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NONLINEAR OSCILLATIONS IN DISCONJUGATE FORCED FUNCTIONAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. For the equation

$$L_n y(t) + a(t)h(y(g(t))) = f(t)$$

where

$$L_n y(t) = p_n(t)\ (p_{n-1}(t)\ (\ldots(p_1(t)\ (p_0(t)y(t))')'\ldots)')'$$

sufficient conditions have been found for all of its solutions to be oscillatory. The conditions found also lead to growth estimates for the nonoscillatory solutions.

KEY WORDS AND PHRASES. Oscillatory, nonoscillatory, disconjugate, nonlinear oscillations.

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1. INTRODUCTION.

Our main purpose in this paper is to study the oscillatory phenomenon associated with the equation

$$L_n y(t) + a(t)h(y(g(t))) = f(t) \tag{1.1}$$

Where $n \geq 2$ and L_n is a disconjugate differential operator defined by

$$L_n y(t) = p_n(t)(p_{n-1}(t)(\ldots(p_1(t)(p_0(t)(y)t))')'\ldots)')'. \tag{1.2}$$

Following our work (Singh and Kusano [6]), it is assumed that:

(i) $p_i \in C([\alpha, \infty), (o, \infty))$, $o \leq i \leq n$

(ii) $a, f, g \in C([\alpha, \infty), R)$, there exists a $t_0 > \alpha$ such that $0 < g(t) \leq t$ for $t \geq t_0$, and $g(t) \to \infty$;

(iii) $h \in C(R,R)$, h is nondecreasing and sign $h(y)$ = sign y.

We introduce the notation:

$$L_0y(t) = p_0y(t),\ L_iy(t) = p_i(t)(L_{i-1}(y(t)))',\ 1 \leq i \leq n. \tag{1.3}$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $y : [T_y, \infty) \to R$ such that $L_iy(t)$, $0 \leq i \leq n$, exist and are continuous on $[T_y, \infty)$. In what follows by a "solution" of equation (1.1) we mean a function $y \in D(L_n)$ which is nontrivial in any neighborhood of ∞ and satisfies (1.1) for all sufficiently large t. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise the solution is called nonoscillatory.

A great many oscillation criteria are known for an equation of the form

$$(r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = 0 \tag{1.4}$$

For this we refer the reader to Onose [2], Singh [3] and Kusano and Onose [1]. A recently published Russian book by Shevelov [7] gives a detailed list of references on the subject. Obtaining an oscillation criterion for the forced equation

$$(r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = f(t) \tag{1.5}$$

is not so simple. To the best of this author's knowledge, the first attempt to obtain conditions for the oscillation of the equation

$$(r(t)y'(t))' + a(t)h(y(g(t))) = f(t) \tag{1.6}$$

was made by Kusano and Onose [1], and later by other authors including this one [4]. The main technique employed rendered the forced equation into an almost homogeneous equation, i.e. a function $\lambda(t)$ was sought such that $\lambda^{(i)}(t) \to 0$ as $t \to \infty$ for $i = 0, 1, \ldots, n$; and $(r(t)\lambda'(t))^{(n-1)} = f(t)$.

In this work, we shall present a new but elementary technique to obtain an oscillation criterion for the equation (1.1).

In order to shorten long expressions we introduce the following notations: For any function $Q(t) \in C[\alpha,\infty)$ and $t, s \in [\alpha,\infty)$, define

$$I_0(Q(r),r) = p_n^{-1}(r)Q(r) \tag{1.7}$$

and

$$I_k(Q(t),t,s;\, p_n,p_{n-1}, \cdots, p_{n-k}) = \int_s^t p_{n-k}^{-1}(r)\ I_{k-1}(Q,t,r;p_n,\cdots,p_{n-k+1})dr \tag{1.8}$$

for $1 \leq k \leq n-1$.

Any solution $y(t)$ of equation (1.1) which is continuous in a finite interval can be indefinitely extended to the right of α provided the coefficients are continuous. In fact following our proof of Theorem 3.1 in [5] we can state the following theorem:

THEOREM 1.1 The continuity of $p_1, p_2, \ldots, p_n$; and that of h, g, a and f guarantee that any solution of equation (1.1) continuous in a finite interval $[\alpha,\tau]$ can be continuously extended to all of $[\alpha,\infty)$.

