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CONTACT CO-ISOTROPIC CR SUBMANIFOLDS OF A PSEUDO-SASAKIAN MANIFOLD

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ABSTRACT. It is proved that any co-isotropic submanifold M of a pseudo-Sasakian manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ is a CR submanifold (such submanifolds are called CICR submanifolds) with involutive vertical distribution $D^{\perp}$. The leaves $M^{\perp}$ of $D^{\perp}$ are isotropic and M is $D^{\perp}$-totally geodesic. If M is foliate, then M is almost minimal. If M is Ricci $D^{\perp}$-exterior recurrent, then M receives two contact Lagrangian foliations. The necessary and sufficient conditions for M to be totally minimal is that M be contact $D^{\perp}$-exterior recurrent.

KEY WORDS AND PHRASES. *CR submanifold, CICR submanifold, pseudo-Sasakian manifold, para f-structure, transversal quadratic vectorial form, mixed isotropic manifold, index of relative nullity, contact Lagrangian distribution, almost mean curvature vector field, Ricci $D^{\perp}$-exterior recurrent submanifold, totally minimal submanifold, contact $D^{\perp}$-exterior recurrent submanifold.*
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1. INTRODUCTION.

Many papers have been recently concerned with *Sasakian* manifolds $\tilde{M}(\phi,\xi,\tilde{\eta},\tilde{g})$ and contact CR submanifolds of M (see for example Yano and Kon [1]; Kobayashi [2]. *Pseudo-Sasakian* manifolds $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ were developed by Rosca [3].

The purpose of the present paper is to study *co-isotropic* submanifolds M of $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ (since $\tilde{g}$ is pseudo-Riemannian, M are real). If $T_p(M)$ and $T_p^{\perp}(M)$ are the tangent and normal spaces of M at a point $p \in M$, M is co-isotropic if and only if $T_p^{\perp}(M) \subset T_p(M)$. It is proved that any co-istropoic submanifold M is a contact CR submanifold and such kind of CR submanifolds is called *CICR submanifolds*. If M is a horizontal CICR submanifold, then the canonical vector field ξ belongs to the *horizontal* distribution

$D: p \to D_p \subset T_p(M)$ (see Kobayashi [2]), and the *vertical* distribution $D^{\perp}_p (D^{\perp}: p \to D^{\perp}_p \subset T_p(M))$ coincides with $T^{\perp}_p(M)$.

The following basic properties are proved: $D^{\perp}$ is always *involutive* (as in the case of a proper immersion), the leaves $M^{\perp}$ of $D^{\perp}$ are isotropic, and M is both $D^{\perp}$*-totally geodesic* and *mixed totally geodesic* (Bejancu [4]).

In addition, the *almost mean curvature vector* $\Gamma^{\perp}$ (which is defined) of $M^{\perp}$ is a geodesic section on $M^{\perp}$ and $M^{\perp}$ is of constant almost mean curvature.

In Section 3 we study *foliate* (Kobayashi [2]) CICR submanifolds. For this purpose we define a *transversal quadratic vectorial form* II_t associated with $x: M \to \tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$. The following results are proved:

(i) the necessary and sufficient conditions for M to be foliate is that $II_t(X,UY) = II_t(UX,Y)$ for any $X,Y \in D$;

(ii) any foliate CICR submanifold is *almost minimal*.

There is a class of foliate CICR submanifolds for which the simple unit form which corresponds to $D^{\perp}$ is exterior recurrent (Datta [5]). Such submanifolds are said to be *Ricci* $D^{\perp}$*-exterior recurrent* and, if the recurrence 1-form is conformal to η, then M is said to be *contact* $D^{\perp}$*-exterior recurrent*. The following result is proved: the necessary and sufficient condition for a CICR submanifold M to be *minimal* is that M be contact $D^{\perp}$-exterior recurrent.

Finally in Section 4 we discuss the case when M is a contact CICR submanifold of $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ which is ξ*-vertical* (Kobayashi [2]). In this case the leaves $M^{\perp}$ of $D^{\perp}$ are *mixed isotropic* (Rosca [6]; Goldberg and Rosca [7]) submanifolds and such submanifolds M can not be foliate.

2. PRELIMINAIRES.

Let $\tilde{M}$ be a $(2m+1)$-dimensional pseudo-Riemannian manifold of index $m+1$, i.e. of signature $(m+1,m)$. At each point $\tilde{p} \in \tilde{M}$ one has the standard decomposition (see Rosca [3]; Libermann [9]):

$$T_{\tilde{p}}(\tilde{M}) = H_{\tilde{p}}(\tilde{M}) \oplus \mathcal{T}_{\tilde{p}}(\tilde{M}) \qquad (2.1)$$

where $T_{\tilde{p}}$, $H_{\tilde{p}}$, and $\mathcal{T}_{\tilde{p}}$ are the tangent space at p, a $(2m)$-dimensional *neutral* vector space, and a *time-like* line orthogonal to $H_{\tilde{p}}$ respectively.

