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PATH DECOMPOSITIONS OF CHAINS AND CIRCUITS

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ABSTRACT. Expressions for the path polynomials (see Farrell [1]) of chains and circuits are derived. These polynomials are then used to deduce results about node disjoint path decompositions of chains and circuits. Some results are also given for decompositions in which specific paths must be used.

KEY WORDS AND PHRASES. Path polynomial, path decomposition, incorporated graph, restricted decomposition.

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1. INTRODUCTION.

The graphs considered here will be finite, undirected and without loops or multiple edges. Let G be such a graph. With every path α in G, let us associate the weight w_α. With every spanning subgraph C of G consisting of paths $\alpha_1,\alpha_2,\ldots,\alpha_k$, let us associate the monomial

$$w(C) = \prod_{i=1}^{k} w_{\alpha_i} .$$

The path polynomial of G is

$$P(G;\underline{w}) = \Sigma w(C),$$

where the summation is taken over all the path covers (spanning subgraphs consisting of paths only) of G. The vector $\underline{w}$ is called the general weight vector. In this paper we will assign weights to paths according to the number of nodes which they contain. Therefore, we will have $\underline{w} = (w_1,w_2,\ldots,w_k,\ldots)$, and $P(G;\underline{w})$ will be a polynomial in the indeterminates w_1,w_2, etc. If we put $w_i = w$, for all i, then the resulting polynomial $P(G;w)$ will be called the simple path polynomial of G. The path polynomial is a special F-polynomial (see [1]).

By path incorporating an edge x in G we will mean that x is distinguished in some manner and required to belong to every path cover of G that we consider. An incorporated graph will be a graph whose edges are all incorporated. A graph containing an incorporated subgraph will be called a restricted graph. Sometimes it will be convenient to shrink an incorporated subgraph to a "node". We will call such a "node" a compound node. A graph G containing an incorporated subgraph or a compound node i.e. a restricted graph, will normally be denoted by G*.

A tree with nodes of valencies 1 and 2 only will be called a chain. The chain with n nodes will be denoted by P_n. By attaching a chain P_n to a nonempty graph G we will mean that a terminal node of P_n is identified with a node of G, so as to form a connected component in which G and P_n are subgraphs with exactly one node in common. By adding a chain P_n to a graph G, we will mean that the two terminal nodes of P_n are attached to different nodes of G (N.B. In this case, G must have at least two nodes). We refer the reader to Harary [2] for definitions of the standard terms used in graph theory.

Upper limits of summations will be infinity unless otherwise specified. A node disjoint path decomposition of G is another name for a path cover of G. The name island decomposition was used by Goodman and Hedetniemi [3], who derived an efficient algorithm for finding hc(T) - the minimum number of edges needed to be added to a tree T in order to make it Hamiltonian. hc(G) is called the Hamiltonian completion number of G. It will be denoted by $\zeta(G)$. Some results on $\zeta(G)$ were obtained by Boesch, Chen, and McHugh [4]. Some results for trees were obtained by Slater [5].

In this paper we will use some of the basic results on path polynomials in order to derive the path polynomials of chains and circuits. From these polynomials, we will deduce results about node disjoint path decompositions of the graphs. Throughout the paper "decomposition" will mean "node disjoint path decomposition" and "cover" will mean "path cover".

2. BASIC PRELIMINARY RESULTS.

By putting the covers of G into classes according to whether or not they contain a specified edge, we obtain the following theorem, called the fundamental theorem for path polynomials.

THEOREM 1. Let G be a graph and e an (unincorporated) edge of G. Let G" be the graph obtained from G by deleting e and G*, the graph obtained from G by path incorporating e. Then

$$P(G;\underline{w}) = P(G';\underline{w}) + P(G^*;\underline{w}).$$

(N.B. Throughout this paper, we will refer to graphs obtained by deleting edges as G′ and graphs obtained by incorporating edges as G*, whenever Theorem 1 is applied.)

The fundamental algorithm for path polynomials consists of repeated applications of Theorem 1, until graphs G_i are obtained for which $P(G_i;\underline{w})$ are known. The following lemmas imply some useful simplifications to the fundamental algorithm (also called the reduction process).

