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# SUPREMUM NORM DIFFERENTIABILITY

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ABSTRACT. The points of Gateaux and Fréchet differentiability of the norm in $C(T, E)$ are obtained, where $T$ is a locally compact Hausdorff space and $E$ is a real Banach space. Applications of these results are given to the space $\bar{\ell}_{\infty}(E)$ of all bounded sequences in $E$, and to the space $B(\ell_1, E)$ of all bounded linear operators from $\ell_1$ into $E$.

KEY WORDS AND PHRASES. *Banach spaces, continuous functions, vector-valued functions, supremum norm, smooth points.*

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## 1. INTRODUCTION.

In [1], Banach proved that if $T$ is a compact metric space and $C(T)$ is the Banach space of all continuous real valued functions on $T$, with the supremum norm, then

$$\lim_{\lambda \to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda}$$

exists for all $g \in C(T)$ if and only if there exists a $t_0 \in T$ such that $|f(t_0)| > |f(t)|$ for all $t \in T$, $t \neq t_0$.

This theorem, however, is no longer true if $T$ is a locally compact, non-compact, Hausdorff space; as can easily be seen by considering the Banach space $\ell_\infty$ of all bounded real valued sequences with the supremum norm.

In fact, if $\mathbb{N}$ is the set of positive integers equipped with the discrete topology, then $\ell_\infty = C(\mathbb{N})$ , the space of all bounded continuous functions on $\mathbb{N}$ . If we let $x = \{x_n\}_{n\geq 1} \in \ell_\infty$ , where $x_1 = 1$ and $x_n = \frac{n-1}{n}$ for $n > 1$ , then $x$ peaks at $n = 1$ , but because of the behaviour of $x$ at infinity and the existence of Banach limits, it is possible to find two distinct support functionals to the ball in $\ell_\infty$ at $x$ , so that $x$ is not a smooth point.

In this note, we characterize the points of Gateaux and Fréchet differentiability of the norm function in $C(T, E)$ , the space of all bounded continuous $E$-valued functions on the locally compact Hausdorff space $T$ , where $E$ is a real Banach space.

Two applications of these results are given. The first is to the space $\ell_\infty(E)$ of all bounded sequences in $E$ , and the second to the space $B(\ell_1, E)$ of all bounded linear operators from $\ell_1$ into $E$ .

## 2. DEFINITIONS AND NOTATION.

In the following, $E$ denotes a real Banach space and $E^*$ denotes the dual of $E$ . The unit ball of $E$ is $B_E = \{x \in E \mid \|x\| \leq 1\}$ and its boundary $S_E = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of $E$ .

A Banach space $E$ is said to be smooth at $x \in E \sim \{0\}$ if and only if there exists a unique hyperplane of support to $B_E$ at $\frac{x}{\|x\|}$ ; that is, there exists only one continuous linear functional $\phi \in E^*$ with $\|\phi\| = 1$ such that $\phi(x) = \|x\|$ . Such a linear functional $\phi \in E^*$ is called the support functional to $B_E$ at $\frac{x}{\|x\|}$ , and $\phi^{-1}(\{1\})$ is called the hyperplane of support to $B_E$ at $\frac{x}{\|x\|}$ . A Banach space $E$ is said to be a smooth Banach space if it is smooth at every $x \in S_E$ .

The norm function $\|.\| : E \to \mathbb{R}^+$ is said to be Gateaux differentiable at $x \in E \sim \{0\}$ if and only if there exists a functional $\phi \in E^*$ with

$$\lim_{\lambda \to 0} \left| \frac{\|x + \lambda h\| - \|x\|}{\lambda} - \phi(h) \right| = 0 , \qquad (*)$$

for every $h \in E$. The functional $\phi$ is called the Gateaux derivative of the norm at $x \in E$. The norm function $\|.\| : E \to \mathbb{R}^+$ is said to be Fréchet differentiable at $x \in E \sim \{0\}$ if and only if there exists a functional $\phi \in E^*$ such that

$$\lim_{\|h\|\to 0} \frac{|\, \|x + h\| - \|x\| - \phi(h)\, |}{\|h\|} = 0 , \qquad (**)$$

that is, the limit in (*) exists uniformly for $h \in B_E$.

