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THE REGULARITY SERIES OF A CAUCHY SPACE

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ABSTRACT. This study extends the notion of regularity series from convergence spaces to Cauchy spaces, and includes an investigation of related topics such as that T_2 and T_3 modifications of a Cauchy space and their behavior relative to certain types of quotient maps. These concepts are applied to obtain a new characterization of Cauchy spaces which have T_3 completions.

KEY WORDS AND PHRASES. *Cauchy space, R-Series, Regular Cauchy space, C_3 Cauchy space, cauchy retriction, Wyler modification.*
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0. INTRODUCTION.

In two earlier papers ([5] and [7]), the author and Gary Richardson introduced the regularity series (or R-series) of a convergence space. Because the ideas and results developed in these papers have been fruitful in later investigations, it seems appropriate to extend at least some of them to the realm of Cauchy spaces. This is especially natural in view of the key role which regularity plays in the theory of Cauchy space completions.

We shall make one significant deviation from the notation of [5] and [7]; a convergence space will be denoted by "(X,q)" (where X is the underlying set and q the convergence structure) rather than by "X". The usual notation for a Cauchy space is "$(X,\mathfrak{C})$", where $\mathfrak{C}$ is the Cauchy structure. This will enable us to make an easy transition from a convergence structure to the associated Cauchy structure (or vice versa) on the same underlying set.

In Section 1, we define the R-series and W-series of an arbitrary Cauchy space and show that their properties are analogous to those of the R-series for a convergence space discussed in [7]. Section 2 considers certain Cauchy quotient maps relative to which the R- and W-series are well behaved. The next section considers the T_2 and T_3 modifications of a Cauchy space. In Section 4 we consider *admissible convergence structures* (those which admit a Cauchy structure); the interaction between admissibility and regularity is studied, and the *symmetric* *series* for an arbitrary convergence space is introduced. Section 5 introduces a generalization of the Wyler completion [2] which is called the *Wyler modification*. The concluding section extends a set function originally defined in [4] and makes uses of the results of preceding sections to characterize Cauchy spaces having T_3 completions.

1. THE CAUCHY R-SERIES.

Let X be a set, F(X) the set of all filters on X. The fixed ultrafilter generated by $x \in X$ will be denoted by $\dot{x}$. If $\mathcal{F}, \mathcal{G} \in F(X)$ and $F \cap G \neq \phi$ for all $F \in \mathcal{F}$, $G \in \mathcal{G}$, we shall write "$\mathcal{F} \vee \mathcal{G}$".

The term *convergence space* will mean a "Limitierung" in the sense of Fischer [1]. For any convergence space (X,q), "cl_q^n" denotes the nth iteration of the *q-closure operator*. We shall also introduce a *weak q-closure operator* defined as follows: If $A \subseteq X$, then $wcl_q A = \{x \in X : \dot{y} \to X \text{ for some } y \in A\}$. The nth iteration of the weak closure operator is denoted "wcl_q^n". A convergence space (X,q) is *regular* (respectively, *weakly regular*) if $cl_q \mathcal{F} \to x$ (respectively, $wcl_q \mathcal{F} \to x$) whenever $\mathcal{F} \to x$. A convergence space T_2 (respectively, T_1) if each filter converges to at most one point (respectively, each fixed ultrafilter converges to exactly one point).

Starting with a set X, a subset of $\mathfrak{C}$ of F(X) is called a *Cauchy structure* on X if the following conditions are satisfied:

(c_1) $\dot{x} \in \mathfrak{C}$, for all $x \in X$

(c_2) $\mathcal{F} \in \mathfrak{C}$ and $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{G} \in \mathfrak{C}$

(c_3) $\mathcal{F}, \mathcal{G} \in \mathfrak{C}$ and $\mathcal{F} \vee \mathcal{G}$ implies $\mathcal{F} \cap \mathcal{G} \in \mathfrak{C}$.

If $(X,\mathfrak{C})$ is a Cauchy space, the *induced convergence structure* $q_{\mathfrak{C}}$ on x is defined by : $\mathcal{F} \to x$ if $\mathcal{F} \cap \dot{x} \in \mathfrak{C}$. A Cauchy space is *regular* (respectively, *weakly regular*) if $\mathcal{F} \in \mathfrak{C}$ implies $cl_{q_{\mathfrak{C}}} \mathcal{F} \in \mathfrak{C}$ (respectively $wcl_q \mathcal{F} \in \mathfrak{C}$). A Cauchy space is T_2 if $\dot{x} \cap \dot{y} \in \mathfrak{C}$ implies $x = y$. A T_2 Cauchy space is obviously weakly regular; a T_2 regular Cauchy space (or convergence space) is said to be T_3. A Cauchy space $(X,\mathfrak{C})$ is *complete* if each $\mathcal{F} \in \mathfrak{C}$ is q-convergent: if every ultrafilter in F(X) belongs to $\mathfrak{C}$, then $\mathfrak{C}$ is said to be *totally bounded*.

A function $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ is *Cauchy continuous* if $\mathcal{F} \in \mathfrak{C}_1$ implies $f(\mathcal{F}) \in \mathfrak{C}_2$; a Cauchy continuous function will henceforth be called a *map*. The Cauchy structures on a set X are partially ordered by the relation $\mathfrak{C}_1 \leq \mathfrak{C}_2$ iff the identity function $id_X : (X,\mathfrak{C}_2) \to (X,\mathfrak{C}_1)$ is a map.

It is well known that, for any Cauchy space $(X,\mathfrak{C})$ there is a finest regular Cauchy structure $r\mathfrak{C}$ on X which is coarser than $\mathfrak{C}$; $r\mathfrak{C}$ is called the *regular modification* of C. Similarly, there is a finest weakly regular Cauchy structure $w\mathfrak{C}$ coarser than $\mathfrak{C}$ which is called the *weakly regular modification* of $\mathfrak{C}$. Starting with a Cauchy space $(X,\mathfrak{C})$ we shall construct two series of Cauchy structures which terminate, respectively, in the regular and weakly regular modifications of $\mathfrak{C}$.

Every collection $\mathfrak{A}$ of subsets of X generates a unique Cauchy structure in a manner which we shall now describe. First, we say that a finite set of filters

$\{\mathcal{F}_1,\ldots,\mathcal{F}_n\}$ is *linked* if they can be arranged (by renumbering, if necessary) in such a way that $\mathcal{F}_1 \vee \mathcal{F}_2, \mathcal{F}_2 \vee \mathcal{F}_3,\ldots,\mathcal{F}_{n-1} \vee \mathcal{F}_n$. Starting with an arbitrary collection $\mathbf{A}$ of subsets of X, define $\mathbf{A}' = \mathbf{A} \cup \{\dot{x} : x \in X\}$, and let $\mathfrak{C}_{\mathbf{A}} = \{\mathcal{G} \in F(X) :$ there are linked filters $\mathcal{F}_1,\ldots,\mathcal{F}_n$ in $\mathbf{A}'$ such that $\mathcal{G} \geq \bigcap_{i=1}^{n} \mathcal{F}_i\}$.

PROPOSITION 1.1. In the terminology of the preceding paragraph, $\mathfrak{C}_{\mathbf{A}}$ is the finest Cauchy structure on X which contains all members of the collection $\mathbf{A}$.

We shall refer to $\mathfrak{C}_{\mathbf{A}}$ as the *Cauchy structure generated by* $\mathbf{A}$.