LEMMA 1. (i) If an incorporated edge creates an incorporated circuit in G*, then

$$P(G^*;\underline{w}) = 0.$$

(ii) If an incorporated edge creates a node of valency 3 in G*, then

$$P(G^*;\underline{w}) = 0.$$

In any practical application of the reduction process, we could incorporate an edge e, by identifying the nodes at the ends of e as described in Read [6] for chromatic polynomials. However, we must keep track of the number of nodes in the incorporated subgraph associated with the new node formed by identification.

Lemma 1 suggests that we can delete from G*, all edges which will finally complete circuits with incorporated edges. Also, if two edges have been incorporated at node x, then all unincorporated edges at x can be deleted.

The following result is called the Component Theorem. It is also given in [1] for general F-polynomials. It can be easily proved.

THEOREM 2. Let G be a graph consisting of n components $H_1, H_2, \ldots, H_n$. Then

$$P(G;\underline{w}) = \prod_{i=1}^{n} P(H_i;\underline{w}).$$

3. PATH POLYNOMIALS OF CHAINS.

The notation G*[n] will be used for a restricted graph containing an incorporated chain with n edges. The following theorem gives a recurrence relation for the path polynomial of chains.

THEOREM 3.

$$P(P_n;\underline{w}) = \sum_{i=1}^{n} w_i \, P(P_{n-i};\underline{w}).$$

PROOF. Apply the reduction process to P_n by deleting a terminal edge. Then G' will be a graph containing two components, an isolated node and P_{n-1}. G* will be $P_n*[1]$. Apply the process to $P_n*[1]$ by deleting the terminal edge adjacent to the incorporated edge. In this case, the graph G' will be a graph with two components, an incorporated edge, having a weight w_2 and P_{n-2}. G* will be $P_n*[2]$. Continue the reduction process in this manner to subsequent graphs $P_n*(k)$ $(k > 2)$, until the entire chain is incorporated. The result then follows by adding the contributions of the graphs G' formed during the process and the final incorporated graph $P_n*[n-1]$, having a weight w_n.

For brevity, we will write P(n) for $P(P_n;\underline{w})$, when it would lead to no confusion. Conventionally, we write P(0) = 1. The following table gives values of P(n) for n = 1,2,3,4,5 and 6. Observe that the sum of the coefficients of the polynomial in the n^{th} row is 2^{n-1}.

The following theorem gives a generating function $P(P_n;\underline{w},t)$ for P(n).

THEOREM 4.

$$P(P_n;\underline{w},t) = [1 - (w_1 t + w_2 t^2 + w_3 t^3 + \ldots)]^{-1}.$$

PROOF.

$$P(n) = \sum_{i=1}^{n} w_i \, P(n-i)$$

$$= \text{coefficient of } t^n \text{ in } \sum_{s=1} w_s t^s . \sum_{k=0} P(k) t^k.$$

$$\text{Put } w(t) = \sum_{s=1} w_s t^s \text{ and } P(t) = \sum_{k=0} P(k) t^k.$$

Table 1

Path Polynomials of Chains

n	P(n)
1	w_1
2	$w_1^2 + w_2$
3	$w_1^3 + 2w_1w_2 + w_3$
4	$w_1^4 + 3w_1^2w_2 + 2w_1w_3 + w_2^2 + w_4$
5	$w_1^5 + 4w_1^3w_2 + 3w_1^2w_3 + 3w_1w_2^2 + 2w_1w_4 + 2w_2w_3 + w_5$
6	$w_1^6 + 5w_1^4w_2 + 4w_1^3w_3 + 6w_1^2w_2^2 + 3w_1^2w_4 + 6w_1w_2w_3 + 2w_1w_5 + w_2^3 + 2w_2w_4 + w_3^2 + w_6$

Then

$$\sum_{n=1} P(n)t^n = w(t)\, P(t) .$$

$$\Rightarrow \qquad P(t) - P(0) = w(t)\, P(t).$$

The result therefore follows.