It is well known, Mazur [8], that $E$ is smooth at $x \in E \sim \{0\}$ if and only if $\lim_{\lambda\to 0} \frac{\|x + \lambda h\| - \|x\|}{\lambda}$ exists for all $h \in E$, if and only if the norm function $\|.\|: E \to \mathbb{R}^+$ is Gateaux differentiable at $x$.

3. SMOOTH POINTS IN $C(T, E)$.

If $T$ is a topological space and $E$ is a real Banach space, then $C(T, E)$ denotes the Banach space of all bounded continuous E-valued functions on $T$, with the supremum norm; that is,

$$C(T, E) = \{f: T \to E \mid f \text{ is bounded and continuous}\} ,$$

and $\|f\| = \sup\{\|f(t)\| : t \in T\}$, for $f \in C(T, E)$.

As mentioned earlier, Banach [1] proved that if $T$ is a compact metric space, then $C(T) = C(T, \mathbb{R})$ is smooth at $f \neq 0$ if and only if $f$ is a peaking function, that is, there exists a point $t_0 \in T$ such that $\|f\| = |f(t_0)| > |f(t)|$ for all $t \in T$, $t \neq t_0$.

Kondagunta [6], and Cox and Nadler [3], have characterized the points of Gateaux and Fréchet differentiability of the norm in $C(T, E)$ when $T$ is compact Hausdorff. Cox and Nadler, in the same paper, give a characterization of the points of Fréchet differentiability of the norm in $C(T, \mathbb{R})$ when $T$ is locally compact Hausdorff.

In this section, we generalize these results to the space $C(T, E)$ when $T$ is locally compact Hausdorff. The techniques used by Cox and Nadler will not work in this case, since, in general, the range of $f \in C(T, E)$ is no longer relatively compact and hence no extension to an $\hat{f} \in C(\beta T, E)$ is possible. However, a slight modification of the argument given in Köthe [7], (page 352), for the corresponding result in $\ell_\infty$, does work.

As usual, the results on smoothness are the more difficult, and the results on Fréchet differentiability follow as a corollary.

THEOREM 3.1. Let $T$ be a locally compact Hausdorff space and $E$ a Banach space. A point $f \in C(T, E)$, $f \neq 0$, is a smooth point of $C(T, E)$ if and only if

(i) there exists a $t_0 \in T$ such that $\|f(t_0)\| > \|f(t)\|$ for all $t \in T$, $t \neq t_0$,

(ii) there exists a compact neighborhood $K$ of $t_0$ such that
$$\sup_{t \in T \sim K} \|f(t)\| < \|f\| ,$$

(iii) $f(t_0)$ is a smooth point of $E$.

PROOF.

A. Assume first that the norm $\|.\| : C(T, E) \to \mathbb{R}^+$ is Gateaux differentiable at $f \in C(T, E)$, where we may (and do) assume that $\|f\| = 1$.

(I) We show first that if the mapping $\|f(.)\| : T \to \mathbb{R}^+$ achieves its maximum at $t_0 \in T$, then $t_0$ is unique. Suppose there exist $t_0, t_1 \in T$, $t_0 \neq t_1$, such that $\|f(t_0)\| = \|f(t_1)\| = 1$. For $t \in T$ and $\phi \in E^*$, let $\delta_{\phi,t} \in C(T, E)^*$ denote the evaluation functional given by $\delta_{\phi,t}(g) = \phi(g(t))$ for $g \in C(T, E)$; then $\|\delta_{\phi,t}\| = 1$ for all $t \in T$, $\phi \in S_{E^*}$. Using the Hahn-Banach theorem, choose $\phi_0, \phi_1 \in E^*$ with $\|\phi_0\| = \|\phi_1\| = 1$ such that $\phi_0(f(t_0)) = \phi_1(f(t_1) = 1$. Then $\delta_{\phi_0,t_0}$ and $\delta_{\phi_1,t_1}$ are distinct support functionals to the ball in $C(T, E)$ at $f$, which contradicts the fact that $f$ is a smooth point in $C(T, E)$.

(II) We show next that given any compact set $K \subseteq T$, either $\sup_{t \in T \sim K^\circ} \|f(t)\| < 1$, or there exists a $t_0 \in T \sim K^\circ$ such that $\|f(t_0)\| = 1$.

