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A FIXED POINT THEOREM FOR CONTRACTION MAPPINGS

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ABSTRACT. Let S be a closed subset of a Banach space E and $f: S \to E$ be a strict contraction mapping. Suppose there exists a mapping $h: S \to (0,1]$ such that $(1 - h(x))x + h(x)f(x) \in S$ for each $x \in S$. Then for any $x_0 \in S$, the sequence $\{x_n\}$ in S defined by $x_{n+1} = (1 - h(x_n))x_n + h(x_n)f(x_n)$, $n \geq 0$, converges to a $u \in S$. Further, if $\sum h(x_n) = \infty$, then $f(u) = u$.

KEY WORDS AND PHRASES. Contraction mapping

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1. INTRODUCTION.

In a recent paper [1], Ishikawa proved the following result.

THEOREM. Let S be a closed subset of a Banach space E and let f be a nonexpansive mapping from S into a compact subset of E. Suppose there exists a real sequence $\{h_n\}$, $0 \leq h_n \leq b < 1$ and an $x_0 \in S$ such that $x_{n+1} = (1 - h_n)x_n + h_n fx_n \in S$ for each $n > 0$. If $\sum h_n = \infty$, then the sequence $\{x_n\}$ converges to a fixed point of f.

In this note, we investigate the above result when f therein is a contraction mapping (for some α, $0 < \alpha < 1$, $\| fx - fy \| \leq \alpha \| x - y \|$, for all $x, y \in S$) but does not necessarily have a precompact range. We show that if $0 < h_n \leq 1$, then the sequence $\{x_n\}$ above converges to a $u \in S$ and if $\sum h_n = \infty$ then $fu = u$. The proof is much less computational in this case.

2. MAIN RESULT.

Throughout, let E denote a Banach space. The main result is

THEOREM 1. Let S be a closed subset of E and $f: S \to E$ be a contraction mapping satisfying the condition: there exists a mapping $h: S \to (0,1]$ such that for each $x \in S$,

$$(1 - h(x))x + h(x)f(x) \in S. \tag{1.1}$$

If $x_0 \in S$ and the sequence $\{x_n\}$ in S is defined by

$$x_{n+1} = (1 - h(x_n))x_n + h(x_n)f(x_n),\ n \geq 0, \tag{1.2}$$

then (a) the sequence $\{x_n\}$ converges to a $u \in S$ and (b) if $\sum h(x_n) = \infty$, then u in (a) is the unique fixed point of f.

The following result (see Knopp [2], Theorem 4, p. 220) is used in the proof of Theorem 1.

PROPOSITION 1. Let $\{a_n\}$ be a sequence of reals with $0 \leq a_n < 1$. Then the sequence $\{\prod_{i=1}^{n} (1 - a_i)\} \to b > 0$ iff $\sum a_n < \infty$.

Proof of Theorem 1. Let $h_n = h(x_n)$. It follows by (2) that

$$x_{n+1} - x_n = h_n(fx_n - x_n), \tag{1.3}$$

$$\text{and } fx_n - x_{n+1} = (1 - h_n)(fx_n - x_n). \tag{1.4}$$

Thus, for each positive integer n,

$$\begin{aligned} || fx_n - x_n || &\leq || fx_n - fx_{n-1} || + || fx_{n-1} - x_n|| \\ &\leq \alpha|| x_n - x_{n-1} || + (1 - h_{n-1}) || fx_{n-1} - x_{n-1}||. \end{aligned}$$

Therefore, it follows by (1.3) that

$$\begin{aligned} &|| fx_n - x_n || \\ &\leq (\alpha h_{n-1} + 1 - h_{n-1})||fx_{n-1} - x_{n-1} || = (1 - (1 - \alpha)h_{n-1})|| fx_{n-1} - x_{n-1} ||. \end{aligned}$$

Thus $\{|| fx_n - x_n ||\}$ is a decreasing sequence of non-negative reals. Furthermore, it follows by successive iterations on the last inequality that for any $n > 0$,

$$|| fx_n - x_n || \leq \prod_{i=0}^{n-1} (1 - (1 - \alpha)h_i)||fx_0 - x_0 || \leq || fx_0 - x_0 ||. \tag{1.5}$$

Set $u_i = (1 - \alpha)h_i$. Since $0 < u_i < 1$, $\{\prod_{i=0}^{n} (1 - u_i)\}$ is a decreasing sequence of positive reals and hence there is a $b \geq 0$ such that $\prod_{i=0}^{n} (1 - u_i) \to b$. We consider two cases (i) $b > 0$ and (ii) $b = 0$. If $b > 0$, then by Proposition 1, $\sum(1 - \alpha)h_i < \infty$ and hence $\sum h_i < \infty$. Consequently, by (1.3) and (1.5),

$$\sum|| x_{n+1} - x_n || \leq || fx_0 - x_0 || \sum h_n < \infty.$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence in S and hence there is a $u \in S$ such that $\{x_n\} \to u$. Thus (a) holds in this case. If b = 0 then is follows by (1.5) that

$$\| x_n - fx_n \| \to 0. \tag{1.6}$$

Since for any $m \geq n$,

$$\begin{aligned} \| x_m - x_n \| &\leq \| x_m - fx_m \| + \| fx_m - fx_n \| + \| fx_n - x_n \| \\ &\leq \alpha \| x_m - x_n \| + 2 \| x_n - fx_n \|, \end{aligned}$$

it follows that $\| x_m - x_n \| \leq 2(1 - \alpha)^{-1} \| x_n - fx_n \| \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence and hence converges to a $u \in S$. Furthermore, it follows by (1.6) that u = fu. This establishes (a). Now, if $\sum h(x_n) = \infty$ then $\sum (1 - \alpha)h_n = \infty$ and hence by Proposition 1, $b = \prod_{i=0}^{\infty} (1 - u_i) = 0$. Consequently, by case (ii) the sequence $\{x_n\} \to u$ and fu = u. The uniqueness is obvious for such mappings.