We will now consider chains containing incorporated subchains. The covers of these restricted chains will be covers in which a particular path must appear. When the incorporated subchain contains r edges and contains a node of valency 1 of the restricted chain, the graph will be denoted by $P_n*[r]$, where n is the number of nodes in the nodes in the unincorporated subchain (including the node common to both subchains). The path polynomial of $P_n*[r]$ is given in the following theorem.

THEOREM 5.

$$P(P_n*[r];\underline{w}) = \sum_{i=1}^{n} w_{r+i}\, P(n-i).$$

PROOF. Apply the reduction process to $P_n*[r]$ by always deleting the edge which links the restricted edges to the unrestricted ones. The result then follows.

When the incorporated subchain does not include a node of valency 1 of the restricted chain, the graph will be denoted by $P^*_{n_1+n_2}[r]$ where n_1 and n_2 $(n_1,n_2 > 1)$ are the number of nodes in the unrestricted subchains. The path polynomial of this graph is given in the following result.

THEOREM 6.

$$P(P^*_{n_1+n_2}[r];\underline{w}) = \sum_{s=1}^{n_1} \sum_{t=1}^{n_2} w_{r+s+t-1}\, P(n_1-s)\, P(n_2-t) \quad (n_1,n_2 > 1).$$

PROOF. Apply the reduction process to the graph, by deleting the edge of the subchain P_{n_1}, which is adjacent to an incorporated edge. The graph G_1' will consist of two components P_{n_1-1} and $P^*_{n_2}[r]$. G_1* will be $P^*_{(n_1-1)+n_2}[r+1]$. Apply the reduction process to G_1^*, and to subsequent G_i^*'s by deleting the edge of the unrestricted subchain of P_{n_1} which is adjacent to an incorporated edge. By adding the contributions of the G_i''s and the final restricted graph $P_{n_2}^*[n_1 - 1 + r]$ we get

$$P(P_{n_1+n_2}^{**}[r];\underline{w}) = \sum_{s=1}^{n_1} P(n_1 - s)\ P(P_{n_2}^*[r+s-1];\underline{w}).$$

The result follows by using Theorem 5.

When the incorporated chain is attached to a node of valency 2 of the chain, the graph will be denoted by $P_{n_1,n_2}^*[r]$, where n_1 and n_2 $(n_1,n_2 > 1)$ are the number of nodes in the subchains separated by the node of valency 3. The result for this graph is given in the following theorem.

THEOREM 7.

$$P(P_{n_1,n_2}^*[r];\underline{w}) = P(n_1-1) \sum_{i=1}^{n_2} w_{r+i}\ P(n_2-i) + P(n_2-1) \sum_{i=1}^{n_{i-1}} w_{r+j+1}\ P(n_1-j-1) \quad (n_1,n_2 > 1).$$

PROOF. Let x be the node of valency 3 and y and z the nodes of the subchains P_{n_1} and P_{n_2} respectively, which are adjacent to x. Apply the reduction process by deleting the edge xy. G will be the graph consisting of components P_{n_1-1} and $P_{n_2}^*[r]$. G^* will consist of components P_{n_2-1} and $P_{n_1-1}^*[r+1]$. Hence we get

$$P(P_{n_1,n_2}^*[r];\underline{w}) = P(n_1 - 1)\ P(P_{n_2}^*[r];\underline{w}) + P(n_2 - 1)\ P(P_{n_1-1}^*[r+1];\underline{w}).$$

The result then follows by using Theorem 5.

4. SIMPLE PATH POLYNOMIALS OF CHAINS.

The factor $(1+w)$ appears quite frequently in the simple path polynomials of chains and circuits. We will therefore replace it by z. Let us assume that

$$P(G;w) = \sum_{i=1} A_i z^i,$$

where the A_i's are constants. By putting $w = 0$, we get $\sum_{i=1} A_i = 0$, since $P(G;w)$ can have no constant term. Also, the number of Hamiltonian paths in G (denoted by $H(G)$) will be the coefficient of w in $P(G;w)$, which is $\sum_{i=1} i\ A_i$. Hence we have

LEMMA 2. Let $P(G;w) = \sum_{i=1} A_i\ z^i$, where $z = 1 + w$. Then

(i) $\sum_{i=1} A_i = 0$

and (ii) $H(G) = \sum_{i=1} i\ A_i$.