To the contrary, suppose there exists a compact set $K \subseteq T$, such that $\sup_{t \in T \sim K^\circ} \|f(t)\| = 1$ and $\|f(t)\| < 1$, for all $t \in T \sim K^\circ$. Let $F_0(t) = \|f(t)\|$, for all $t \in T$. Then $F_0$ is a bounded continuous function on $T$ and thus has an extension, $F$, to $\beta T$, the Stone-Čech compactification of $T$. Since $\sup_{t \in T \sim K^\circ} \|f(t)\| = 1$, we have $A = F^{-1}(1) \cap (\beta T \sim T) \neq \phi$. Also
$$A = \bigcap_{n=1}^{\infty} \{t \in T \sim K : F(t) > 1 - \tfrac{1}{n}\}$$

and thus $A$ is a $G_\delta$ set in $\beta T$. By Čech [2], singletons in $\beta T \sim T$ are not $G_\delta$ sets, so $A$ contains at least two distinct points $p$ and $q$.

Let $\{p_\mu\}$ and $\{q_\nu\}$ be disjoint nets contained in $T$ such that $p_\mu \to p$ and $q_\nu \to q$ in $\beta T$. For each $\mu$, let $\phi_\mu \in E^*$, with $\|\phi_\mu\| = 1$ and $\phi_\mu(f(p_\mu)) = F(p_\mu)$. Also, for each $\nu$, choose $\psi_\nu \in E^*$, with $\|\psi_\nu\| = 1$ and $\psi_\nu(f(q_\nu)) = F(q_\nu)$. Let $\Phi_\mu = \delta_{\phi_\mu, p_\mu}$ and $\Psi_\nu = \delta_{\psi_\nu, q_\nu}$, for each $\mu$ and $\nu$. Then $\Phi_\mu$ and $\Psi_\nu \in C(T, E)^*$ and $\|\Phi_\mu\| = \|\Psi_\nu\| = 1$, for each $\mu$ and $\nu$.

Since the ball in $C(T, E)^*$ is w*-compact, there exist $\Phi, \Psi \in C(T, E)^*$, with $\|\Phi\| \leq 1$ and $\|\Psi\| \leq 1$, such that $\Phi$ is a w*-accumulation point of the net $\{\Phi_\mu\}$ and $\Psi$ is a w*-accumulation point of the net $\{\Psi_\nu\}$. By construction, $\Phi(f) = \Psi(f) = 1$ and, thus, $\Phi$ and $\Psi$ are support functionals to the ball in $C(T, E)$ at $f$. Since $f$ is assumed to be a smooth point in $C(T, E)$, it must be that $\Phi = \Psi$. We will show that this is impossible. Let $P = \{p_\mu\} \cup \{p\}$ and $Q = \{q_\nu\} \cup \{q\}$. Then $P$ and $Q$ are disjoint closed subsets of $\beta T$, which is a compact Hausdorff space and therefore normal. Let $h_1, h_2 \in C(\beta T)$ with $0 \leq h_1, h_2 \leq 1$, $h_1 + h_2 = 1$ and $h_1(P) = h_2(Q) = 0$. Use $h_i$ for the restriction of $h_i$ to $T$, as well.

Clearly, if $g \in C(T, E)$, then $h_1 g \in \ker \Phi$ and $h_2 g \in \ker \Psi$. Since $\Phi = \Psi$, and $g \in C(T, E)$ can be written as $g = h_1 g + h_2 g$, we have $\Phi = \Psi = 0$. But, this contradicts $\|\Phi\| = \|\Psi\| = 1$. Therefore, we must have that either $\sup_{t \in T \sim K^\circ} \|f(t)\| < 1$, or there exists $t_0 \in T \sim K^\circ$ with $\|f(t_0)\| = 1$, for any compact set $K \subsetneq T$.

(III) Finally we show that (i), (ii), (iii) hold. Taking $K = \emptyset$ in (II), since $\|f\| = 1$, we see that there exists a $t_0 \in T$ with $\|f(t_0)\| = 1$; and from (I), $\|f(t_0)\| > \|f(t)\|$ for all $t \neq t_0$. Again by (II), if $K \subsetneq T$ is a compact set with $t_0 \in K^\circ$, then $\sup_{t \in T \sim K} \|f(t)\| < 1$. If there exist distinct functionals $\phi_1, \phi_2 \in E^*$ with $\|\phi_1\| = \|\phi_2\| = 1$ such that $\phi_1(f(t_0)) = \phi_2(f(t_0)) = 1$, then this implies that $\delta_{\phi_1, t_0}, \delta_{\phi_2, t_0} \in C(T, E)^*$ are distinct support functionals to the ball in $C(T, E)$ at $f$, which contradicts the fact that $f$ is a smooth point. Therefore $f(t_0)$ is a smooth point of $E$.

