Internat. J. Math. & Math. Sci.
Vol. 7 No. 1 (1984) 89-95

COMMON FIXED POINTS FOR FAMILY OF MAPPINGS

D.E. ANDERSON and K.L. SINGH

Department of Mathematical Sciences
University of Minnesota, Duluth
Duluth, Minnesota 55812 U.S.A.

J.H.M. WHITFIELD

Department of Mathematical Sciences
Lakehead University
Thunder Bay, Ontario P7B 5E1 Canada

(Received March 14, 1983 and in revised form August 16, 1983)

ABSTRACT. The main aim of the present paper is to prove the existence of common fixed points for mappings which are not necessarily continuous. Our results, which are primarly motivated by investigation of Husain and Sehgal (Bull. Austral. Math. Soc. 13 (1975), 261-267), generalize the results of Husain and Sehgal, Sehgal, Kannan, Reich, Hardy and Rogers, and others.

KEY WORDS AND PHRASES. Fixed points, common fixed points, upper semicontinuous mapping, asymptotically regular mapping.
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 47H10,52H25.

1. INTRODUCTION.

In [1], Iseki gave some sufficient conditions for the existence of common fixed points for a sequence of self-mappings of a complete metric space. The results of [1] were extended further in Husain and Sehgal [2] and Singh and Meade [3]. For a single mapping, Theorem 1 [2] was further extended by Husain and Sehgal [4]; and for a pair of mappings, it was extended by Kasahara [5]. The purpose of this paper is to obtain some common fixed-point theorems for a family of mappings under conditions that are considerably weaker than considered in [2]. The results herein improve the results in [1], [2], [3], [5], [6], [7], [8], and several known results.

2. PRELIMINARIES AND BASIC DEFINITIONS.

Throughout this paper, let (X,d) be a complete metric space and let R^+ be the nonnegative reals. Let ψ denote a family of mappings such that each $\phi \in \psi$, $\phi:(R^+)^5 \to R^+$, and ϕ is upper semicontinuous and nondecreasing in each coordinate variable. Also, let

$\gamma(t) = \phi\{t,t,a_1t,a_2t,t\}$, where γ is a function $\gamma:R^+ \to R^+$, where $a_1 + a_2 = 2$, $a_i \in \{0,1,2\}$. We will need the following:

LEMMA 2.1 [Lemma 1 (3)]. For any $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$. The following is proved in [2]:

THEOREM 2.1. Let f,g be self-mappings of a complete metric space X. Suppose there exists a $\phi \in \psi$ such that for each $x,y \in X$,

$$d(fx,gy) \le \phi(d(x,fx),d(y,gy),d(x,gy),d(y,fx),d(x,y)), \tag{2.1}$$

where ϕ satisfies the condition: for any $t > 0$,

$$\phi(t,t,a_1t,a_2t,t) < t, \quad a_i \in \{0,1,2\} \text{ with } a_1 + a_2 = 2. \tag{2.2}$$

Then, there exists a $u \in X$ such that

(a) $fu = gu = u$ and

(b) u is the unique fixed point of each f and g.

REMARK 2.1. The condition ϕ to be continuous was weakened by the upper semicontinuity of ϕ in [3].

The following example shows that if we replace (2.2) in Theorem 2.1 by

$$\phi(t,t,t,t,t) < t, \tag{2.2)'$$

then the conclusion of Theorem 2.1 is no longer true.

EXAMPLE 2.1. Let $X = \{1,2,3,4\}$, $d(1,2) = d(3,4) = 2$, and $d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1$. Define $f:X \to X$ by $f1 = f4 = 2$, $f2 = f3 = 1$, and define $g:X \to X$ by $g1 = g3 = 4$, $g2 = g4 = 3$. Then, $d(fx,gy) \le 1/2 \max\{d(x,fx),d(y,gy),d(x,gy),d(y,fx),d(x,y)\}$. Taking $k = 3/4$, we see that f,g satisfy Ciric's condition [9], but are without fixed points.

REMARK 2.2. The above example answers in the negative a question raised by Ciric [9].