We can obtain the simple path polynomial of P_n from Theorem 4 by putting $w_i = w$ for all i, then equating coefficients of t^n. However, the result can be more easily obtained by a straightforward combinatorial argument. A cover of P_n with cardinality k can be obtained by deleting $k-1$ edges for any $0 < k \le n$. Hence, we have

THEOREM 8.

$$P(P_n;w) = w(1+w)^{n-1} = z^n - z^{n-1}.$$

The following corollaries are immediate from Theorems 5 and 6. They can also be independently proved, by recognizing that in the case of the simple path polynomial of restricted chains, we can "shrink" the incorporated subchains to a node.

COROLLARY 5.1. For all r,

$$P(P_n{*}[r];w) = P(P_n;w) = z^n - z^{n-1}.$$

COROLLARY 6.1. For all r,

$$P(P_{n_1+n_2}^{**}[r];w) = P(P_{n_1+n_2-1};w) = z^{n_1+n_2-1} - z^{n_1+n_2-2}.$$

We can obtain the simple path polynomial of $P_{n_1,n_2}^*[r]$ by direct substitution into Theorem 7; however, this will be a bit tedious. It is better to obtain the result independently by using the reduction process. The result is given in the following corollary, for which an independent proof is given.

COROLLARY 7.1.

$$P(P_{n_1,n_2}^*[r];w) = z^{n_1+n_2-4}(z-1)^2(z+1).$$

PROOF. Apply the reduction process to the graph, just as we did in the proof of Theorem 7. G will consist of components P_{n_1-1} and $P_{n_2}{*}[r]$. But from Corollary 5.1,

$$P(P_{n_2}{*}[r];w) = P(P_{n_2};w).$$

G* will consist of components P_{n_2-1} and $P_{n_1-1}^*[r+1]$. But

$$P(P_{n_1-1}^*[r+1];w) = P(P_{n_1-1};w).$$

Hence we get

$$P(P_{n_1,n_2}^*[r];w) = P(P_{n_1} - 1)\ P(n_2) + P(n_2 - 1)\ P(n_1 - 1)$$

$$= (z^{n_1-1} - z^{n_1-2})(z^{n_2} - z^{n_2-1}) + (z^{n_2-1} - z^{n_2-2})(z^{n_1-1} - z^{n_1-2})$$

The result then follows after simplification.

Let us denote by P_n^{k*}, the restricted chain formed by attaching to a chain P_n, k incorporated paths ($k < [n/2]$) such that no two are attached to adjacent nodes of P_n. We will also assume that none of the end-nodes of P_n are used, for if it is, then it can be treated as an ordinary node, as implied by Corollary 5.1. The simple path polynomial of P_n^{k*} is given in the following theorem.

THEOREM 9.

$$P(P_n^{k*};w) = z^{n-2k-1}(z-1)^{k+1}(z+1)^k.$$

PROOF BY INDUCTION ON k. Corollary 7.1 verifies the case for $k = 1$. Let us assume that the statement holds for $k = r$. Then

$$P(P_n^{r*};w) = z^{n-2r-1}(z-1)^{r+1}(z+1)^r.$$

We can apply the reduction process to $P_n^{(r+1)*}$ by deleting the edge which is adjacent to "last" incorporated path from the end of the restricted chain, such that G would consist of (i) a restricted chain containing r incorporated paths, and (ii) a restricted chain with one end-node attached to an incorporated path. Thus the components of G will be P_s^{r*} and P_t*, with $s+t=n$. Since the incorporated path is attached to an end-node of P_t*, $P(P_t*;w) = P(P_t;w)$. Hence

$$P(G\ ;w) = z^{s-2r-1}(z-1)^{r+1}(z+1)^r \quad z^{t-1}(z-1)$$
$$= z^{n-2r-2}(z-1)^{r+2}(z+1)^r.$$

The graph G* will contain two components P_{s-1}^{r*} and P_t. Hence

$$P(G*;w) = z^{s-2r-2}(z-1)^{r+1}(z+1)^r \quad z^{t-1}(z-1)$$
$$= z^{n-2r-3}(z-1)^{r+2}(z+1)^r.$$