The R-Series $\{r_\beta\mathfrak{C}\}$ for $(X,\mathfrak{C})$ is constructed as follows:

$r_0\mathfrak{C} = \mathfrak{C}$

$r_1\mathfrak{C}$ is the Cauchy structure on X generated by $\{\mathrm{cl}^n_{q_{\mathfrak{C}}} \mathcal{F} : \mathcal{F} \in \mathfrak{C},\ n \in N\}$

⋮

$r_\beta\mathfrak{C}$ is the Cauchy structure on X generated by $\{\mathrm{cl}^n_{q_{\beta-1}} \mathcal{F} : \mathcal{F} \in \mathfrak{C},\ n \in N\}$

if β is a non-limit ordinal where $q_{\beta-1}$ is the convergence structure compatible with $r_{\beta-1}$.

$r_\beta\mathfrak{C} = \bigcup\{r_\alpha\mathfrak{C} : \alpha < \beta\}$ if β is a limit ordinal.

The W-series $\{w_\beta\mathfrak{C}\}$ is obtained by repeating the preceding construction using the weak closure operator in place of the closure operator.

Let $\ell_R\mathfrak{C}$ (respectively, $\ell_w\mathfrak{C}$) be the least ordinal number γ such that $r_\gamma\mathfrak{C} = r_{\gamma+1}\mathfrak{C}$ (respectively, $w_\gamma\mathfrak{C} = w_{\gamma+1}\mathfrak{C}$). The next result resembles Proposition 2.1, 7 ; all parts of this proposition are either obvious or straightforward, and we omit the proof.

PROPOSITION 1.2. Let $(X,\mathfrak{C})$ be an arbitrary Cauchy space; let β, γ be ordinal numbers with $\beta < \gamma$.

(1) $r\mathfrak{C} \leq r_\gamma\mathfrak{C} \leq r_\beta\mathfrak{C} \leq \mathfrak{C}$

(2) $w\mathfrak{C} \leq w_\gamma\mathfrak{C} \leq w_\beta\mathfrak{C} \leq \mathfrak{C}$

(3) $r_\gamma\mathfrak{C} \leq w_\gamma\mathfrak{C}$

(4) $r_\gamma\mathfrak{C} = r\mathfrak{C}$ iff $\gamma \geq \ell_R\mathfrak{C}$

(5) $w_\gamma\mathfrak{C} = w\mathfrak{C}$ iff $\gamma \geq \ell_w\mathfrak{C}$

PROPOSITION 1.3. If $(X,\mathfrak{C})$ is a complete Cauchy space, then $(X,r_\beta\mathfrak{C})$ and $(X,w_\beta\mathfrak{C})$ are complete for all ordinal numbers β.

Proof. Let $\mathcal{G} \in r_1\mathfrak{C}$; then there are $\mathcal{F}_1,\ldots,\mathcal{F}_n$ in $\mathfrak{C}$ and $m \in N$ such that $\mathrm{cl}^m_{q_{\mathfrak{C}}} \mathcal{F}, \ldots, \mathrm{cl}^m_{q_{\mathfrak{C}}} \mathcal{F}_n$ are linked and $\mathcal{G} \geq \bigcap_{i=1}^{n} \mathrm{cl}^m_{q_{\mathfrak{C}}} \mathcal{F}_i$. Since $(X,\mathfrak{C})$ is complete, there are x_i, $i = 1, \ldots, n$, such that $\mathcal{F}_i \cap \dot{x}_i \in \mathfrak{C}$. Thus $\mathcal{G} \cap \dot{x}_i \in r_1\mathfrak{C}$ for $i = 1, \ldots, n$, and it follows that $r_1\mathfrak{C}$ is complete. This reasoning extends by transfinite induction to all ordinal numbers β. □

PROPOSITION 1.4. Let $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ be a map. In the following diagrams, in which all vertical arows are f and all horizontal arrows are the respective identity functions, each arrow is a map.

$$\begin{array}{ccccccccc}
(X,\mathfrak{C}) & \to & (X,r_1\mathfrak{C}) & \to \dots \to & (X,r_\beta\mathfrak{C}) & \to \dots \to & (X,r\mathfrak{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(Y,\mathfrak{D}) & \to & (Y,r_1\mathfrak{D}) & \to \dots \to & (Y,r_\beta\mathfrak{D}) & \to \dots \to & (Y,r\mathfrak{D})
\end{array}$$

$$\begin{array}{ccccccccc}
(X,\mathfrak{C}) & \to & (X,w_1\mathfrak{C}) & \to \dots \to & (X,w_\beta\mathfrak{C}) & \to \dots \to & (X,w\mathfrak{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(Y,\mathfrak{D}) & \to & (Y,w_1\mathfrak{D}) & \to \dots \to & (Y,w_\beta\mathfrak{D}) & \to \dots \to & (Y,w\mathfrak{D})
\end{array}$$

PROOF. From the original assumption, it follows that $f(cl^n_{q_\mathfrak{C}} \mathcal{F}) \supseteq cl^n_{q_\mathfrak{C}} f(\mathcal{F})$ for all $\mathcal{F} \in F(X)$. It follows easily that $f : (X,r_1\mathfrak{C}) \to (Y,r_1\mathfrak{D})$ is a map. This reasoning can be extended by induction to the remaining vertical arrows in the first diagram. Similar reasoning can be applied to the second diagram. □

2. QUOTIENT MAPS

A Cauchy quotient map $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ is an onto map such that $\mathfrak{C}_2$ is the finest Cauchy structure on X_2 relative to which f is continuous. In other words, f is a Cauchy quotient map iff $\{f(\mathcal{F}) : \mathcal{F} \in \mathfrak{C}_1\}$ generates $\mathfrak{C}_2$.

If $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ is a map and there is a map $g : (X_2,\mathfrak{C}_2) \to (X_1,\mathfrak{C}_1)$ such that $f \circ g = id_{X_2}$, then f is called <u>Cauchy retraction</u> and $(X_2,\mathfrak{C}_2)$ is a <u>Cauchy retract</u> of $(X_1,\mathfrak{C}_1)$. It is easy to see that a Cauchy retraction is a Cauchy quotient map, and that g is a Cauchy embedding of $(X_2,\mathfrak{C}_2)$ into $(X_1,\mathfrak{C}_1)$. Thus a Cauchy retract is both a quotient space and a subspace of the domain space.

If $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ is an onto map with the property that $\mathcal{G} \in \mathfrak{C}_2$ implies $f^{-1}(\mathcal{G}) \in \mathfrak{C}_1$, then f is called a Cauchy initial map; in this case $\mathfrak{C}_1$ is called the <u>initial Cauchy structure</u> on X_1 determined by f and $(X_2,\mathfrak{C}_2)$. If $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ is an initial Cauchy map, let $x_y \in f^{-1}(y)$ be chosen arbitrarily for each $y \in X_2$, and define $g : (X_2,\mathfrak{C}_2) \to (X_1,\mathfrak{C}_1)$ by $g(y) = x_y$. Clearly g is a map, and $f \circ g = id_{X_2}$. Thus each Cauchy initial map is a Cauchy retraction.

PROPOSITION 2.1. The image of a complete Cauchy space under a Cauchy quotient map is complete.