PROOF. Same as that of Theorem 3.1 in [5] with minor changes.

2. MAIN RESULTS.

THEOREM 2.1 Suppose there exists a function $\psi(t)$ such that

$$\psi(t) \in C^{(n)}[R],\ \psi(t) > o \text{ and } (\psi' p_1(t))' \geq o \text{ for } t \geq \alpha. \tag{2.1}$$

Further suppose that $\psi(t)$ satisfies

$$\int_\alpha^\infty 1/\psi^2(t)dt < \infty \tag{2.2}$$

$$\limsup_{t\to\infty} \int_\alpha^t 1/\psi^2(s) \int_\alpha^s \psi(x) I_{n-3}(f(x),x,\alpha;p_n,p_{n-1},\ldots,p_3)p_2^{-1}(x)dxds = \infty \tag{2.3}$$

$$\liminf_{t\to\infty} \int_\alpha^t 1/\psi^2(s) \int_\alpha^s \psi(x) I_{n-3}(f(x),x,\alpha;p_n,p_{n-1},\ldots,p_3)p_2^{-1}(x)dxds = -\infty \tag{2.4}$$

$$\int_\alpha^\infty \frac{1}{\psi^2(s)} \int_\alpha^s \psi(x)p_2^{-1}(x) \int_\alpha^x p_3^{-1}(x_3) \int_\alpha^{x_3} p_4^{-1}(x_4)\ldots \int_\alpha^{x_{n-2}} p_{n-1}^{-1}(x_{n-1}) dx_{n-1}\, dx_{n-2},\, dx_{n-3}\ldots dx < \infty \tag{2.5}$$

$$a(t) \geq o \text{ and } p_1'(t) \leq o \text{ for } t \geq \alpha. \tag{2.6}$$

Then all solutions of equation (1.1) are oscillatory.

PROOF. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of equation (1.1). Let $T \geq \alpha$ be large enough so that $y(t)$ and $y(g(t))$ are of the same sign for $t \geq T$. Without any loss of generality, let $y(t) > o$ and $y(g(t)) > o$ for $t \geq T$. On repeated integration for $t \geq T$, we obtain from equation (1.1)

$$(p_1(t)(p_0(t)y(t))')' - p_2^{-1}(t)L_2y(T) - (p_2^{-1}(t)\int_T^t p_3^{-1}(x)dx)L_3y(T)$$

$$- (p_2^{-1}(t)\int_T^t p_3^{-1}(s)\int_T^s p_4^{-1}(x)dxds)L_4y(T) - \dots$$

$$-L_{n-1}y(t)\, p_2^{-1}(t)\int_T^t p_3^{-1}(x_3)\int_T^{x_3} p_4^{-1}(x_4)\cdot\dots\int_T^{x_{n-2}} p_{n-1}^{-1}(x_{n-1})dx_{n-1}dx_{n-2}dx_{n-3}\dots dx_3$$

$$+p_2^{-1}(t)\, I_{n-3}\,(a(t)h(y(g(t))),t,T;p_n,\dots p_3$$

$$= p_2^{-1}(t)\, I_{n-3}(f(t),t,T;p_n,p_{n-1},\dots,p_3) \tag{2.7}$$

Multiplying (15) by $\psi(x)$ and rearranging terms we have

$$(p_1(t)\psi(t)(p_0(t)y(t))')' - \psi'(t)p_1(t)(p_0(t)y(t))' - \psi(t)p_2^{-1}(t)L_2y(T)$$

$$-\psi(t)((p_2^{-1}(t)\int_T^t p_3^{-1}(x)dx)L_3y(T)-\dots-\psi(t)L_{n-1}y(T)p_2^{-1}(t)\int_T^t p_3^{-1}(x_3)\dots$$

$$\dots\int_T^{x_{n-2}} p_{n-1}^{-1}(x_{n-1})dx_{n-1}\dots dx_3 + \psi(t)p_2^{-1}(t)\, I_{n-3}(ah,t,T;p_n,\dots,p_3)$$