It follows that

$$P(P_n^{(r+1)*};w) = z^{n-2r-2}(z-1)^{r+2}(z+1)^r + z^{n-2r-3}(z-1)^{r+2}(z+1)r.$$

On simplification, we get

$$P(P_n^{(r+1)*};w) = z^{n-2r-3}(z-1)^{r+2}(z+1)^{r+1}.$$

Hence the statement holds for $k = r+1$, when it is assumed for $k = r$. By the Principle of Induction, it holds for all r.

5. PATH DECOMPOSITION OF CHAINS.

Let $\pi = (n_1^{k_1}, n_2^{k_2}, \ldots, n_r^{k_r})$ be a partition of n. We will denote by $N_G(\pi)$, the number of decompositions of the graph G into k_i paths with n_i nodes, for $i = 1,2,\ldots,r$. N_G will denote the total number of decompositions of G into paths. It is clear that $N_G(\pi)$ is the coefficient of $w_{n_1}^{k_1} w_{n_2}^{k_2} \ldots w_{n_r}^{k_r}$ in $P(G;\underline{w})$.

The following corollary is immediate from Theorem 8. It confirms the observation made about the sum of the coefficients of the polynomials in Table 1 above.

COROLLARY 8.1.

$$N_{P_n} = 2^{n-1}.$$

It can be observed from Table 1, that all partitions of the integer n appear as subscripts of w. $[w_1^k = w_1 w_1 \ldots$ (k times)]. This suggests the following theorem which is otherwise obvious.

THEOREM 10. For every partition $\pi = (d_1, d_2, \ldots, d_k)$ of n, a decomposition of P_n into paths containing d_i nodes ($i = 1, 2, \ldots, k$).

The following corollary of Theorem 3 gives a recurrence which is useful for finding the number of decompositions of a chain into paths with specified lengths.

COROLLARY 3.1.

$$N_{P_n}(n_1^{k_1}, n_2^{k_2}, \ldots, n_r^{k_r}) = \sum_{i=1}^{r} N_{P_{n-n_i}}(n_1^{k_1}, n_2^{k_2}, \ldots, n_{i-1}^{k_{i-1}}, n_i^{k_i-1}, n_{i+1}^{k_{i+1}}, \ldots, n_r^{k_r}).$$

PROOF. From Theorem 3, we have

$$P(n) = w_1 P(n-1) + w_2 P(n-2) + \ldots + w_n.$$

$N_{P_n}(n_1^{k_1}, n_2^{k_2}, \ldots, n_r^{k_r})$ is the coefficient of $w_{n_1}^{k_1} w_{n_2}^{k_2} \ldots w_{n_r}^{k_r}$ in $P(n)$. The only terms on the R.H.S. which contribute to this coefficient are $w_{n_1} P(n - n_1)$, $w_{n_2} P(n - n_2)$, ... and $w_{n_r} P(n - n_r)$. The result therefore follows.

Corollary 3.1 suggests an algorithm for finding the number $N_{P_n}(n_1^{k_1}, n_2^{k_2}, \ldots, n_r^{k_r})$. The following lemma is quite useful when applying Corollary 3.1. Its proof is straightforward.

LEMMA 3.

(i) $N_{P_n}(1^{n-2}, 2) = n - 1.$

(ii) $N_{P_n}(1^{n-3}, 3) = n - 2.$

(iii) $N_{P_n}(1, n-1) = 2.$

(iv) $N_{P_n}((n/k)^k) = 1$, where k divides n.

ILLUSTRATION.

$$\begin{aligned} N_{P_6}(1^2, 2^2) &= N_{P_5}(1, 2^2) + N_{P_4}(1^2, 2) \\ &= N_{P_4}(2^2) + N_{P_3}(1, 2) + N_{P_3}(1, 2) + N_{P_2}(1^2) \\ &= 1 + 2 + 2 + 1 \\ &= 6, \text{ in agreement with Table 1.} \end{aligned}$$

By a restricted decomposition of a graph G, we mean a decomposition in which particular paths must be included. Clearly, the decompositions of restricted graphs will be restricted.