The following example shows that the condition $\phi(t,t,t,t,t) < t$ is necessary for the existence of a fixed point.

EXAMPLE 2.2. Let $X = [1,\infty)$ with the usual metric. Define $T:X \to X$ by $T(x) = x + \frac{1}{x}$ and $\phi(t,t,t,t,t) = t + \frac{1}{t}$. Clearly, ϕ is continuous (and, hence, upper semicontinuous) and nondecreasing. Moreover, T satisfies the condition $d(Tx,Ty) \le \phi(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))$. Let $m = \min\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$. Without loss of generality, we may assume $x \le y$ in the contractive definition. If x and y satisfy $y \ge 1 + \frac{1}{x}$, then $m = \min\{d(y,Tx),d(y,Ty)\}$, and $d(Tx,Ty) < \phi(m,m,m,m,m)$. If $y < x + \frac{1}{x}$, then $m = \min\{d(y,Tx),d(x,y)\}$, and again $d(Tx,Ty) < \phi(m,m,m,m,m)$. However, $\phi(t,t,t,t,t) \not< t$ and T is without a fixed point. It is clear from Example 2.1 that in order to ensure the existence of a common fixed point for control function ϕ of (2.2)', we must impose some additional condition. One such possible condition is the following:

DEFINITION 2.1. A pair $\{f,g\}$ of mappings is <u>asymptotically</u> <u>regular</u> at

x_0 if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, where $x_1 = f(x_0)$, $x_2 = g(x_1)$, ..., $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1})$.

Other necessary conditions for ensuring the existence of common fixed points for the pair of mappings f, g are given by Kasahara [5], Park [10], Park and Rhoades [11], and Rhoades [12]. In all of these papers, the commutativity of f and g is assumed.

3. MAIN RESULTS.

THEOREM 3.1. Let f and g be two self-mappings of a complete metric space X. Suppose there exists a $\phi \in \Psi$ such that for each $x, y \in X$, $d(fx,gy) \le \phi(d(x,fx),d(y,gy),d(x,gy),d(y,fx),d(x,y))$, where for any $t > 0$, ϕ satisfies (2.2)'. Suppose that the pair $\{f,g\}$ is asymptotically regular at $x_0 \in X$; then, there exists a $u \in X$ such that

(a) $gu = fu = u$ and

(b) u is the unique fixed point of f and g.

PROOF. Define the sequence $\{x_n\}$, respectively, by $x_1 = f(x_0)$, $x_2 = g(x_1)$, ..., $x_{2n+1} = f(x_{2n})$, and $x_{2n+2} = g(x_{2n+1})$. Let $d_n = d(x_n, x_{n+1})$. By the asymptotic regularity of the pair $\{f,g\}$, it follows that

$$d_n = d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}$$

We show that $\{x_n\}$ is a Cauchy sequence. It is enough to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there is an $\epsilon > 0$ such that for each integer $2k$, $k \in I^+$, there exist integers $2n(k)$ and $2m(k)$ with $2k \le 2n(k) < 2m(k)$ such that

$$d(x_{2n(k)}, x_{2m(k)}) > \epsilon. \tag{3.2}$$

Let, for each positive integer $2k$, $k \in I^+$, $2m(k)$ be the least integer exceeding $2n(k)$ satisfying (2.2); that is,

$$d(x_{2n(k)}, x_{2m(k)-2}) \le \epsilon \quad \text{and} \quad d(x_{2n(k)}, x_{2m(k)}) > \epsilon. \tag{3.3}$$

Then, for each integer $2k$, $k \in I^+$,

$\epsilon < d(x_{2n(k)}, x_{2m(k)}) \le d(x_{2n(k)}, x_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}$. Therefore, by (3.1) and (3.2), we obtain

$$d(x_{2n(k)}, x_{2m(k)}) \to \epsilon \quad \text{as} \quad k \to \infty. \tag{3.4}$$