$$= \psi(t)p_2^{-1}(t)\, I_{n-3}(f(t),t,T;p_n,p_{n-1},\dots,p_3) \tag{2.8}$$

Integrating (2.8) and dividing by $\psi^2(t)$ we get

$$\frac{p_1(t)(p_0y)'}{\psi(t)} - \frac{p_1(T)\psi(T)(p_0y)'(T)}{\psi^2(t)} - \frac{1}{\psi^2(t)}\int_T^t \psi'(x)p_1(x)(p_0y)'dx$$

$$-\left[\frac{1}{\psi^2(t)}\int_T^t \psi(x)p_2^{-1}(x)dx\right]L_2y(T)-\left[\frac{1}{\psi^2}\int_T^t \psi(x)p_2^{-1}(x)\int_T^x p_3^{-1}(s)dsdx\right]L_3y(T)$$

$$-\dots\frac{1}{\psi^2(t)}\left[\int_T^t \psi(x)p_2^{-1}(x)\int_T^t p_3^{-1}(x_3)dx_3\dots\int_T^{x_{n-2}} p_{n-1}^{-1}(x_{n-1})dx_{n-1}\dots dx_3dx\right]L_{n-1}(T)$$

$$+\frac{1}{\psi^2(t)}\int_T^t \psi(x)p_2^{-1}(x)\, I_{n-3}(a(x)h(y(g(x))),x,T;p_n,\dots,p_3)dx$$

$$= \frac{1}{\psi^2(t)}\int_T^t \psi(x)p_2^{-1}(x)\, I_{n-3}\,(f(x),x,T;p_n,p_{n-1},\dots,p_3)dx. \tag{2.9}$$

In (2.9), we integrate the third term by parts to obtain

$$\frac{p_1(t)(p_0y)'}{\psi(t)} - \frac{K_0}{\psi^2(t)} - \frac{p_1(t)p_0y\psi'}{\psi^2} + \frac{1}{\psi^2(t)}\int_T^t (\psi'(x)p_1(x))'p_0y(x)dx$$

$$- \left[\frac{1}{\psi^2(t)} \int_T^t \psi(x) p_2^{-1}(x) dx \right] L_2 y(T) - \ldots .$$

$$- \ldots - \left[\frac{1}{\psi^2(t)} \int_T^t \psi(x) p_2^{-1}(x) \int_T^x p_3^{-1}(x_3) \ldots . \int_T^{x_{n-2}} p_{n-1}^{-1}(x_{n-1}) dx_{n-1} \ldots dx_3 dx \right]$$

$$+ \frac{1}{\psi^2(t)} \int_T^t \psi(x) p_2^{-1}(x) I_{n-3}(a(x)h(y(g(x))), x, T; p_n, \ldots, p_3) dx$$

$$= \frac{1}{\psi^2(t)} \int_T^t \psi(x) p_2^{-1}(x) \; I_{n-3}(f(x), x, T; p_n, \ldots, p_3) dx \qquad (2.10)$$

where

$$K_0 = p_1(T)\psi(T)(p_0 y)'(T) + p_1(T)p_0(T)y(T)\psi'(T)$$

Integrating first term in (2.10) again by parts and observing that

$$\int_T^t \frac{p_1(x)p_0(x)y(x)\psi'(x)}{\psi^2(x)} dx$$

$$= - \int_T^t \left(\frac{p_1(x)}{\psi(x)} \right)' p_0(x) y(x) dx \; + \int_T^t \frac{p_1'(x)p_0(x)y(x)}{\psi(x)} dx \qquad (2.11)$$

We get

$$\frac{p_1(t)p_0(t)y(t)}{\psi(t)} - K_1 - \int_T^t \frac{p_1'(x)p_0(x)y(x)}{\psi(x)} + \int_T^t \frac{1}{\psi^2(x)} \int_T^x (\psi' p_1)' p_0(s) y(s) . ds dx$$

$$- L_2 y(T) \int_T^t \frac{1}{\psi^2(x)} \int_T^t \psi(s) p_2^{-1}(s) ds - \ldots .$$

$$\ldots - \int_T^t \frac{1}{\psi^2(s)} \int_T^s \psi(x) p_2^{-1}(x) \int_T^x p_3^{-1}(x_3) \ldots \int_T^{x_{n-2}} p_{n-1}^{-1}(x_{n-1}) dx_{n-1} \ldots dx_3 dx ds$$