The following corollaries can be easily deduced from the results given in Section 4 by putting w = 1.

COROLLARY 5.2. The number of restricted decompositions of the chain P_{n+k} in which a particular terminal subchain P_k is always included, is 2^{n-1}.

COROLLARY 6.2. The number of restricted decompositions of $P_{n_1+n_2+n_3}$ $(n_1, n_2, n_3 > 1)$ in which a particular subchain P_{n_3} (which does not include a terminal edge) is always included, is $2^{n_1+n_2-2}$, where n_1 and n_2 are the numbers of nodes in the subchains separated by P_{n_3}.

COROLLARY 7.2. Let G be a graph consisting of 3 chains P_{n_1}, P_{n_2} and P_{n_3} $(n_1, n_2, n_3 > 1)$ attached to a single node. The number of restricted decompositions of G, in which the chain P_{n_3} is always included, is $3.2^{n_1+n_2-4}$.

COROLLARY 9.1. Let G be the graph formed by attaching k chains to non-adjacent nodes of valency 2 in P_n. The number of restricted decompositions of G, in which the k chains are always included, is $3^k \cdot 2^{n-2k-1}$.

By extracting the coefficient of w^s in the expression given for $P(P_{n_1,n_2}^{*}[r];w)$ in Corollary 7.1 and the expression for $P(P_n^{k*};w)$ given in Theorem 9, we obtain the following results.

COROLLARY 7.3. Let G be a graph consisting of 3 chains P_{n_1}, P_{n_2} and P_{n_3} $(n_1,n_2,n_3 > 1)$ attached to a node. The number of restricted decompositions of G, with cardinality $s(s > 1)$, in which the chain P_{n_3} is always included, is

$$\begin{bmatrix} N-3 \\ s-2 \end{bmatrix} + \begin{bmatrix} N-4 \\ s-2 \end{bmatrix}$$

where $N = n_1 + n_2$.

COROLLARY 9.2. Let G be a graph formed by attaching k chains to non-adjacent nodes of valency 2 of a chain P_n. The number of restricted decompositions of G, with cardinality $s(s > k)$, in which the k chains are always included is

$$\sum_{r=0}^{s} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} n-2k-1 \\ s-k-r-1 \end{bmatrix} 2^{k-r} .$$

6. PATH POLYNOMIALS OF CIRCUITS.

The notation C_n will be used for a circuit with n nodes. The following theorem gives an expression for the path polynomial of C_n.

THEOREM 11.

$$P(C_n;\underline{w}) = \sum_{r=1}^{n} r w_r \, P(n-r).$$

PROOF. Apply the reduction process to C_n by deleting an edge. The graph G′ will be P_n. G* will be $C_n^{*}[1]$. Continue to apply the reduction process to the restricted graphs formed, by always deleting an edge which is adjacent to an incorporated edge. The graphs G′ will be restricted chains. The graph formed by incorporating the final edge of C_n cannot contribute to $P(C_n;\underline{w})$ since it contains an incorporated circuit.

By adding the contributions of the graphs G′ formed during the process, we get

$$P(C_n;\underline{w}) = \sum_{r=0}^{n-1} P(P_{n-r}^{*}[r];\underline{w}) \quad (\text{with } P_n^{*}[0] \equiv P_n).$$

By using Theorem 5, we get

$$P(C_n;\underline{w}) = \sum_{r=0}^{n-1} \sum_{i=1}^{n-r} w_{r+i} \, P(n-r-i)$$

$$= \sum_{r=1}^{n} r \, w_r \, P(n-r), \text{ as required.}$$

The following table gives values of $C(n)$ $(\equiv P(C_n;\underline{w}))$ for n = 1,2,3,4,5, and 6. Observe that the sum of the coefficients of the polynomial in the n^{th} row is $2^n - 1$.