PROOF. Let $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ be a Cauchy quotient map, and $(X,\mathfrak{C})$ a complete Cauchy space. Let $\mathcal{G} \in \mathfrak{D}$; then there are $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathfrak{C}$ such that $f(\mathcal{F}_1), \dots, f(\mathcal{F}_n)$ are linked filters and $\mathcal{G} \geq \bigcap_{i=1}^{n} f(\mathcal{F}_i)$. Since $(X,\mathfrak{C})$ is complete, there is $x_i \in X_i$ such that $\mathcal{F}_i \cap \dot{x}_i \in \mathfrak{C}$ for $i = 1, \dots, n$. Then $\mathcal{G} \cap f(\dot{x}_i) \in \mathfrak{D}$ for $i = 1, \dots, n$, and it follows that $(Y,\mathfrak{D})$ is complete. □

PROPOSITION 2.2. Let $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ be a Cauchy retraction map. If $(X_1,\mathfrak{C}_1)$ is regular, weakly regular, or T_2, then $(X_2,\mathfrak{C}_2)$ has the same property. For any ordinal number β, $f : (X_1 r_\beta \mathfrak{C}_1) \to (X_2, r_\beta \mathfrak{C}_2)$ and $f : (X_1, w_\beta \mathfrak{C}_1) \to (X_2, w_\beta \mathfrak{C}_2)$ are Cauchy retraction maps.

PROOF. The first assertion follows from the fact that $(X_2,\mathfrak{C}_2)$ is Cauchy-homeomorphic to a subspace of $(X_1,\mathfrak{C}_1)$. The second is obtained by applying Proposition 1.4 to both f and the associated map $g : (X_2,\mathfrak{C}_2) \to (X_1,\mathfrak{C}_1)$. □

PROPOSITION 2.3. If $f : (X_1,\mathfrak{C}_1) \to (X_2,\mathfrak{C}_2)$ is a Cauchy initial map, then $f : (X_1,r_\beta\mathfrak{C}_1) \to (X_2,r_\beta\mathfrak{C}_2)$ and $f : (X_1,w_\beta\mathfrak{C}_1) \to (X_2,w_\beta\mathfrak{C}_2)$ are Cauchy initial maps for all ordinal numbers β.

PROOF. First note that a Cauchy initial map is a proper map (see [7]) between the associated convergence spaces (which we denote by (X_1,q_1) and (X_2,q_2)); this implies that $f(\mathrm{cl}^n_{q_1} A) = \mathrm{cl}^n_{q_2} f(A)$ for all $n \in N$ and $A \subseteq X_1$. Assume that $\mathcal{G} = \mathrm{cl}^m_{q_2} \mathcal{G}_1 \cap \ldots \cap \mathrm{cl}^m_{q_2} \mathcal{G}_n \in r_1\mathfrak{C}_2$, where $\mathcal{G}_1, \ldots, \mathcal{G}_n \in \mathfrak{C}_2$ and $\mathrm{cl}^m_{q_2} \mathcal{G}_1, \ldots, \mathrm{cl}^m_{q_2}\mathcal{G}_n$ are linked. It follows that $f^{-1}(\mathrm{cl}^m_{q_2} \mathcal{G}_i) = \mathrm{cl}^m_{q_1} f^{-1}(\mathcal{G}_i)$ for $i = 1, \ldots, n$ and $\mathrm{cl}^m_{q_1} f^{-1}(\mathcal{G}_1), \ldots, \mathrm{cl}^m_{q_1} f^{-1}(\mathcal{G}_n)$ are also linked. Thus $f^{-1}(\mathcal{G}) = \mathrm{cl}^m_{q_1} f^{-1}(\mathcal{G}_1) \cap \ldots \cap \mathrm{cl}^m_{q_1} f^{-1}(\mathcal{G}_n) \in r_1\mathfrak{C}_1$. This establishes the result for $\beta = 1$. The argument extends easily by induction to an arbitrary ordinal number β. □

3. THE T_2 AND T_3 MODIFICATIONS.

Let $(X,\mathfrak{C})$ be a weakly regular Cauchy space, and define the equivalence relation $x \sim y$ iff $\dot{x} \cap \dot{y} \in \mathfrak{C}$. Let $[x] = \{y \cap X : \dot{x} \cap \dot{y} \in \mathfrak{C}\}$, and let $X^\wedge = \{[x] : x \in X\}$. Let $\psi : X \to X^\wedge$ be the canonical map defined by $\psi(x) = [x]$, and let $\mathfrak{C}^\wedge$ be the quotient Cauchy structure on $X^\wedge$ induced by $\psi : (X,\mathfrak{C}) \to (X^\wedge,\mathfrak{C}^\wedge)$.

LEMMA 3.1. If $A \subseteq X$, where $(X,\mathfrak{C})$ is a weakly regular Cauchy space, then $\mathrm{wcl}_{q_\mathfrak{C}} A = \psi^{-1}(\psi A)$.

PROOF. $Y \in \mathrm{wcl}_{q_\mathfrak{C}} A \Leftrightarrow \exists\, x \in A$ such that $\dot{x} \to y \Leftrightarrow \exists\, x \in A$ such that $y \in [x] \in \psi(A) \Leftrightarrow y \in \psi^{-1}(\psi(A))$. □

PROPOSITION 3.2. Let $(X,\mathfrak{C})$ be a weakly regular Cauchy space. Then $\psi : (X,\mathfrak{C}) \to (X^\wedge,\mathfrak{C}^\wedge)$ is a Cauchy initial map, and $(X^\wedge,\mathfrak{C}^\wedge)$ is T_2.

PROOF. If $\mathcal{F} \in \mathfrak{C}$, then by Lemma 3.1, $\mathrm{wcl}_{q_\mathfrak{C}} \mathcal{F} = \psi^{-1}\psi(\mathcal{F})$. Since filters of the form $\psi(\mathcal{F})$, $\mathcal{F} \in \mathfrak{C}$, generate $\mathfrak{C}^\wedge$ and $\mathrm{wcl}_{q_\mathfrak{C}} \mathcal{F} \in \mathfrak{C}$, it follows that ψ is a Cauchy initial map. To show that $(X^\wedge,\mathfrak{C}^\wedge)$ is T_2, suppose $\dot{a} \cap \dot{b} \in \mathfrak{C}^\wedge$, where $a = [\dot{x}]$ and $b = [\dot{y}]$. Then $\psi^{-1}(\dot{a} \cap \dot{b}) = [\dot{x}] \cap [\dot{y}] \in \mathfrak{C}$. This means $[x] \cap [y] \neq \emptyset$, which implies $[x] = [y] = a = b$. □

Next, let $(X,\mathfrak{C})$ be an arbitrary Cauchy space. Define $(X_h,\mathfrak{C}_h) = (X^\wedge,(w\mathfrak{C})^\wedge)$ and $(X_\tau,\mathfrak{C}_\tau) = (X^\wedge,(r\mathfrak{C})^\wedge)$. By Propositions 2.3 and 3.2, it follows that the former Cauchy space is T_2 and the latter is T_3; we shall call these the T_2 and T_3 modifications, respectively, of $(X,\mathfrak{C})$. Consider the following diagram (HT):

$$
\begin{array}{ccccccc}
(X,\mathfrak{C}) & \xrightarrow{\mathrm{id}_X} & (X,w\mathfrak{C}) & \xrightarrow{\mathrm{id}_X} & (X,r\mathfrak{C}) & \xrightarrow{\psi_{r\mathfrak{C}}} & (X_\tau,\mathfrak{C}_\tau) \\
 & & \downarrow \psi_{w\mathfrak{C}} & & \downarrow \psi_{w\mathfrak{C}} & & \downarrow \theta \\
 & & (X_h,\mathfrak{C}_h) & \xrightarrow{\mathrm{id}_{X_h}} & (X_h,r\mathfrak{C}_h) & \xrightarrow{\mathrm{id}_{r\mathfrak{C}_h}} & ((X_h)_\tau, (r\mathfrak{C}_h)_\tau),
\end{array}
$$

where $\theta : (X_\tau, \mathfrak{C}_\tau) \to ((X_h)_\tau, (r\mathfrak{C}_h)_\tau)$ is defined by $\theta([x]_\tau) = \psi_{r\mathfrak{C}_h}(\psi_{w\mathfrak{C}}(x))$.