It follows from the triangular inequality that

$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \le d_{2m(k)-1}$ and

$|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \le d_{2m(k)-1} + d_{2n(k)}$. Hence, by (3.4), as $k \to \infty$, $d(x_{2n(k)}, x_{2m(k)-1}) \to \epsilon$ and $d(x_{2n(k)+1}, x_{2m(k)-1}) \to \epsilon$. Now, let $p(2k) = d(x_{2n(k)}, x_{2m(k)})$, $q(2k) = d(x_{2n(k)}, x_{2m(k)-1})$, and $r(2k) = d(x_{2n(k)+1}, x_{2m(k)-1})$. Then, $p(2k) \le d_{2n(k)} + d(fx_{2n(k)}, gx_{2m(k)-1})$ $\le d_{2n(k)} + \phi(d_{2n(k)}, d_{2m(k)-1}, p(2k), r(2k), q(2k))$. Since ϕ is upper semicontinuous, as $k \to \infty$, it follows

$\epsilon \le \phi(0,0,\epsilon,\epsilon,\epsilon) \le \phi(\epsilon,\epsilon,\epsilon,\epsilon,\epsilon) < \epsilon$, a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence; and, hence, by completeness, there exists $u \in X$ such that $x_n \to u$. We show that u is a common fixed point of f and g. Now, since $x_{2n} = g(x_{2n-1})$, $d(fu,x_{2n}) \le \phi(d(u,fu),d_{2n-1},d(u,x_{2n}),d(x_{2n-1},fu),d(x_{2n-1},u))$. Taking the limit as $n \to \infty$, we obtain $d(fu,u) \le \phi(d(u,fu),0,0,d(u,fu),0) < d(u,fu)$, a contradiction, unless $u = fu$. A similar argument applied to $d(x_{2n+1},gu)$ yields $gu = u$. To show the uniqueness, suppose there is a $v \ne u$ for which $gv = v$. Let $r = d(u,v) > 0$. Then, $r = d(fu,gu) \le \phi(0,0,r,r,r) < r$, a contradiction. Thus, $u = v$.

LEMMA 3.1. Mappings satisfying conditions of Theorem 2.1 are asymptotically regular.

PROOF. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows: Let $x_1 = f(x_0)$, $x_2 = g(x_1)$; and, inductively, for each $n \in I^+$ (positive integers), let $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1})$. We claim that $d(x_1,x_2) \le d(x_0,x_1)$. Suppose it isn't. Then, $d(x_0,x_1) < d(x_1,x_2)$. Let $r = d(x_1,x_2)$. Then, $r = d(x_1,x_2) = d(fx_0,gx_1)$
$\le \phi(d(x_0,x_1),d(x_1,x_2),d(x_0,x_2),d(x_1,x_1),d(x_0,x_1))$
$\le \phi(d(x_1,x_2),d(x_1,x_2),2d(x_1,x_2),0,d(x_1,x_2)) \le \phi(r,r,2r,0,r) < r$, a contradiction. Therefore,
$d(x_1,x_2) \le \phi(d(x_0,x_1),d(x_0,x_1),2d(x_0,x_1),0,d(x_0,x_1)) = \gamma(d(x_0,x_1))$.
Similarly, $d(x_2,x_3) \le \gamma(d(x_1,x_2)) \le \gamma^2(d(x_0,x_1))$; and in general, $d(x_n,x_{n+1}) \le \gamma^n(d(x_0,x_1))$. Since $\lim_{n\to\infty} \gamma^n(t) = 0$ for $t > 0$ (Lemma 2.1), it follows that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$; i.e., the pair $\{f,g\}$ is asymptotically regular.

COROLLARY 3.1. Let X be a complete metric space, and let $f,g:X \to X$ be two mappings satisfying
$d(fx,gy) \le k \max\{d(x,y),d(x,fx),d(y,gy),d(x,gy),d(y,fx)\}$ for all $x,y \in X$ and for some k, $0 \le k < 1$. Suppose that the pair $\{f,g\}$ is asymptotically regular at x_0. Then, f and g have a common fixed point.

PROOF. Define $\phi:(R^+)^5 \to R^+$ as follows:
$\phi(t_1,t_2,t_3,t_4,t_5) = k \max\{t_1,t_2,t_3,t_4,t_5\}$. Then, $\phi \in \psi$ and ϕ,f,g satisfy the hypothesis of Theorem 3.1.