Table 2

Path Polynomials of Circuits

n	C(n)
1	w_1
2	$w_1^2 + 2w_2$
3	$w_1^3 + 3w_1w_2 + 3w_3$
4	$w_1^4 + 4w_1^2w_2 + 4w_1w_3 + 2w_2^2 + 4w_4$
5	$w_1^5 + 5w_1^3w_2 + 5w_1^2w_3 + 5w_1w_2^2 + 5w_2w_3 + 5w_5$
6	$w_1^6 + 6w_1^4w_2 + 6w_1^3w_3 + 9w_1^2w_2^2 + 6w_1^2w_4 + 12w_1w_2w_3 + 6w_1w_5 + 2w_2^3 + 6w_2w_4 + 3w_3^2 + 6w_6$

A generating function for $P(C_n;\underline{w})$ can be obtained by using a technique similar to that used in the proof of Theorem 4. It is given in the following theorem.

THEOREM 12.

$$P(C_n;\underline{w},t) = \frac{w_1t + 2w_2t^2 + 3w_3t^3 + \ldots}{1 - (w_1^t + w_2t^2 + w_3t^3 + \ldots)} = \frac{t\,P'(t)}{P(t)},$$

where $P(t) \equiv P(P_n;\underline{w},t)$.

Let $C^*_{n+r-1}[r]$ be a restricted circuit consisting of an incorporated path with r edges and a path with n nodes. Let us apply the reduction process to $C^*_{n+r-1}[r]$ as we did in the proof of Theorem 11 for the restricted circuits. This will yield

$$P(C^*_{n+r-1}[r];\underline{w}) = \sum_{i=0}^{n-1} P(P^*_{n-i}[r+i];\underline{w}).$$

By applying Theorem 5, we get the following result.

THEOREM 13.

$$P(C^*_{n+r-1}[r];\underline{w}) = \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} w_{r+i+j}\, P(n-i-j).$$

Consider now the case in which the incorporated chain P_r^* is attached to C_n. We will denote this graph by $C_n^{**}[r-1]$. If we apply the reduction process to this graph, by deleting an unincorporated edge which is adjacent to an incorporated edge, the graph G' will be $P^*_{q+r-1}[r-1]$ and G^* will be $P^*_{q+r-1}[r]$. Hence we obtain the following result by using Theorem 5.

THEOREM 14.

$$P(C_n^{**}[r-1];\underline{w}) = \sum_{i=1}^{n+r-1} (w_{r+i} + w_{r+i-1})\, P(n+r-1-i).$$

7. SIMPLE PATH POLYNOMIALS OF CIRCUITS.

The simple path polynomial of C_n can be easily deduced from Theorems 8 and 11, or by elementary combinatorial argument. It is given in the following theorem.

THEOREM 15.

$$P(C_n;w) = (1 + w)^n - 1 = z^n - 1.$$

The restricted graph $C^*_{n+r-1}[r]$ can be treated as C_{n-1} by "shrinking" the incorporated path to a node. Thus we have

COROLLARY 13.1. $P(C^*_{n+r-1}[r];w) = P(C_{n-1} i\ w) = z^{n-1} - 1.$

Instead of deriving results for $C^{**}_n[r-1]$, we would solve the general problem in which k incorporated chains are attached to non-adjacent nodes of $C_n (k \le [\frac{n}{2}])$. Let us denote this graph by C^k_n*.

We will consider 2 cases: (1) C^k_n* contains a pair of consecutive incorporated chains which are separated by a path of length greater than 2, and (2) C^k_n* has no such pair of incorporated chains. In case (1), we can apply the reduction process to C^k_n*, by deleting an edge which is adjacent to an incorporated chain and which belongs to a separating path of length greated than 2. G' will be $P_n{}^{(k+1)}*$, since the end-node to which an incorporated chain is attached, can be treated as an ordinary node. $G*$ will be $P_{n-1}{}^{(k-1)}*$. Hence from Theorem 9, we get

$$\begin{aligned} P(C^k_n*;w) &= z^{n-2k-1}\ (z-1)^k\ (z+1)^{k-1} + z^{n-2k}\ (z-1)^k\ (z+1)^{k-1} \\ &= z^{n-2k}\ (z^2 - 1)^k. \end{aligned}$$

In case (2), G' will be $P_{n-1}{}^{(k-1)}*$ and $G*$ will be $P_{n-1}{}^{(k-2)}*$. It follows that

$$\begin{aligned} P(C^k_n*;w) &= z^{n-2k-1}\ (z-1)^k\ (z+1)^{k-1} + z^{n-2k+2}\ (z-1)^{k-1}\ (z+1)^{k-2} \\ &= z^{n-2k-1}\ (z-1)^{k-1}\ (z+1)^{k-2}\ (z^3 + z^2 - 1). \end{aligned}$$

Hence we have the following theorem.