PROPOSITION 3.3. The diagram (HT) is commutative. All identity functions are maps, and all other functions are Cauchy initial maps. Furthermore, θ is a Cauchy homeomorphism.

PROOF. All parts of this proposition are clear except, perhaps, the following: (a) $\psi_{w_{\mathfrak{C}}} : (X, r\mathfrak{C}) \to (X_h, r\mathfrak{C}_h)$ is a Cauchy initial map; (b) θ is a Cauchy homeomorphism.

Statement (a) is a consequence of Proposition 2.3. To prove (b), let a = b in X_τ, where $a = \psi_{r\mathfrak{C}}(x)$, $b = \psi_{r\mathfrak{C}}(y)$. Then $\dot{x} \cap \dot{y} \in r\mathfrak{C}$ since $\psi_{r\mathfrak{C}}$ is a Cauchy initial map; thus $\theta(a) = \theta(b)$, and θ is well defined. Next, let $\theta(a) = \psi_{r\mathfrak{C}_h}(\psi_{w\mathfrak{C}}(\psi^{-1}_{r\mathfrak{C}}(\theta))$ in $(X_h)_\tau$, where $\theta(a) = \psi_{r\mathfrak{C}_h}(\psi_{w\mathfrak{C}}(y))$, $\theta(b) = \psi_{r\mathfrak{C}_h}(\psi_{w\mathfrak{C}}(y))$. Since both maps involved in the latter equations are Cauchy initial maps, $\dot{x} \cap \dot{y} \in r\mathfrak{C}$, and it follows that $a = \psi_{r\mathfrak{C}}(x) = \psi_{r\mathfrak{C}}(y) = b$; thus θ is one-to-one. If $\mathfrak{A} \in \mathfrak{C}_\tau$, then one can show by a direct agrument that $\theta(\mathfrak{A}) = \psi_{r\mathfrak{C}_h}(\psi_{w\mathfrak{C}}(\psi^{-1}_{r\ \mathfrak{C}}(\mathfrak{A})))$; the latter filter is in $\psi(r\mathfrak{C}_h)_\tau$ since all maps involved are Cauchy initial maps. A similar argument shows that θ^{-1} is a map, and the proof of (b) is complete. $\square$

Let $f : (X, \mathfrak{C}) \to (Y, \mathfrak{D})$ be a Cauchy map, and define $f_h : (X_h, \mathfrak{C}_h) \to (Y_h, \mathfrak{D}_h)$ by $f_h([x]_h) = [f(x)]_h$ and $f_\tau : (X_\tau, \mathfrak{C}_\tau) \to (Y_\tau, \mathfrak{D}_\tau)$ by $f_\tau([x]_\tau) = [f(x)]_\tau$. If $[x]_h = [y]_h$ in X_h, then $\dot{x} \cap \dot{y} \in w\mathfrak{C}$, and so $f(\dot{x} \cap \dot{y}) = \dot{f}(x) \cap \dot{f}(y) \in w\mathfrak{D}$ by Proposition 1.4; thus $[f(x)]_h = [f(y)]_h$, and f_h is well-defined. A similar argument shows that f_τ is well-defined.

PROPOSITION 3.4. If $f : (X, \mathfrak{C}) \to (Y, \mathfrak{D})$ and $g : (Y, \mathfrak{D}) \to (Z, \mathfrak{E})$ are maps, then f_h, g_h, f_τ, and g_τ (defined in the preceding paragraph) are maps, $(gf)_h = g_h f_h$, and $(gf)_\tau = f_\tau g_\tau$.

PROOF. To prove that f_h is a map, let $\mathfrak{A} \in \mathfrak{C}_h$; then $\psi^{-1}_{w\mathfrak{C}}(\mathfrak{A}) \in w\mathfrak{C}$, and so $f_h(\mathfrak{A}) \geq \psi_{w\mathfrak{D}}(f(\psi^{-1}_{w\mathfrak{C}}(\mathfrak{A}))) \in \mathfrak{D}_h$, which establishes the desired result. A similar argument shows that f_τ is a map. Finally, note that $(g_h \circ f_h)[x]_h = g_h([f(x)]_h) = [gf(x)]_h = (g \circ f)_h[x]$; a similar argument establishes that $g_\tau \circ f_\tau = (gf)_\tau$. $\square$

PROPOSITION 3.5. If $f : (X, \mathfrak{C})$ $(Y, \mathfrak{D})$ is a Cauchy retraction (respectively, a Cauchy initial map) the f_h and f_τ are Cauchy retractions (respectively, Cauchy initial maps).

PROOF. If f is a Cauchy retraction map, then there is a map $g : (Y, \mathfrak{D}) \to (X, \mathfrak{C})$ such that $f \circ g = id_Y$. Then $f_h : (X_h, \mathfrak{C}_h) \to (Y_h, \mathfrak{D}_h)$ and $g_h : (Y_h, \mathfrak{D}) \to (X_h, \mathfrak{C}_h)$ are both maps, and $(f \circ g)_h = f_h \circ g_h = (id_Y)_h = id_{Y_h}$, which implies that f_h is a Cauchy retraction map. A similar argument establishes that f_τ is a Cauchy retraction map.

Next, assume that f is a Cauchy initial map.

$$\begin{array}{ccccc} (X,\mathfrak{C}) & \xrightarrow{id_X} & (X, w\mathfrak{C}) & \xrightarrow{\psi_{w\mathfrak{C}}} & (X_h, \mathfrak{C}_h) \\ \downarrow f & & \downarrow f & & \downarrow f_h \\ (Y,\mathfrak{D}) & \xrightarrow{id_Y} & (Y, w\mathfrak{D}) & \xrightarrow{\psi_{w\mathfrak{D}}} & (Y_h, \mathfrak{D}_h). \end{array}$$

Let $A \in \mathcal{D}_h$. By commutativity of the preceding diagram, $f^{-1}(\psi_{w\mathcal{D}}^{-1}(A)) = \psi_{w\mathcal{C}}^{-1}(f_h^{-1}(A))$. Since $f:(X,w\mathcal{C}) \to (Y,w\mathcal{D})$ and $\psi_{w\mathcal{D}}$ are Cauchy initial maps, $f^{-1}(\psi_{w\mathcal{D}}^{-1}(A)) \in w\mathcal{C}$, and so $f_h^{-1}(A) = \psi_{w\mathcal{C}}\psi_{w\mathcal{C}}^{-1}(f_h^{-1}(A)) \in \mathcal{C}_h$. A similar argument shows that f_τ is a Cauchy initial map. □

4. ADMISSIBLE CONVERGENCE STRUCTURES.

A convergence space (X,q) is *Cauchy-admissible* if there is a Cauchy structure $\mathcal{C}$ on X such that $q = q_{\mathcal{C}}$. "Cauchy-admissible" will be shortened to *admissible*. Admissible convergence structures have been characterized by H. Keller [3]. We denote by $F_q(x)$ the set of all filters on X which q-converge to x.

PROPOSITION 4.1. The following statements about a convergence space (X,q) are equivalent.

(1) (X,q) is admissible.

(2) If $F_q(x) \cap F_q(y) \neq \emptyset$, then $F_q(x) = F_q(y)$.

(3) $\mathcal{C}^q = \{\mathcal{F} \in F(X) : \mathcal{F}\ q\text{-converges}\}$ is a Cauchy structure on X.