For x, $y \in E$, let $[x,y] = \{z \in E: z = (1 - h)x + hy, 0 \leq h \leq 1\}$. Let $(x,y) = [x,y] \setminus \{x,y\}$. As an application of Theorem 1, we have

COROLLARY 1. Let S be a closed subset of E and f: $S \to E$ be a contraction mapping. If for each $x \in S$, there exists a $y \in [x,fx] \cap S$ such that $fy \in S$, then f has a fixed point.

PROOF. Define h: $S \to (0,1]$ as follows. If $fx \in S$, let h(x) = 1 and if $fx \notin S$, then choose a $y \in [x,fx] \cap S$ with $fy \in S$ (such a y exists by hypothesis). Clearly, $y \neq x$ and $y = (1 - h)x + hfx$ for some h with $0 < h < 1$. Let h(x) = h in this case. Thus (1.1) holds. Note that if $f(x) \notin S$ then h(y) = 1. Now, for any $x_0 \in S$ and the sequence $\{x_n\}$ defined by (1.2) that is, $x_{n+1} = (1 - h(x_n))x_n + h(x_n)f(x_n)$, either $h(x_n) = 1$ or $h(x_{n+1}) = 1$ according as $fx_n \in S$ or $fx_n \notin S$. In either case $\sum h(x_n) = \infty$. Thus by Theorem 1, f has a fixed point.

It is known (see [3]) that if S is a closed subset of E and x, $y \in E$ such that x is an interior point of S and $y \notin S$, then there $z \in (x,y) \cap \partial S$. As a consequence of this result and Corollary 1, we have

COROLLARY 2. Let S be a closed subset of E and f: $S \to E$ be a contraction mapping. If $f(\partial S) \subseteq S$ then f has a fixed point.

PROOF. If for $x \in S$, $fx \in S$, then y = x satisfies the condition in Corollary

1 and if $fx \notin S$ then by hypothesis $x \notin \partial S$. Consequently, there is a $y \in (x,fx) \cap \partial S$ with $fy \in S$. Thus by Corollary 1, f has a fixed point.

We now give two examples. Example 1 shows that Corollary 2 is indeed a special case of Theorem 1. In Example 2, we show that if $\sum h(x_n) < \infty$ in Theorem 1, then the sequence $\{x_n\}$ may not converge to a fixed point.

EXAMPLE 1. Let $S = \{0, 2^{-n}: n \geq 0\}$. Define a mapping $f: S \to R$ (reals) by

$$f(2^{-n}) = 3\cdot 2^{-(n+3)}, \quad n \geq 0,$$
$$f(0) = 0.$$

It is clear that any $x, y \in S$, $|| fx - fy || \leq (3/8) || x - y ||$. Let $h: S \to (0,1]$ be defined by $h(0) = 1$ and $h(x) = (4/5)$ for $x \neq 0$. It is easy to verify that for $x = 2^{-n}$, $(1 - h(x))x + h(x)f(x) = 2^{-(n+1)}$, while for $x = 0$, it is clearly 0. Thus (1.1) holds. Further, if $x_0 = 1$, then by (1.2), $x_n = 2^{-n}$ and since $\sum h(x_n) = \infty$, Theorem 1 implies the existence of a $u \in S$ with $fu = u$ (which is 0 in this case). Note that $f(\partial S)$ is not a subset of S.

EXAMPLE 2. Let $\{a_n\}$ be a sequence of reals defined by $a_1 = 1$ and $a_n = \prod_{i=2}^{n} (1 = 2^{-i})$ for $n \geq 2$. Since $\sum 2^{-i} < \infty$, it follows by Proposition 1 that $\{a_n\} \to b > 0$. Let

$$S = [0,b] \cup \{a_n: n \geq 1\}.$$

Let $fx = 2^{-1}\cdot x$ for each $x \in S$. Define $h: S \to (0,1]$ by

$$h(x) = 1 \text{ if } x \in [0,b]$$
$$= 2^{-n}, \text{ if } x = a_n,\ n \geq 1.$$

Then for any $n \geq 1$, $a_{n+1} = (1 - h(a_n))a_n + h(a_n)f(a_n)$. Since $f[0,b] \subseteq [0,b]$, it follows that f satisfies (1.1). Also, if $x_0 = 1$, and the sequence $\{x_n\}$ is as constructed in (1.2), then $x_n = a_n$ and $\{x_n\} \to b$ but $f(b) \neq b$. Note that $\sum h(x_n) = \sum (x^{-n}) < \infty$ in this case.

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