LEMMA 3.2. Let f be a self-mapping of a complete metric space X satisfying (2.1) (with $f = g$) and (2.2)'. If $\sup\{d(x_0,f^n x_0):n \in \omega - \{0\}\} < \infty$ for some $x_0 \in X$, then f is asymptotically regular, where ω (omega) is the set of all nonnegative integers.

PROOF. Let $\delta_n = \sup\{d(fx_n,fx_{n+1})\}$. By hypothesis, δ_n is finite for each $n \in \omega$. Since $\delta_{n+1} \le \delta_n$ for any $n \in \omega$, $\{\delta_n\}_{n\in\omega}$ converges to some $\delta \ge 0$. We claim that $\delta = 0$. Suppose it isn't; that is, $\delta > 0$.
$d(fx_n,fx_{n+1}) \le \phi(d(x_n,x_{n+1}),d(x_n,fx_n),d(x_{n+1},fx_{n+1}),d(x_n,fx_{n+1}),d(x_{n+1},fx_n))$

$\leq \phi(\delta_{n-1},\delta_{n-1},\delta_{n-1},\delta_{n-1},\delta_{n-1})$, and, hence, we have $\delta_n \leq \phi(\delta_{n-1},\delta_{n-1},\delta_{n-1},\delta_{n-1},\delta_{n-1})$ for all $n \in \omega$. Using upper semicontinuity of ϕ, it follows that $\delta \leq \phi(\delta,\delta,\delta,\delta,\delta) < \delta$, a contradiction. Thus, $\lim_{n\to\infty} d(fx_n,fx_{n+1}) = 0$; i.e., f is asymptotically regular.

REMARK 3.1. Using Lemmas 3.1 and 3.2, we get results of Husain and Sehgal [2], Singh and Meade [3], and Husain and Sehgal [4], respectively, as corollaries of our theorem 2.1.

REMARK 3.2. Special cases of mappings satisfying conditions (2.1) and (2.2)' have been discussed by Rakotch [13], Boyd and Wong [14], Bianchini [15], Kannan [16 and 6], Reich [7 and 8], Rus [17], Sehgal [18], Rhoades [19], Chatterjea [20], Hardy and Rogers [21], Ciric [22 and 9], Massa [23], Zamfirescu [24], and others. Theorem 3.1 is a generalization of results of Massa, Ciric, Kannan, Reich, Rhoades, Bianchini, Hardy and Rogers, Husain and Sehgal, Singh and Meade, Kurepa, Rakotch, Boyd and Wong, Rus, Zamfirescu, and others.

The following example shows that a mapping satisfying condition (2.1) and (2.2)' $(f = g)$ need not satisfy any condition considered by the above authors.

EXAMPLE 3.1. Let $X = [0,\infty)$ with the usual metric. Define $T:X \to X$ by $T(x) = \frac{x}{1 + x}$ and $\phi:(R^+)^5 \to R$ by $\phi(t_1,t_2,t_3,t_4,t_5) = \frac{t}{1 + t}$, where $t = \max\{t_1,t_2,t_3,t_4,t_5\}$. Then, T satisfies our condition with 0 only as a fixed point. Indeed, for any $x,y \in X$, $d(Tx,Ty) \leq \frac{|x - y|}{1 + x + y + xy} \leq \frac{|x - y|}{1 + |x - y|} \leq \phi(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))$. However, T does not satisfy any other condition. Indeed, for $y = 0$ and any $x \in X$, we have $d(Tx,To) = \frac{x}{1 + x} \leq k \max\{0,\frac{x^2}{1 + x},x,\frac{x^2}{1 + x},x\}$. For any $x > 0$, $\frac{x^2}{1 + x} \leq x$, and $\frac{x}{1 + x} \leq x$, we have $\frac{x}{1 + x} \leq kx$; that is, $\frac{1}{1 + x} \leq k$.