(4) $\mathcal{F} \to x$ iff there are linked filters $\mathcal{F}_1 \to x_1, \ldots, \mathcal{F}_n \to x_n$ such that $x \in \{x_1, \ldots, x_n\}$ and $\mathcal{F} \geq \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_n$.

Given an arbitrary convergence structure q on X, define αq as follows: $\mathcal{F} \to x$ in $(X,\alpha q)$ if there are linked filters $\mathcal{F}_1 \to x_1, \ldots, \mathcal{F}_n \to x_n$ in (X,q) such that $x \in \{x_1, \ldots, x_n\}$ and $\mathcal{F} \geq \bigcap_{i=1}^{n} \mathcal{F}_i$. It is obvious that αq is the finest admissible convergence structure on X coarser than q; we shall call αq the *admissible modification* of q. We omit the straightforward proof of the next proposition.

PROPOSITION 4.2. If $f : (X,q) \to (Y,p)$ is continuous, then $f : (X,\alpha q) \to (Y,\alpha p)$ is continuous.

It is clear that "admissible convergence space" and "complete Cauchy space" are the same mathematical notion, formulated in slightly different ways. An R-series for convergence spaces is developed in [5] and [7], and in this section we consider the relationship between this notion and the R-series for Cauchy spaces developed in Section 1 of this paper. We begin this investigation by studying the relationship between "admissibility" and "regularity".

Starting with a convergence space (X,q), the R-series $\{r_\alpha q\}$ of (X,q) is a family of convergence structures defined on the set X as follows: $r_0 q = q$; $\mathcal{F} \to x$ relative to $r_1 q$ iff there exist $n \in N$ and $\mathcal{G} \to x$ relative to q such that $\mathcal{F} \geq \mathrm{cl}_q^n \mathcal{G}$; for ordinal $\alpha > 0$, $\mathcal{F} \to x$ relative to $r_\alpha q$ iff there exist $n \in N$, $\mathcal{G} \to x$ relative to q, and $\beta < \alpha$ such that $\mathcal{F} \geq \mathrm{cl}_{r_\beta q}{}^n \mathcal{G}$.

LEMMA 4.3. For any convergence space (X,q) and $A \subseteq X$, $\mathrm{cl}_q A \subseteq \mathrm{cl}_{\alpha q} A \subseteq \mathrm{cl}_{r_2 q}^2 A$.

PROOF. The first inclusion is obvious. Let $x \in \mathrm{cl}_{\alpha q} A$; then there is $\mathcal{F} \to x$ in $(X,\alpha q)$ such that $A \in \mathcal{F}$. Also, there are linked filters $\mathcal{F}_1 \to x_1, \ldots, \mathcal{F}_n \to x_n$ in (X,q) such that $\mathcal{F} \geq \bigcap_{i=1}^{n} \mathcal{F}_i$ and $x = x_k$ for some $k \leq n$. Clearly, there is ℓ, $1 \leq \ell \leq n$, such that $\mathcal{F}_\ell \vee \dot{A}$, where $\dot{A}$ is the filter of all oversets of A. Thus $x_\ell \in \mathrm{cl}_q A$. Because the filters $\{\mathcal{F}_i\}$ are linked, it follows that $\dot{x}_\ell \geq \mathrm{cl}_{r_1 q}^n \dot{x}_k$. Thus $\dot{x}_\ell \to x_k$ in $(X,r_2 q)$, which implies $x_k \in \mathrm{cl}_{r_2 q}\{x_\ell\}$. Thus $x_k = x \in \mathrm{cl}_{r_2 q}^2 A$. □

PROPOSITION 4.4. If (X,q) is regular, then αq is regular.

PROOF. Let $\mathcal{F} \to x$ in $(X,\alpha q)$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be as in the preceding proof. Then $cl_{\alpha q}\mathcal{F} \geq cl_{\alpha q}(\bigcap_{i=1}^{n} \mathcal{F}_i) \geq cl_q^2(\bigcap_{n=1}^{n} \mathcal{F}_i) = \bigcap_{i=1}^{n} cl_q^2 \mathcal{F}_i$, where Lemma 4.3 is used for the second inequality. Since q is regular, $cl_q^2 \mathcal{F}_i \to x_i$ in (X,q), and so $cl_{\alpha q}\mathcal{F} \to x$ in $(X,\alpha q)$. □

A convergence space is defined in [5] to be *symmetric* it it is regular and $\mathcal{F} \to x$ wherever $\mathcal{F} \to y$ and $\dot{y} \to x$.

PROPOSITION 4.5. (X,q) is symmetric iff it is admissible and regular.

PROOF. A regular, admissible space is obviously symmetric. Conversely, let $\mathcal{F} \geq \bigcap_{i=1}^{n} \mathcal{F}_i$, where $\mathcal{F}_i \to x_i$ in (X,q) for $i = 1, \ldots, n$, and the $\{\mathcal{F}_i\}$ are linked. Because the $\{\mathcal{F}_i\}$ are linked and q is regular, $\dot{x}_i \to x_j$ for $i,j \in \{1, \ldots, n\}$. Since q is symmetric, $\mathcal{F}_i \to x_j$ for $i,j \in \{1, \ldots, n\}$. Thus $\bigcap_{i=1}^{n} \mathcal{F}_i \to x_j$ for $j = 1, \ldots, n$, and so q is admissible. □

For a convergence space (X,q), let rq denote the regular modification of q (i.e., the terminal element of the R-series of (X,q)). It is clear from the preceding propositions that αrq is the finest symmetric convergence structure on X coarser than q (i.e., the *symmetric modification* of q); in accordance with the notation of [5], we introduce the notation $\sigma q = \alpha rq$. A *symmetric series* $\{\sigma_\beta q\}$ for a convergence space (X,q) can be constructed as follows:

$$\sigma_0 q = q$$
$$\sigma_1 q = \alpha r_1 q$$
$$\vdots$$
$$\sigma_\beta q = \sigma(\sigma_{\beta-1} q), \text{ if } \beta \text{ is a non-limit ordinal}$$
$$\sigma_\beta(q) = \inf\{\sigma_\gamma(q) : \gamma < \beta\}, \text{ if } \beta \text{ is a limit ordinal.}$$

If (X,q) is an admissible convergence space, then we can identify q with the complete Cauchy structure $\mathfrak{C}^q = \{\mathcal{F} \in F(X) : \mathcal{F} \text{ q-converges}\}$. A comparison of the respective definitions leads to the following result.

PROPOSITION 4.6. If (X,q) is an admissible convergence space, then $\sigma_\beta q = r_\beta \mathfrak{C}^q$ for all ordinal numbers β. In particular, $\sigma q = r\mathfrak{C}^q$.

Starting with an arbitrary convergence space (X,q), we define the *T_2-modification* $(X_h,q_h) = (X_h,(\mathfrak{C}^{\alpha q})_h)$ and the *T_3-modification* $(X_\tau,(\mathfrak{C}^{\alpha q})_\tau)$. The latter notion was previously discussed in [7]; the former is apparently new. Given a continuous function $f : (X,q) \to (Y,p)$, let $f_h : (X_h,q_h) \to (Y_h,p_h)$ and $f_\tau : (X_\tau,q_\tau) \to (Y_\tau,p_\tau)$ be defined as in the paragraph preceding Proposition 3.4. The next proposition is clear.

PROPOSITION 4.7. If $f : (X,q) \to (Y,p)$ is a continuous function, then $f_\tau : (X_\tau,q_\tau) \to (Y_\tau,p_\tau)$ and $f_h : (X_h,q_h) \to (Y_h,p_h)$ are also continuous.