$$+ \int_T^t \frac{1}{\psi^2(s)} \int_T^s \psi(x) \; p_2^{-1}(x) \; I_{n-3}(a(x)h(y(g(x))), x, T, p_n, \ldots, p_3) dx ds$$

$$= \int_T^t \frac{1}{\psi^2(s)} \int_T^s \psi(x) p_2^{-1}(x) I_{n-3}(f(x), x, T; p_n, p_{n-1}, \ldots, p_3) dx ds \qquad (2.12)$$

where

$$K_1 = \frac{p_1(T)p_0(T)y(t)}{\psi(T)}$$

The terms on the left hand side of (2.12) are either positive or finite. Since the right hand side oscillates between $-\infty$ and ∞, a contradiction is apparent. The theorem is proved.

EXAMPLE 1. Consider the equation

$$(t(\tfrac{1}{t}y')')' + y(t-3\pi/2) = t^8 \sin t,\ t \geq \frac{3\pi}{2} \tag{2.13}$$

Here $p_0(t) \equiv 1$, $p_1(t) \equiv 1/t$, $p_2(t) \equiv t$ and $p_3(t) \equiv 1$.
Choosing $\psi(t) = t^3$ for $t \geq \alpha$, it is easily verified that all conditions of the theorem are satisfied. Hence all solutions of equation (1.1) are oscillatory.

THEOREM 2.2 Suppose conditions (2.2) - (2.6) of theorem 2.1 hold, let $\psi(t)$ be a nonnegative solution of the equation

$$(p_1(t)y'(t))' - a(t)h(y(g(t))) = o. \tag{2.14}$$

Then all solutions of equation (1.1) are oscillatory.

PROOF. Since $(p_1(t)\psi(t))' \geq o$, all conditions of theorem 2.1 are satisfied. The proof is complete.

COROLLARY 2.1 For the equation

$$y^{(n)}(t) + a(t)h(y(g(t))) = f(t) \tag{2.15}$$

Conditions (2.3) - (2.5) respectively reduce to

$$\liminf_{t\to\infty} \int_\alpha^t 1/\psi^2(s) \int_\alpha^s \psi(x) \int_\alpha^x (x-u)^{n-3} f(u)dudxds = -\infty \tag{2.16}$$

$$\limsup_{t\to\infty} \int_\alpha^t 1/\psi^2(s) \int_\alpha^s \psi(s) \int_\alpha^x (x-u)^{n-3} f(u)dudxds = \infty \tag{2.17}$$

and

$$\lim_{t\to\infty} \int_\alpha^t 1/\psi^2(s) \int_\alpha^s \psi(x)\, x^{n-4}\, dxds < \infty. \tag{2.18}$$

Thus subject to conditions (2.2), (2.6) and (2.15) - (2.18) all solutions of equation (2.15) are oscillatory.

EXAMPLE 2. Consider the equation

$$y^{(iv)}(t) + e^{2\pi}y(t-\pi) = -63\ e^{2t} \sin(2t),\ t > \pi \tag{2.19}$$

If we choose $\psi(t) = e^t$, then conditions of corollary 2.1 can be easily verified. Hence all solutions of equation (2.19) are oscillatory. In fact $y(t) = e^{2t} \sin(2t)$ is one such solution.

For the bounded solutions of equation (1.1), the condition $a(t) \geq o$ can be improved as the following theorem shows:

THEOREM 2.3 Suppose there exists a function $\psi(t)$ such that $\psi(t) \in C^{(n)}[R]$, $\psi(t) > o$, $(\psi' p_1)' \geq o$ and $p_1'(t) \leq o$ for $t \geq \alpha$. Further suppose that $\psi(t)$ satisfies conditions (2.2) - (2.5) of Theorem 2.1 and the condition

$$\int_\alpha^\infty 1/\psi^2(t) \int_\alpha^t \psi(s)\, I_{n-3}(|a(s)|, s, a; p_n, p_{n-1}, \ldots, p_3) p_2^{-1}(s)\, ds\, dt < \infty \tag{2.20}$$

Then all bounded solutions of equation (1.1) are oscillatory.