4. SEQUENCE OF MAPPINGS.

In this section, we prove common fixed-point theorems for a sequence of mappings. These results include results of Husain and Sehgal [2], Iseki [1], and others as a particular case.

THEOREM 4.1. Let g and a sequence $\{f_n\}$ be self-mappings of X such that $f_n \to g$ uniformly. Suppose for each $n \geq 1$, f_n has a fixed point x_n and g satisfies the condition: for all $x,y \in X$,

$$d(gx,gy) \leq \phi(d(x,gx),d(y,gy),d(x,gy),d(y,gx),d(x,y)) \tag{4.1}$$

for some $\phi \in \psi$ satisfying (2.2)'. If x_0 is the fixed point of g and $\sup d(x_n,x_0) < \infty$, then $x_n \to x_0$.

PROOF. Note that x_0 is a unique fixed point of g. Since $f_nx_n = x_n$ and $f_n \to g$ uniformly, it follows that $d(f_nx_n,gx_n) = d(x_n,gx_n) \to 0$ as $n \to \infty$. Let $r = \lim\sup d(x_n,x_0)$. Then since $d(gx_n,x_0) \leq d(gx_n,x_n) + d(x_n,x_0)$, it follows by (4.1) that

$d(x_n,x_0) \le d(x_n,gx_n) + d(gx_n,gx_0)$
$\le d(x_n,gx_n) + \phi(d(x_n,gx_n),d(x_0,gx_0),d(x_n,gx_0),d(x_0,gx_n),d(x_n,x_0))$
$\le d(x_n,gx_n) + \phi(d(x_n,gx_n),0,d(x_n,x_0),d(x_0,x_n) + d(x_n,gx_n),d(x_n,x_0))$. This implies that $r \le \phi(0,0,r,r,r) < r$; hence, $r = 0$ and, consequently, $x_n \to x_0$.

REMARK 4.1. A special case of Theorem 4.1 is Theorem 3 [2]. If in Theorem 4.1 condition (4.1) is replaced by
$d(gx,gy) \le \alpha[d(x,gx) + d(y,gy)] + \beta[d(x,gy) + d(y,gx)] + \gamma d(x,y)$, where α,β,γ are some nonnegative reals with $2\alpha + 2\beta + \gamma < 1$, then it is easy to show [1] that $\sup d(x_n,x_0) < \infty$. Thus, Theorem 4.1 also improves Theorem 2 in [1].

THEOREM 4.2. Let $\{f_n\}$ be a sequence of self-mappings of X satisfying the condition: there is a $\phi \in \psi$ satisfying (2.2)' such that for all $x,y \in X$ and $n \ge 1$,
$d(f_nx,f_ny) \le \phi(d(x,f_nx),d(y,f_ny),d(x,f_ny),d(y,f_nx),d(x,y))$ and each mapping is asymptotically regular. Let x_n be the fixed points of f_n (given by Theorem 3.1), and let $g:X \to X$ such that $f_n \to g$. If x_0 is any cluster point of the sequence $\{x_n\}$, then $gx_0 = x_0$.

PROOF. Let $x_{n_i} \to x_0$. Since $f_n \to g$, $d(f_{n_i}x_0,gx_0) \to 0$. Furthermore, for each $i \ge 1$,
$d(x_{n_i},f_{n_i}x_0) = \alpha_i \le d(x_{n_i},x_0) + d(x_0,gx_0) + d(gx_0,f_{n_i}x_0) \to d(x_0,gx_0)$ and
$d(x_0,f_{n_i}x_0) = \beta_i \le d(x_0,gx_0) + d(gx_0,f_{n_i}x_0) \to d(x_0,gx_0)$. Thus, for each $i \ge 1$, $d(x_0,gx_0) \le d(x_0,x_{n_i}) + d(f_{n_i}x_{n_i},f_{n_i}x_0) + d(f_{n_i}x_0,gx_0)$
$\le d(x_0,x_{n_i}) + \phi(0,\beta_i,\alpha_i,d(x_{n_i},x_0),d(x_{n_i},x_0)) + d(f_{n_i}x_0,gx_0)$. Therefore, as $i \to \infty$, $d(x_0,gx_0) \le \phi(0,d(x_0,gx_0),d(x_0,gx_0),0,0)$, which implies $gx_0 = x_0$.