5. THE WYLER MODIFICATION AND COMPLETION.

Let $(X,\mathfrak{C})$ be a Cauchy space. If $\mathcal{F},\mathfrak{G}$ are filters in $\mathfrak{C}$ such that $\mathcal{F} \cap \mathfrak{G} \in \mathfrak{C}$, then $\mathcal{F}$ and $\mathfrak{G}$ are defined to be $\mathfrak{C}$-*equivalent*. Let $X^* = \{<\mathcal{F}> : \mathcal{F} \in \mathfrak{C}\}$ be the set of all $\mathfrak{C}$-equivalence classes, and let $j : X \to X^*$ be the canonical function $j(x) = <\dot{x}>$. Let $\mathfrak{C}^*$ be the Cauchy structure on X^* generated by $\{j(\mathcal{F}) \cap <\dot{\mathcal{F}}> : \mathcal{F} \in \mathfrak{C}\}$. We shall call $(X^*,\mathfrak{C}^*)$ the *Wyler modification* of $(X,\mathfrak{C})$.

PROPOSITION 5.1. (a) For any Cauchy space $(X,\mathfrak{C})$, $(X^*,\mathfrak{C}^*)$ is a complete Cauchy space, $j : (X,\mathfrak{C}) \to (X^*,\mathfrak{C}^*)$ is a map, and $j(X)$ is dense in X^*.

(b) If $(X,\mathfrak{C})$ is weakly regular, then $(X^*,\mathfrak{C}^*)$ is T_2.

PROOF. All parts of (a) follow directly from the definition of $\mathfrak{C}^*$. To prove (b), assume $\mathfrak{A} \cap \dot{a} \cap \dot{b} \in \mathfrak{C}^*$. $\mathfrak{A} \in \mathfrak{C}^*$ implies there are filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ in $\mathfrak{C}$ such that $j(\mathcal{F}_1) \cap <\dot{\mathcal{F}}_1>, \ldots, j(\mathcal{F}_n) \cap <\dot{\mathcal{F}}_n>$ are linked, $\mathfrak{A} \geq \cap \{j(\mathcal{F}_i) \cap <\dot{\mathcal{F}}_i> : i = 1, \ldots, n\}$, and $a, b \in \{<\mathcal{F}_i> : i = 1, \ldots, n\}$. It follows that $j^{-1}j(\mathcal{F}_1), \ldots, j^{-1}j(\mathcal{F}_n)$ are linked in X. But $j^{-1}j(\mathcal{F}_i) = \mathrm{wcl}_q \mathcal{F}_i$ for $i = 1, \ldots, n$; since $(X,\mathfrak{C})$ is weakly regular, it follows that $<\mathcal{F}_i> = \ldots = <\mathcal{F}_n> = a = b$, and hence $(X^*,\mathfrak{C}^*)$ is T_2. □

For a weakly regular Cauchy space $(X,\mathfrak{C})$, let $\psi : (X,\mathfrak{C}) \to (X^\wedge,\mathfrak{C}^\wedge)$ be the quotient map defined at the beginning of Section 3. Define $j^\wedge : (X^\wedge,\mathfrak{C}^\wedge) \to (X^*,\mathfrak{C}^*)$ by $j^\wedge([x]) = <\dot{x}>$. Note that $[x] = [y] \Leftrightarrow \dot{x} \cap \dot{y} \in \mathfrak{C} \Leftrightarrow <\dot{x}> = <\dot{y}>$; thus $j^\wedge$ is one-to-one. Also, $\mathcal{F} \in \mathfrak{C}$ implies $j^\wedge(\psi(\mathcal{F})) = j(\mathcal{F})$. A comparison of the definitions of $\mathfrak{C}^\wedge$ and $\mathfrak{C}^*$ leads immediately to the following result.

PROPOSITION 5.2. For a weakly regular Cauchy space $(X,\mathfrak{C})$, the following diagram is commutative, and $j^\wedge$ is a dense Cauchy embedding.

$$\begin{array}{ccc} (X,\mathfrak{C}) & \xrightarrow{\ j\ } & (X^*,\mathfrak{C}^*) \\ \downarrow \psi & \nearrow_{\, j^\wedge} & \\ (X^\wedge,\mathfrak{C}^\wedge) & & \end{array}$$

For a weakly regular Cauchy space $(X,\mathfrak{C})$, the quotient space $(X^\wedge,\mathfrak{C}^\wedge)$ is the T_2 modification $(X_h,\mathfrak{C}_h)$ of $(X,\mathfrak{C})$ defined in Section 3. Thus we obtain:

COROLLARY 5.3. If $(X,\mathfrak{C})$ is a weakly regular Cauchy space, then $((X^*,\mathfrak{C}^*),j^\wedge)$ is a T_2 completion of $(X_h,\mathfrak{C}_h)$.

In case $(X,\mathfrak{C})$ is T_2, $(X,\mathfrak{C})$ coincides with $(X_h,\mathfrak{C}_h)$ and $j^\wedge$ with j; in this case $((X^*,\mathfrak{C}^*), j)$ is called *the Wyler completion* of $(X,\mathfrak{C})$. The Wyler completion is the finest in standard form (See [2], [6]). Any T_2 completion $((X_1,\mathfrak{C}_1), h)$ of $(X,\mathfrak{C})$ such that any map f from $(X,\mathfrak{C})$ to a complete space $(Y,\mathfrak{D})$ can be lifted to map $f_1 : (X_1,\mathfrak{C}_1) \to (Y,\mathfrak{D})$ such that $f_1 \cdot h = f$ is then equivalent to the Wyler completion.

We next examine the extension properties of the Wyler modification. If $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ is a map, define $f^* : (X^*,\mathfrak{C}^*) \to (Y^*,\mathfrak{D}^*)$ by $f^*(<\mathcal{F}>_{\mathfrak{C}}) = <f(\mathcal{F})>_{\mathfrak{D}}$. If $<\mathcal{F}>_{\mathfrak{C}} = <\mathfrak{G}>_{\mathfrak{C}}$, then $\mathcal{F} \cap \mathfrak{G} \in \mathfrak{C}$ implies $f(\mathcal{F}) \cap f(\mathfrak{G}) \in \mathfrak{D}$; thus $<f(\mathcal{F})>_{\mathfrak{D}} = <f(\mathfrak{G})>_{\mathfrak{D}}$, and so f^* is well defined. Also, one easily verifies that if $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ and $g : (Y,\mathfrak{D}) \to (Z,\mathfrak{B})$ are maps, then $(g \cdot f)^* = g^* \cdot f^* : (X^*,\mathfrak{C}^*) \to (Z^*,\mathfrak{B}^*)$.

