \sum |x_{i+1} - x_i|^p.$$

Define $S = \beta^{-1}(m + 2\alpha)^{-1/p}T$. Then S is an isometry on $s_p(\alpha)$ if and only if

$$\frac{n + 2\alpha}{m + 2\alpha}\left(\frac{\gamma}{\beta}\right)^p = \alpha. \tag{2.3}$$

But then for any $\alpha > 0$, any $m \geq 1$, $n \geq 1$, S will be an isometry on $s_p(\alpha)$ provided γ/β satisfies (2.3). Since (2.3) is inconsistent for $\alpha = 0$, we see that this S is not a Lamperti isometry. Thus each $s_p(\alpha)$ has isometries which are not isometries for any other value of α.

3. FINITE CODIMENSION.

The isometry in both Example 1 and Example 2 has a range of infinite codimension. This suggests that perhaps there are no isometries on $s_p(\alpha)$ which have a range with finite codimension. This section will show that there are no Lamperti isometries of finite codimension that are no surjective. The key will be the following fact from [3]. For $x, y \in s_p(\alpha)$, let xy be the sequence $(xy)_i = x_i y_i$. Let $Vx = \{0, x_o, x_1, \ldots\}$. Then, for any isometry T on $s_p(\alpha)$,

$$\left.\begin{aligned} xy &= 0 \\ x(Vy) &= 0 \\ (Vx)y &= 0 \end{aligned}\right\} \quad \text{implies that} \quad \left\{\begin{aligned} (Tx)(Ty) &= 0 \\ (Tx)(VTy) &= 0\, . \\ (VTx)(Ty) &= 0 \end{aligned}\right. \tag{3.1}$$

Let e_i be the standard basis for ℓ_p. Suppose that T is an isometry of finite codimension. Let E_i be the support of Te_i, that is, $E_i = \{k | (Te_i)_k \neq 0\}$. By (3.1) $E_i \cap E_{i+k} = \emptyset$ if $i \geq 0$, $k \geq 2$. Note in Example 2 that $E_i \cap E_{i+1} \neq \emptyset$. We shall say E_i, E_j adjoin if there exists $i_1 \in E_i$, $j_1 \in E_j$ such that $|i_1 - j_1| = 1$. For any isometry T, the following hold:

(i) E_i and E_{i+1} either adjoin or intersect

(ii) E_i and E_{i+k}, $k \geq 2$ neither adjoin nor intersect.

(ii) follows immediately from (3.1). To see (i), suppose that (i) does not hold. If $i \geq 1$,

$$\|\beta e_i + e_{i+1}\|^p = |\beta|^p + 1 + \alpha(|\beta|^p + |1 - \beta|^p + 1) \tag{3.2}$$

whereas

$$\|\beta Te_i + Te_{i+1}\|^p = |\beta|^p\|Te_i\|^p + \|Te_{i+1}\|^p = |\beta|^p(1 + 2\alpha) + 1 + 2\alpha. \tag{3.3}$$

But $\|\beta e_i + e_{i+1}\| = \|T(\beta e_i + e_{i+1})\|$, so that (3.2) and (3.3) are equal. Since (3.2) depends on β, whereas (3.3) depends on $|\beta|$, we have a contradiction if $\alpha > 0$. A similar proof works if $i = 0$.

THEOREM 1. If T is a Lamperti Isometry of finite codimension in $s_p(\alpha)$, then T is surjective.

PROOF. Since T is a Lamperti Isometry, all but a finite number of the E_i are singletons. What's more, since E_i must adjoin, but not intersect E_{i-1}, E_{i+1}, after some index, the sets are listed in order. Let E_j be the last set which is not a singleton. Let $(h_o, \ldots, h_k, 0, \ldots) = Te_j$, where $h_k \neq 0$ and $E_{j+1} = \{k + 1\}$. We allow some $h_r = 0$ if $r < k$. Now, if $j \geq 1$, $\|\beta e_j + e_{j+1}\|^p = \|\beta Te_j + Te_{j+1}\|^p$, or

$$|\beta|^p + 1 + \alpha(|\beta|^p + |1 - \beta|^p + 1) = 1 + \sum_{i=0}^{k} |\beta|^p|h_i|^p$$

$$+ \alpha\left(\sum_{i=0}^{k-1} |\beta|^p|h_{i+1} - h_i|^p + |\beta h_k - 1|^p + 1\right)$$

$$= 1 + \sum_{i=0}^{k} |\beta|^p|h_i|^p + \alpha\left(\sum_{i=0}^{k-1} |\beta|^p|h_{i+1} - h_i|^p + |\beta|^p|h_k|^p\right)$$

$$-\alpha|\beta|^p|h_k|^p + \alpha|\beta h_k - 1|^p + \alpha$$

$$= 1 + |\beta|^p\|Te_j\|^p - \alpha|\beta|^p|h_k|^p + \alpha|\beta h_k - 1|^p + \alpha$$

$$= 1 + |\beta|^p(1 + 2\alpha) - \alpha|\beta|^p|h_k|^p + \alpha|\beta h_k - 1|^p + \alpha.$$

Working with the first and last terms gives $\alpha|1-\beta|^p = \alpha|\beta|^p - \alpha|\beta|^p|h_k|^p + \alpha|\beta h_k - 1|^p$ or

$$|1 - \beta|^p = |\beta|^p - |\beta|^p|h_k|^p + |\beta h_k - 1|^p. \tag{3.4}$$

Equation (3.4) holds for all β, and both sides of (3.4) are differentiable with respect to β, except at $\beta = 0, 1, 1/h_k$. Differentiating with respect to β gives for $p > 1$, and $\beta > 0$, β small,

$$-p|1 - \beta|^{p-1} = p|\beta|^{p-1} - p|\beta|^{p-1}|h_k|^p - ph_k|\beta h_k - 1|^{p-1}. \tag{3.5}$$

But (3.5) holds in an interval of the form $0 < \beta < \varepsilon$. Hence, it holds for $\beta \to 0^+$. Thus $-p = -ph_k$ or $h_k = 1$. But T is also an isometry in ℓ_p so that $\sum_{i=0}^{k} |h_i|^p = 1$. Hence, $E_j = \{k\}$, which is a contradiction. The proof for $j = 0$ is similar. □

We conjecture that Theorem 1 is true also for non-Lamperti isometries on $s_p(\alpha)$, but we have been unable to prove it. With minor modifications, one can proceed exactly as in Theorem 1 (to get the E_i, for i greater than some k, singletons is not too hard). The difficulty is that $h_k = 1$ no longer provides any contradiction that we can see. On the other hand, numerical counterexamples seem quite messy.

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