PROOF. We proceed as in Theorem (2.1) and arrive at (2.12). If y(t) is bounded then there exists a constant $C_y > o$ such that

$$|h(y(g(t)))| \leq C_y. \tag{2.21}$$

In equation (2.12) the last term on the left hand side

$$\left|\int_T^t 1/\psi^2(s) \int_T^s \psi(x) p_2^{-1}(x)\, I_{n-3}(a(x)h(y(g(x))), x, T; p_n, \ldots p_3)\, dx\, ds\right|$$

$$\leq Cy \int_T^\infty 1/\psi^2(t) \int_T^t \psi(x) p_2^{-1}(x) I_{n-3}(|a(x)|, x, T; p_n, \ldots, p_3)\, dx\, dt < \infty. \tag{2.22}$$

Hence all terms on the left hand side of equation (2.12) are either finite or non-negative. The proof now follows as in theorem 2.1.

REMARK. When n=2,3, condition (2.18) reduces to condition (2.2) with obvious changes in (2.16) and (2.17).

Under the conditions of theorem 2.3, equation (1.1) may possess unbounded nonoscillatory solutions as the following example shows:

EXAMPLE 3. The equation

$$y''(t) + 50 \sin t\, y(t) = (4 + 50 \sin t)\, e^{2t} \tag{2.23}$$

has $y(t) = e^{2t}$ as a nonoscillatory solution.

If we choose $\phi(t) = e^t$ we have from (2.20)

$$\left| \int_\pi^\infty e^{-2t} \int_\pi^t e^s \sin s\, ds\, dt \right| < \infty \tag{2.24}$$

Conditions (2.16) and (2.17) yield

$$\liminf_{t\to\infty} \int_{\pi}^{t} e^{-2t} \int_{\pi}^{s} e^{3s}(4 + 50 \sin s)\,ds\,dt = -\infty$$

and

$$\limsup_{t\to\infty} \int_{\pi}^{t} e^{-2t} \int_{\pi}^{s} e^{3s}(4 + 50 \sin s)\,ds\,dt = \infty$$

Thus all conditions of theorem 2.3 are satisfied. Hence all bounded solutions of equation (2.23) are oscillatory even though it has an unbounded nonoscillatory solution.

COROLLARY 2.2. Subject to the conditions of theorem 2.3 all nonoscillatory solutions of equation (1.1) are unbounded.

Corollary 2.2 leads to the following theorem which gives a growth condition for the nonoscillatory solutions of equation (1.1).

THEOREM 2.4 Suppose conditions of theorem 2.3 hold. Let $y(t)$ be a nonoscillatory solution of equation (1.1). Then

$$\limsup_{t\to\infty} (|y(t)|\,p_n^{-1}(t)) = \infty \tag{2.25}$$

PROOF. Suppose to the contrary that there exist constant $D_y > o$ and $T > \alpha$ such that

$$|y(g(t))|\,p_n^{-1}(t) \le D_y \tag{2.26}$$

for $t \ge T$. Condition on h implies that there exists a $C_y > o$ such that

$$|h(y(g(t)))| \le C_y\,|y(g(t))| \le C_y D_y \equiv K_y \tag{2.27}$$

Following the proof of theorem 2.3 we see that the constant K_y replaces C_y in equality (2.22) and the proof is complete.

REMARK. It is a matter of general interest to obtain theorems involving operator L_n in its most general form; i.e. when L_n is not necessarily in canonical form. L_n is said to be in canonical form if

$$\int_{\alpha}^{\infty} p_i^{-1}(t)\,dt = \infty, \quad 1 \le i \le n-1. \tag{2.28}$$

Recently Trench [8] has shown that when L_n is not in canonical form, i.e. when (2.28) doesn't hold, it can be put in a canonical form in a unique way with a different set of p s satisfying (2.28). More precisely the operator L_n can be rewritten as

$$L_n y = b_n(b_{n-1}(t)(\ldots(b_1(t)(b_0(t)y(t))')'\ldots)')'$$

where

$$\int_\alpha^\infty b_i^{-1}(t)dt = \infty,\ 1 \le i \le n-1 \tag{2.29}$$

each b_i is obtained from $p_1, p_2, --, p_n$, $i = o, 1, --, n$. The actual computations leading to b s are tedious.

It is interesting and important to note that the results in this work do not depend upon condition (2.28) or (2.29).

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