PROPOSITION 5.4. If $(X,\mathfrak{C})$ and $(Y,\mathfrak{D})$ are Cauchy spaces and $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ is a map, Cauchy quotient map, Cauchy retraction, or Cauchy initial map, then $f^* : (X^*,\mathfrak{C}^*) \to (Y^*,\mathfrak{D}^*)$ has the same property.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} (X,\mathfrak{C}) & \xrightarrow{f} & (Y,\mathfrak{D}) \\ \downarrow j_X & & \downarrow j_Y \\ (X^*,\mathfrak{C}^*) & \xrightarrow{f^*} & (Y^*,\mathfrak{D}^*). \end{array}$$

If $j_X(\mathcal{F}) \cap <\dot{\mathcal{F}}>_{\mathfrak{C}}$ is a generating element for $\mathfrak{C}^*$, then $f^*(j_X(\mathcal{F}) \cap <\dot{\mathcal{F}}>_{\mathfrak{C}}) = j_Y f(\mathcal{F}) \cap < f(\dot{\mathcal{F}})_{\mathfrak{D}} >$ is in $\mathfrak{D}^*$, and so f^* is a map. If f is a Cauchy quotient map and $<\mathfrak{G}>_{\mathfrak{D}} \in Y^*$, then $\mathfrak{G} \in \mathfrak{D}$ implies there are $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathfrak{C}$ such that $f(\mathcal{F}_1), \ldots, f(\mathcal{F}_n)$ are linked and $\mathfrak{G} \geq \cap\{f(\mathcal{F}_i) : i = 1, \ldots, n\}$. Then $f^*(<\mathcal{F}_i>_{\mathfrak{C}}) = <\mathfrak{G}>_{\mathfrak{D}}$ for $i = 1, \ldots, n$, and f^* is onto $\mathfrak{D}^*$. Also, if $\mathfrak{A} = j_Y(\mathfrak{G}) \cap <\dot{\mathfrak{G}}>_{\mathfrak{D}}$ is a generating filter in $\mathfrak{D}^*$, where $\mathfrak{G} \in \mathfrak{D}$, and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are as described above, then $\mathfrak{A} \geq f^*(\cap\{j_X(\mathcal{F}_i) \cap <\dot{\mathcal{F}}_i>_{\mathfrak{C}} : i = 1, \ldots, n\})$; from this it follows that f^* is a Cauchy quotient map.

Next assume that $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ is a Cauchy retraction; then there is a map $g : (Y,\mathfrak{D}) \to (X,\mathfrak{C})$ such that $f \circ g = id_Y$. By the remarks preceding the proposition, $(f \circ g)^* = (id_Y)^* = id_{Y^*} : (Y^*,\mathfrak{D}^*) \to (Y^*,\mathfrak{D}^*)$ and it follows that $f^* : (X^*,\mathfrak{C}^*) \to (Y^*,\mathfrak{D}^*)$ is a retraction map.

Finally, assume that $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ is a Cauchy initial map, and let $\mathfrak{A} = j_Y(\mathfrak{G}) \cap <\dot{\mathfrak{G}}>_{\mathfrak{D}}$ be a generating element of $\mathfrak{D}^*$, where $\mathfrak{G} \in \mathfrak{D}$. By a direct argument, one can show that $(f^*)^{-1}[(j_Y\mathfrak{G}) \cap <\dot{\mathfrak{G}}>_{\mathfrak{C}}] \geq j_X f^{-1}(\mathfrak{G}) \cap < f^{-1}(\dot{\mathfrak{G}}) >_{\mathfrak{C}} \in \mathfrak{C}^*$, and so f^* is also a Cauchy initial map. □

PROPOSITION 5.5. If $(X,\mathfrak{C})$ is a weakly regular Cauchy space, then $((X^*,\mathfrak{C}^*),j^\wedge)$ is the Wyler completion of $(X_h,\mathfrak{C}_h)$.

PROOF. Let $f : (X_h,\mathfrak{C}_h) \to (Y,\mathfrak{D})$ be a map, where $(Y,\mathfrak{D})$ is complete and T_2, and let $\psi : (X,\mathfrak{C}) \to (X_h,\mathfrak{C}_h)$ be the canonical quotient map.

$$\begin{array}{ccc} (X,\mathfrak{C}) & \xrightarrow{f} & (X^*,\mathfrak{C}^*) \\ \psi \downarrow & \nearrow j^\wedge & \downarrow (f \circ \psi)^* \\ (X_h,\mathfrak{C}_h) & \xrightarrow[f]{} & (Y,\mathfrak{D}) \end{array}$$

By Corollary 5.3, $((X^*,\mathfrak{C}^*),j^\wedge)$ is a T_2 completion of $(X_h,\mathfrak{C}_h)$, and by Proposition 5.4, the map $f \circ \psi$ has a continuous extension $(f \circ \psi)^* : (X^*,\mathfrak{C}^*) \to (Y^*,\mathfrak{D}^*) = (Y,\mathfrak{D})$. Since the above diagram is commutative, the completion $((X^*,\mathfrak{C}^*),j^\wedge)$ has the lifting property which characterizes the Wyler competition of $(X_h,\mathfrak{C}_h)$. □

The next proposition follows from Propositions 1.4, 2.2, 2.3, and 5.4.

PROPOSITION 5.6. Let $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ be a map. Then $f^* : (X^*,w_\beta\mathfrak{C}^*) \to (Y^*,w_\beta\mathfrak{D}^*)$ and $f^* : (X^*,r_\beta\mathfrak{C}^*) \to (Y^*,r_\beta\mathfrak{D}^*)$ are maps for all ordinals β. If f is a Cauchy retraction or Cauchy initial map, then both of the maps labeled f^* have the same property.

Let $f : (X,\mathfrak{C}) \to (Y,\mathfrak{D})$ be a map, and consider the diagram (D^*):

$$\begin{array}{ccccccc}
(X,\mathfrak{C}) & \xrightarrow{j_X} & (X^*,\mathfrak{C}^*) & \longrightarrow & (X^*_h,\mathfrak{C}^*_h) & \longrightarrow & (X^*_\tau,\mathfrak{C}^*_\tau) \\
f\downarrow & & f^*\downarrow & & (f^*)_h\downarrow & & (f^*)\downarrow \\
(Y,\mathfrak{D}) & \xrightarrow{j_Y} & (Y^*,\mathfrak{D}^*) & \longrightarrow & (Y^*_h,\mathfrak{D}^*_h) & \longrightarrow & (Y^*_\tau,\mathfrak{D}^*_\tau)
\end{array}$$

where the unlabeled horizontal arrows are the canonical quotient maps. The next proposition is an immediate consequence of Proposition 3.5 and 5.4.

PROPOSITION 5.7. In diagram (D^*), if f is a map (respectively, Cauchy retraction, Cauchy initial map) then each vertical arrow in the diagram is a map (respectively, Cauchy retraction, Cauchy initial map).

6. T_3 COMPLETIONS.

In [6], a Cauchy space which has a T_3 completion is defined to be a $\mathcal{C}_3$ Cauchy space. In this section a characterization of C_3 spaces is given which makes use of the R-series along with a set function Σ defined originally in [4]. The main theorem (Theorem 6.4) is due to G. D. Richardson.

Throughout this section, $(X,\mathfrak{C})$ is assumed to be a T_3 Cauchy space. The Wyler completion of $(X,\mathfrak{C})$ will be denoted by $((X^*,\mathfrak{C}^*),j)$. Let p be the convergence structure on X^* compatible with $\mathfrak{C}^*$, and let p_α denote the convergence structure on X^* compatible with $r_\alpha p$ (see Section 4). As usual, N denotes the set of natural numbers; also recall that $\dot{A}$ denotes the filter consisting of all oversets of A.