REMARK 4.2. A special case of Theorem 4.2 is Theorem 4 [2].

REMARK 4.3. Various kinds of contractive-type mappings which are special cases of our mappings may be found in [19].

ACKNOWLEDGEMENTS. The authors appreciate the referee's valuable suggestions. J. H. M. Whitfield's research was supported in part by a grant from NSERC (Canada).

REFERENCES

1. ISEKI, K. On Common Fixed Points of Mappings, Bull. Australian Math. Soc. 10 (1974), 365-370.

2. HUSAIN, S. A. and SEHGAL, V. M. On Common Fixed Points for a Family of Mappings, Bull. Australian Math. Soc. 13 (1975), 261-267.

3. SINGH, S. P. and MEADE, B. On Common Fixed Point Theorems, Bull. Australian Math. Soc. 16 (1977), 49-53.

4. HUSAIN, S. A. and SEHGAL, V. M. A Fixed Point Theorem With a Functional Inequality, Publ. L' Inst. Math. Beograde 21 (35)(1977), 89-91.

5. KASAHARA, S. On Some Recent Results in Fixed Points, Math. Sem. Notes 6 (1978), 373-382.

6. KANNAN, R. Some Results on Fixed Points II, Amer. Math. Monthly 76 (1969), 405-408.

7. REICH, S. Kannan's Fixed Point Theorem, Bull. U. M. I. 4 (1971), 1-11.

8. REICH, S. Some Remarks Concerning Contraction Mappings, Canad. Math. Bull. 14 (1971), 121-124.

9. CIRIC, L. B. On Common Fixed Points in Uniform Spaces, Publ. L' Inst. Math. Beograde 24 (38)(1978, 39-43.

10. PARK, S. On Generalized Contractive Type Conditions, J. Korean Math. Soc. 17 (1980), 131-140.

11. PARK, S. and RHOADES, B. E. Extensions of Some Fixed Point Theorems of Hegedüs and Kasahara, Math. Sem. Notes 9 (1981), 113-117.

12. RHOADES, B. E. Contractive Definitions Revisited, preprint.

13. RAKOTCH, E. A Note on Contractive Mappings, Proc. Amer. Math. Soc. 13 (1962), 458-465.

14. BOYD, D. W. and WONG, J. S. W. On Nonlinear Contractions, Proc. Amer. Math. Soc. 20 (1969), 458-464.

15. BIANCHINI, R. M. T. Su Un Problema Di S. Reich Riguardante La Teoria Dei Punti Fissi, Bull. U. M. I. 5 (1972), 103-108.

16. KANNAN, R. Some Results on Fixed Points, Bull Calcutta Math. Soc. 60 (1968), 71-76.

17. RUS, I. A. On Common Fixed Points, Studia Univ. Babes-Bolyai Series Math.-Mech. 13 (1973), 31-33.

18. SEHGAL, V. M. Some Fixed and Common Fixed Point Theorems in Metric Spaces, Canad. Math. Bull. 17 (1974), 257-259.

19. RHOADES, B. E. A Comparison of Various Definitions of Contractive Mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290.

20. CHATTERJEA, S. K. Fixed Point Theorems, C. R. Acad. Bulgar. Sci. 25 (1972), 727-730.

21. HARDY, G. E. and ROGERS, T. D. A Generalization of a Fixed Point Theorem of Reich, Canad. Math. Bull. 16 (1973), 201-206.

22. CIRIC, L. B. A Generalization of Banach's Contraction Principle, Proc. Amer. Math. Soc. 45 (1974), 267-273.

23. MASSA, S. Generalized Contractions in Metric Spaces, Bull. U. M. I. 4 (1974), 689-694.

24. ZAMFIRESCU, T. A Theorem on Fixed Points, Acad. Nat. Dei Lincei. Rend. Cl. Sci. Fis. Mat. Nat. 52 (1972), 832-834.