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A NOTE ON A PAPER BY S. HABER

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ABSTRACT. A technique used by S. Haber to prove an elementary inequality is applied here to obtain a more general inequality for convex sequences.

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1. INTRODUCTION.

Let a and b be non-negative. Then the following elementary inequality was proved in [1].

$$\frac{1}{n+1}[a^n + a^{n-1}b + \dots + b^n] \geq \left(\frac{a+b}{2}\right)^n \quad (n=0,1,2,\dots)\dots \tag{1.1}$$

Now this inequality can be obtained at once by taking $f(t) = t^n$ in the well-known result

$$\frac{1}{b-a}\int_a^b f(t)dt \geq f\left(\frac{a+b}{2}\right) \quad \dots\dots \tag{1.2}$$

which holds whenever f is convex in $[a,b]$. However, the method used in [1] to obtain (1.1) is interesting and it is the purpose of the present note to show that it can be used to prove a considerably more general result about sequences. Indeed this more general result will have (1.2) as a consequence.

2. MAIN RESULTS.

A lemma which we shall use is the following

LEMMA. If

$$\beta_0 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_m$$

and

$$\sum_{\nu=0}^{m} \alpha_\nu = 0$$

and if the ordering of the α_ν is such that each positive α precedes all the negative ones, then

$$\sum_{\nu=0}^{m} \alpha_\nu \beta_\nu \geq 0 .$$

This lemma, which is easily proved, is not the one stated by Haber but, essentially, it is what he used. For with b_i defined as in [1]

$$(i = 0\ 1\ 2, \ldots, [\tfrac{n}{2}]: n \text{ even})$$

we do *not* in fact have

$$\sum_{i=0}^{[\frac{n}{2}]} b_i = 0$$

which is what is needed to apply the lemma quoted there.

Our result is the following.

THEOREM. Let $\{u_\nu\}_{\nu=0}^{n}$ be a convex sequence. Then

$$\frac{1}{n+1} \sum_{\nu=0}^{n} u_\nu \geq \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} u_\nu \ \ldots\ldots \tag{2.1}$$

To see that (1.2) is a consequence of (2.1) let the function $f(x)$ be bounded and convex (and hence continuous) on $[a,b]$ and take

$$u_\nu = f(a + \frac{\nu}{n}(b-a)) .$$

Then (2.1) reads

$$\frac{1}{n+1} \sum_{n=0}^{n} f(a + \frac{\nu}{n}(b-a)) \geq \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} f(a + \frac{\nu}{n}(b-a)) \ldots \tag{2.2}$$

On letting $n\to\infty$ the left-hand side here tends to the left-hand side of (1.2). And by virtue of Bernstein's result

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} \binom{n}{\nu} \phi\left(\frac{\nu}{n}\right) x^\nu (1-x)^{n-\nu} = \phi(x) \ \ldots\ldots \tag{2.3}$$

whenever $\phi \in C[0,1]$ we see that the right-hand side of (2.2) tends to $f(\frac{a+b}{2})$. Merely take $\phi(x) = f(a + x(b-a))$ and $x = 1/2$ in (2.3).

We now proceed to prove (2.1).

PROOF. Following Haber let us put $Q = [\frac{n}{2}]$ and write

$$\sum_{\nu=0}^{Q}{}^{*} \gamma_\nu = \begin{cases} \gamma_0+\gamma_1 + \ldots\ldots\ldots\ldots + \gamma_Q & \text{if } n \text{ is odd} \\ \gamma_0+\gamma_1 + \ldots + \gamma_{Q-1} + \frac{1}{2}\gamma_Q & \text{if } n \text{ is even} \end{cases} .$$

Then

$$\frac{1}{n+1} \sum_{\nu=0}^{n} u_\nu - \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} u_\nu = \sum_{\nu=0}^{Q}{}^{*} c_\nu [u_\nu + u_{n-\nu}]$$

where

$$c_\nu = \frac{1}{n+1} - \frac{1}{2^n}\binom{n}{\nu}$$

Since $\{u_\nu\}_{\nu=0}^{n}$ is convex then

$$u_{\nu+1} + u_{n-\nu-1} \le u_\nu + u_{n-\nu} \quad (0 \le \nu \le Q-1)$$

which is to say that the sequence $\{u_\nu + u_{n-\nu}\}_{\nu=0}^{Q}$ is non-increasing. We see too that the sequence $\{c_\nu\}_{\nu=0}^{Q}$ is non-increasing and that $\sum_{\nu=0}^{Q}{}^{*} c_\nu = 0$. Appealing to the Lemma quoted above we find that

$$\sum_{\nu=0}^{Q}{}^{*} c_\nu [u_\nu + u_{n-\nu}] \ge 0$$

and this complets the proof of (2.1).

In conclusion I wish to thank the referee for his helpful advice concerning the lemma used here.

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