We next construct a family of set functions Σ^n_α which, for each $n \in N$ and each ordinal number α, map subsets of X into subsets of X^*. If $\mathcal{F}$ is a filter on X, then $\Sigma^n_\alpha(\mathcal{F})$ is defined to be the filter on X^* generated by $\{\Sigma^n_\alpha(F) : F \in \mathcal{F}\}$. For any subset A of X and $n \in N$, we define:

$$\Sigma^1_0(A) = \{ \langle\mathcal{F}\rangle \in X^* : \text{there is } \mathfrak{G} \in \langle\mathcal{F}\rangle \text{ such that } j(\mathfrak{G}) \vee j(\dot{A})\}$$

$$\vdots$$

$$\Sigma^n_0(A) = \{ \langle\mathcal{F}\rangle \in X^* : \text{there is } \mathfrak{G} \in \langle\mathcal{F}\rangle \text{ such that } j(\mathfrak{G}) \vee \Sigma^{n-1}_0(\dot{A})\}.$$

If α is an ordinal number, $A \subseteq X$, and $n \in N$, we define:

$$\Sigma^1(A) = \{ \langle\mathcal{F}\rangle \in X^* : \text{there is } \beta < \alpha,\ k \in N, \text{ and } \mathfrak{G} \in \langle\mathcal{F}\rangle \text{ such that } \Sigma^k_\beta(\mathfrak{G}) \vee j(\dot{A})\}$$

$$\vdots$$

$$\Sigma^n_\alpha(A) = \{ \langle\mathcal{F}\rangle \in X^* : \text{there is } \beta < \alpha,\ k \in N, \text{ and } \mathfrak{G} \in \langle\mathcal{F}\rangle \text{ such that } \Sigma^k_\beta(\mathfrak{G}) \vee \Sigma^{n-1}_\alpha(\dot{A})\}.$$

PROPOSITION 6.1. For all $n \in N$ and $A \subseteq X$, $\Sigma^n_0(A) = cl^n_p j(A)$.

PROOF. Assume that the equality holds for n. If $\langle\mathcal{F}\rangle \in \Sigma^{n+1}_0(A)$, then there is $\mathfrak{G} \in \langle\mathcal{F}\rangle$ such that $j(\mathfrak{G}) \vee \Sigma^n_0(A)$ exists. Using the induction assumption and the fact that $j(\mathfrak{G})$ p-converges to $\langle\mathcal{F}\rangle$, it follows that $\langle\mathcal{F}\rangle \in cl^{n+1}_p j(A)$. Conversely, if $\langle\mathcal{F}\rangle \in cl^{n+1}_p j(A)$, then there is a filter Φ p-converging to $\langle\mathcal{F}\rangle$ such that $cl^n_p j(A) \in \Phi$.

By construction of $\mathfrak{C}^*$, $\Phi \geq j(\mathcal{G})$ for some $\mathcal{G} \in <\mathfrak{C}>$, and by the induction assumption, $j(\mathcal{G}) \vee \Sigma_p^n(\dot{A})$ exists. Thus $<\mathcal{F}> \in \Sigma_p^{n+1}(A)$. For n = 1, the preceding argument can be applied if we define $\Sigma_0^0(A) = cl_p^0 j(A) = j(A)$. ∎

PROPOSITION 6.2. For all ordinal numbers α, for all $n \in N$, and for all subsets A of X, $\Sigma_\alpha^n(A) = cl_{p_\alpha}^n j(A)$.

PROOF. Consider all pairs P of the form (α,n), where α is an ordinal number and $n \in N$; let P be ordered as follows: $(\beta,m) < (\alpha,n)$ if $\beta < \alpha$ or $\beta = \alpha$ and $m < n$. Since P is obviously well-ordered, the proof will proceed by induction.

Assume that the above equality holds for $(\beta,m) < (\alpha,n)$; in view of the preceding proposition, we may assume $\alpha \geq 1$. If $n > 1$, then the induction assumption states that $\Sigma_\alpha^{n-1}(A) = cl_{p_\alpha}^{n-1} j(A)$ and $\Sigma_\beta^k(A) = cl_{p_\beta}^k j(A)$ for all $\beta < \alpha$ and $k \in N$; using these equalities, the argument used in Proposition 6.1 can be repeated to establish the desired result in this case.

Finally, assume n = 1 and let $<\mathcal{F}> \in \Sigma_\alpha^1(A)$. Then there is $\mathcal{G} \in <\mathcal{F}>$, $k \in N$, and $\beta < \alpha$ such that $\Sigma_\beta^k(\mathcal{G}) \vee j(\dot{A})$. Since $\Sigma_\beta^k(\mathcal{G})$ p_α-converges to $<\mathcal{F}>$, it follows that $<\mathcal{F}> \in cl_{p_\alpha} j(A)$. Conversely, if $<\mathcal{F}> \in cl_{p_\alpha} j(A)$, then some ultrafilter Φ on X* containing $j(A)$ p_α-converges to $<\mathcal{F}>$. By the definition of p and $p_\alpha = r_\alpha p$ (see Section 4), there is $\mathcal{G} \in <\mathcal{F}>$, $\beta < \alpha$, and $k \in N$ such that $\Phi \geq cl_{p_\beta}^k j(\mathcal{G})$. Employing the induction hypothesis, we have $\Sigma_\beta^k(\mathcal{G}) \vee j(\dot{A})$, and hence $<\mathcal{F}> \in \Sigma_\alpha^1(A)$. This completes the proof. ∎

Now let $(X,\mathfrak{C})$ be a T_3 Cauchy space, and consider the following two properties:

(P_1) If $\mathcal{F}, \mathcal{G} \in \mathfrak{C}$ and, for each ordinal α and $n \in N$, $\Sigma_\alpha^n(\mathcal{F}) \vee \Sigma_\alpha^n(\mathcal{G})$, then $\mathcal{F} \cap \mathcal{G} \in \mathfrak{C}$

(P_2) For each ordinal α, $n \in N$, and $\mathcal{F} \in \mathfrak{C}$, $j^{-1}(\Sigma_\alpha^n(\mathcal{F})) \in \mathfrak{C}$.

PROPOSITION 6.3. If $(X,\mathfrak{C})$ is a T_3 Cauchy space which satisfies condition (P_1), then for each ordinal number α, $r_\alpha \mathfrak{C}^*$ is T_2, and hence compatible with $p_\alpha = r_\alpha p$.

PROOF. The fact that p_α is T_2 is an immediate consequence of Proposition 6.2 and property (P_1). Since a T_2 convergence structure is admissible and hence symmetric, $\{p_\alpha\}$ coincides with the symmetric series $\{\sigma_\alpha p\}$, which in turn coincides with the Cauchy R-series $\{r_\alpha \mathfrak{C}^*\}$ by Proposition 4.6. ∎

THEOREM 6.4. The following statements about a T_3 Cauchy space $(X,\mathfrak{C})$ are equivalent:

(1) $(X,\mathfrak{C})$ is a C_3 Cauchy space.

(2) $(X,\mathfrak{C})$ satisfies conditions (P_1) and (P_2).

(3) $((X^*,r\mathfrak{C}^*),j)$ is a T_3 completion of $(X,\mathfrak{C})$.

PROOF. The equivalence of (1) and (3) are well known (see [2] or [6]). If $(X,\mathfrak{C})$ satisfies the two conditions, then the set $\mathfrak{D} = \{\Phi \in F(X^*)$: there is ordinal α, $n \in N$, and $\mathcal{F} \in \mathfrak{C}$ such that $\Phi \geq \Sigma_\alpha^n(\mathcal{F})\}$ is easily seen to be a complete T_3 Cauchy structure on X*; using Propositions 6.2 and 6.3, it follows easily that $\mathfrak{D} = r\mathfrak{C}^*$. Condition ($P_2$) guarantees that $j : (X,\mathfrak{C}) \to (X^*,r\mathfrak{C}^*)$ is a Cauchy embedding. Thus (2) ⇒ (3). Conversely, if (3) is assumed, then (P_1) follows from Proposition 6.2

and the fact that $r\mathfrak{C}^*$ is T_2, while (P_2) follows because the $j : (X,\mathfrak{C}) \to (X^*,r\mathfrak{C}^*)$ is a Cauchy embedding. $\square$

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