B.. Conversely, suppose that $f \in C(T, E)$ , $\|f\| = 1$ , and (i), (ii), and (iii) hold; then there exists a unique $t_0 \in T$ such that $\|f(t_0)\| = 1$ , there exists a compact set $K \subseteq T$ with $t_0 \in K^\circ$ such that $\sup_{t \in T \sim K} \|f(t)\| < 1$ , and $E$ is smooth at $f(t_0)$ .

Let $g \in C(T, E)$ , $g \neq 0$ , and let $\delta > 0$ be such that $\|f(t)\| < \|f(t_0)\| - \delta$ for all $t \in T \sim K$ . If $0 < |\lambda| < \frac{\delta}{2\|g\|}$ , then for $t \in T \sim K$ we have

$$\|f(t) + \lambda g(t)\| \leq \|f(t)\| + |\lambda| \; \|g(t)\| < \|f(t_0)\| + |\lambda| \; \|g\| - \delta < \|f(t_0)\| - \frac{\delta}{2} \; .$$

Thus, $\|f(t) + \lambda g(t)\| < \|f(t_0)\| - \frac{\delta}{2}$ for all $t \in T \sim K$ whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$

On the other hand,

$$\|f + \lambda g\| \geq \|f\| - |\lambda| \, \|g\| = \|f(t_0)\| - |\lambda| \; \|g\| > \|f(t_0)\| - \frac{\delta}{2} \quad \text{for} \quad 0 < |\lambda| < \frac{\delta}{2\|g\|} \; .$$

Therefore, for $0 < |\lambda| < \frac{\delta}{2\|g\|}$ ,

$$\sup_{t \in T} \|f(t) + \lambda g(t)\| = \sup_{t \in K} \|f(t) + \lambda g(t)\| \; .$$

Since $K$ is compact, by Kondagunta's result [6] ,

$$\lim_{\lambda \to 0} \frac{\sup_{t \in K} \|f(t) + \lambda g(t)\| - \sup_{t \in K} \|f(t)\|}{\lambda} = \lim_{\lambda \to 0} \frac{\|f \pm \lambda g\| - \|f\|}{\lambda}$$

exists for all $g \in C(T, E)$ . Hence $C(T, E)$ is smooth at $f$ .

q.e.d.

An analogous result holds for the Fréchet differentiability of the norm in $C(T, E)$ .

COROLLARY 3.2. Let $T$ be a locally compact Hausdorff space and $E$ a Banach space. The norm function $\|.\| : C(T, E) \to \mathbb{R}^+$ is Fréchet differentiable at $f \in C(T, E)$ , $f \neq 0$ , if and only if

(i) there exists a unique $t_0 \in T$ such that $\|f(t_0)\| > \sup_{t \neq t_0} \|f(t)\|$ ;

(ii) $\{t_0\}$ is an open subset of $T$ , that is, $t_0$ is an isolated point of $T$ ;

(iii) the norm function $\|.\| : E \to \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$ .

(Note: (ii) follows from (i) . )

PROOF.

A. Suppose that $\|.\| : C(T, E) \to \mathbb{R}^+$ is Fréchet differentiable at $f \in C(T, E)$, $\|f\| = 1$ ; then the ball in $C(T, E)$ is smooth at $f$ , so there exists a $t_0 \in T$ and a compact neighborhood $K$ of $t_0$ such that

(1) $\|f(t_0)\| > \|f(t)\|$ for all $t \in T$ , $t \neq t_0$ ,

(2) $\sup\limits_{t \in T \sim K} \|f(t)\| < 1$ ,

(3) $E$ is smooth at $f(t_0)$ .

Now

$$\lim_{\lambda \to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda \to 0} \frac{\sup\limits_{t \in K} \|f(t) + \lambda g(t)\| - \sup\limits_{t \in K} \|f(t)\|}{\lambda}$$

exists uniformly for $g \in B_{C(T, E)}$ , and since $K$ is compact, an appeal to the result of Cox and Nadler [3] shows that $\{t_0\}$ is open and $\|.\| : E \to \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$ . Also, since $\{t_0\}$ is open and $K \sim \{t_0\}$ is compact, the uniqueness of $t_0$ shows that $\sup\limits_{t \neq t_0} \|f(t)\| < 1$ .