THEOREM 16. If C^k_n* contains a pair of consecutive incorporated chains, separated by a path in C_n of length greater than 2, (or if k = 1) then

$$P(C^k_n*;w) = z^{n-2k}\ (z^2 - 1)^k;$$

otherwise,

$$P(C^k_n*;w) = z^{n-2k-1}\ (z-1)^{k-1}\ (z+1)^{k-2}\ (z^3 + z^2 - 1).$$

8. PATH DECOMPOSITIONS OF CIRCUITS.

The covers of P_n will also be covers of C_n. Therefore, it is expected that the results similar to those given in Section 5 will also hold for circuits.

The following is an immediate corollary of Theorem 15. It confirms our observation of the sum of the coefficients of the polynomials in Table 2.

COROLLARY 11.1. $N_{C_n} = 2^n - 1.$

Since the covers of P_n will also be covers of C_n, we have from Theorem 10, the following corollary.

COROLLARY 10.1. For every partition $\pi = (d_1, d_2, d_3, \ldots, d_k)$ of n, $\exists$ a decomposition of C_n into paths containing d_i nodes $(i = 1,2,\ldots,k)$.

The following corollary of Theorem 11 is analogous to Corollary 3.1. Its proof is similar.

COROLLARY 11.1.

$$N_{C_n}(n_1^{k_1}, n_2^{k_2}, \ldots, n_r^{k_r}) = \sum_{i=1}^{r} n_i N_{P_{n-n_i}}(n_1^{k_1}, n_2^{k_2}, \ldots n_{i-1}^{k_{i-1}}, n_i^{k_i-1}, n_{i+1}^{k_{i+1}}, \ldots, n_r^{k_r}).$$

The following lemma is analogous to Lemma 3. It will be useful for practical applications of Corollary 11.2.

LEMMA 4. (i) $N_{C_n}(1^{n-2}, 2) = N_{C_n}(1,^{n-3},3) = N_{C_n}(1,n-1) = N_{C_n}(n) = n.$

(ii) $N_{C_n}(\ (n/k)^k) = n/k,$ where k divides n.

The following corollaries are immediate from Corollary 13.1.

COROLLARY 13.2. The number of restricted decompositions of C_{n+k} in which a particular subchain P_{k+2} is always included is $2^{n-1} - 1$.

COROLLARY 13.3. The number of restricted decompositions with cardinality s, of C_{n+k}, in which a particular subchain P_{k+2} is always included is $\binom{n-1}{s}$.

From Theorem 16, we obtain the following corollaries.

COROLLARY 16.1. Let G be a graph consisting of a circuit with k chains attached to it. If G contains a pair of consecutive attached chains which are separated by a path of length greater than 2, then the number of restricted decompositions of G in which all k chains are included is $3^k \cdot 2^{n-2k}$. Otherwise, the number of such restricted decompositions is $11 \cdot 3^{k-2} \cdot 2^{n-2k-1}$.

COROLLARY 16.2. The number of restricted decompositions with cardinality $s (s > k)$ of the graph G of Corollary 16.1 is

$$\sum_{r=0}^{k} \binom{k}{r} \binom{n-2k}{s-k-r} 2^{k-r},$$

when G contains a pair of consecutive attached chains, separated by a path of length greater than 2. Otherwise, the number of such decompositions is

$$\sum_{i=0}^{k-2} \binom{k-2}{i} 2^{k-2-i} \left[\binom{N}{m-2} + 3\binom{N+1}{m} + \binom{N+2}{m+1} \right],$$

where $N = n - 2k - 1$ and $m = s - i - k$.

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