B. Conversely, suppose that (i), (ii), and (iii) hold. Using the previous theorem, we see that $\|.\| : C(T, E) \to \mathbb{R}^+$ is Gateaux differentiable at $f$ , and taking $K = \{t_0\}$ ,

$$\lim_{\lambda \to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda \to 0} \frac{\|f(t_0) + \lambda g(t_0)\| - \|f(t_0)\|}{\lambda}$$

exists uniformly for $g \in B_{C(T, E)}$ since $\|.\| : E \to \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$ .

q.e.d.

## 4. APPLICATIONS.

A. $\ell_\infty(E)$ . If $E$ is a Banach space, then

$$\ell_\infty(E) = \{x = \{x_n\}_{n \ge 1} \mid x_n \in E \text{ for } n \ge 1 \text{ and } \sup_{n \ge 1} \|x_n\| < \infty\}$$

with the supremum norm, $\|x\| = \sup\limits_{n \ge 1} \|x_n\|$ .

THEOREM 4.1. Let $E$ be a Banach space, then the norm function $\|.\| : \ell_\infty(E) \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $x = \{x_n\}_{n\geq 1}$, $x \neq 0$, if and only if

(i) there exists an $n_0$ such that $\|x_{n_0}\| > \|x_n\|$ for $n \neq n_0$,

(ii) $\sup_{n\neq n_0} \|x_n\| < \|x\|$,

(iii) the norm function $\|.\| : E \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $x_{n_0}$.

PROOF. Let $\mathbb{N}$ denote the set of positive integers equipped with the discrete topology, then $\ell_\infty(E) = C(\mathbb{N}, E)$ the space of bounded continuous $E$-valued functions on the locally compact Hausdorff space $N$.

q.e.d.

B. $B(\ell_1, E)$. Let $E$ be a Banach space, let $\ell_1$ be the Banach space of all absolutely summable real valued sequences with $\|a\| = \sum_{n=1}^{\infty} |a_n|$ for $a = \{a_n\}_{n\geq 1} \in \ell_1$, and let $B(\ell_1, E)$ be the space of all bounded linear operators from $\ell_1$ into $E$. For $n \geq 1$, let $\delta^n$ be the $n^{th}$ basis vector in $\ell_1$, that is $\delta^n = \{\delta_k^n\}_{k\geq 1}$.

THEOREM 4.2. Let $E$ be a Banach space, then the norm function $\|.\| : B(\ell_1, E) \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $T \in B(\ell_1, E)$, $T \neq 0$, if and only if

(i) there exists an $n_0$ such that $\|T(\delta^{n_0})\| > \|T(\delta^n)\|$ for $n \neq n_0$;

(ii) $\sup_{n\neq n_0} \|T(\delta^n)\| < \|T\|$;

(iii) the norm function $\|.\| : E \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $T(\delta^{n_0})$.

PROOF. The mapping $\sigma : B(\ell_1, E) \to \ell_\infty(E)$ given by $\sigma(T) = \{T(\delta^n)\}_{n\geq 1}$ for $T \in B(\ell_1, E)$, is a linear isometry of $B(\ell_1, E)$ onto $\ell_\infty(E)$.

REMARKS.

1. In connection with the second example, it should be mentioned that Kheinrikh [5] has given a complete characterization of the points of Gateaux and Fréchet

differentiability of the norm in $K(E, F)$ , the space of compact linear operators from $E$ into $F$ , where $E$ and $F$ are Banach spaces. He has also given a characterization of the points of Fréchet differentiability of the norm in $B(E, F)$ , the space of bounded linear operators from $E$ into $F$ (no proofs are given in this paper). However, the more difficult question of smoothness in $B(E, F)$ is still unanswered.

2. Regarding Theorem 3.1. perhaps this will clear up the popular misconception that, for $T$ locally compact Hausdorff, $C(T, E)$ (or $C(T, \mathbb{R})$) is smooth at $f$ if and only if $f$ peaks at some $t_0 \in T$ . (See e.g. Holmes [4], p. 232, #4.10). A result which is obviously false, as the example in the introduction demonstrates.

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