Let $S_{\tilde{p}}, S^{*}_{\tilde{p}} \subset H_{\tilde{p}}$ be two *self-orthogonal* subspaces (both of dimension m) which define an involutive automorphism U of square $+1$ (U is the para-complex operator defined by Sinha [10]). Let $\xi \in \mathcal{T}_{\tilde{p}}$ and $\eta \in \Lambda^1(\tilde{M})$ be the pairng which defines a contact structure σ_c on $\tilde{M}$ and $\tilde{\nabla}$ be the covariant differentiation operator defined by the metric tensor $\tilde{g}$. Then if for any vector fields $\tilde{Z}$, $\tilde{Z}'$ on $\tilde{M}$ the structure tensors $(U,\xi,\tilde{\eta},\tilde{g})$ satisfy

$$\begin{aligned} &U^2(\tilde{Z}) = \tilde{Z}-\tilde{\eta}(\tilde{Z})\xi, && \tilde{g}(U\tilde{Z},U\tilde{Z}') = -\tilde{g}(\tilde{Z},\tilde{Z}')+\tilde{\eta}(\tilde{Z})\tilde{\eta}(\tilde{Z}'), \\ &\tilde{g}(\tilde{Z},\xi) = \tilde{\eta}(\tilde{Z}), && \tilde{\nabla}_{\tilde{Z}}\xi = U\tilde{Z}, \\ &d\tilde{\eta}(\tilde{Z},\tilde{Z}') = -2\tilde{g}(U\tilde{Z},\tilde{Z}'), && \tilde{\eta}(\xi) = 1, \end{aligned} \qquad (2.2)$$

the manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ has been called by Rosca in [3] *pseudo-Sasakian manifold*.

Since the (1,1)-tensor field U satisfies $U^3-U = 0$, one may say that any pseudo-Sasakian manifold is a *para-f-manifold* (Goldberg and Rosca [11]; Vranceanu and Rosca [12]) (U defines a para f-structure).

In order to study improper immersions in $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$, we consider on $\tilde{M}$ adapted Witt frames (Morvan and Rosca [8]) $W = \{h_A: A,B = 0,1,\ldots,2m\}$ where $\{h_a: a = 1,\ldots,m\} = S_{\tilde{p}}$ and $\{h_a{}^*: a^* = a+m\} = S^*_p$ are null vector fields and $h_0 = \xi$ is the *anisotropic* vector field of W.

As is known (Libermann [9]), one has

$$\tilde{g}(h_a,h_b^*) = \delta_{ab}, \quad \tilde{g}(\xi,h_a) = 0$$
$$\tilde{g}(\xi,h_a^*) = 0, \quad \tilde{g}(\xi,\xi) = 1 \tag{2.3}$$

and

$$Uh_a = h_a, \ Uh_a{}^* = -h_a{}^*, \ U\xi = 0. \tag{2.4}$$

If $\{\tilde{\omega}^A\}$ is the dual basis of W, we set $\tilde{\omega}^0 = \tilde{\eta}$ and the line element $d\tilde{p}$ ($d\tilde{p}$ is a canonical vector 1-form on $\tilde{M}$) is expressed by

$$d\tilde{p} = \tilde{\omega}^A \otimes h_A . \tag{2.5}$$

It follows from (2.3) that the metric tensor $\tilde{g}$ is expressed by

$$\tilde{g} = 2\sum_a \tilde{\omega}^a \otimes \tilde{\omega}^{a^*} + \tilde{\eta}\otimes\tilde{\eta} \tag{2.6}$$

If $\tilde{\theta}^A_B = \tilde{\gamma}^A_{BC}\tilde{\omega}^C$ ($\tilde{\gamma}^A_{BC} \in C^\infty(M)$) and $\tilde{\Theta}^A_B$ are the connection forms and curvature 2-forms on the bundle $W(\tilde{M})$ respectively, then the structure equations (E. Cartan) may be written in indexless form as

$$\tilde{\nabla}h = \tilde{\theta}\otimes h, \tag{2.7}$$

$$d\tilde{\omega} = -\tilde{\theta}\wedge\tilde{\omega}, \tag{2.8}$$

$$d\tilde{\theta} = -\tilde{\theta}\wedge\tilde{\theta} + \tilde{\Theta} . \tag{2.9}$$

Referring to (2.3) and (2.7), one finds

$$\tilde{\theta}^a_b + \tilde{\theta}^{b^*}_{a^*} = 0, \qquad \tilde{\theta}^{a^*}_b = 0, \qquad \tilde{\theta}^a_{b^*} = 0,$$
$$\tilde{\theta}^0_a + \tilde{\theta}^{a^*}_0 = 0, \qquad \tilde{\theta}^a_0 + \tilde{\theta}^0_{a^*} = 0 \tag{2.10}$$

and

$$\tilde{\theta}^0_a = \tilde{\omega}^{a^*}, \qquad \tilde{\theta}^0_{a^*} = -\tilde{\omega}^a . \tag{2.11}$$

The 1-form

$$\gamma = \sum_a \tilde{\theta}^a_a \tag{2.12}$$

is called the *Ricci* 1-form (Rosca [13]). By virtue of (2.7), (2.8), and (2.11) one has

$$d\tilde{\eta} = 2\sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*}, \tag{2.13}$$

$$\tilde{\nabla}\xi = Ud\tilde{p} \Rightarrow \langle\tilde{\nabla}_{\tilde{X}}\xi,\tilde{Y}\rangle + \langle\tilde{\nabla}_{\tilde{Y}}\xi,\tilde{X}\rangle = 0 \tag{2.14}$$

where $\tilde{X}$, $\tilde{Y}$ are any vector fields on $\tilde{M}$ ((2.14) proves in intrinsic manner that ξ is a Killing vector field as in the case of Sasakian manifolds).

Further we recall (see Yano and Kon [1]) that a submanifold M of $\tilde{M}$ is called a *contact CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $D: p \to D_p \subset T_p(M)$ (one denotes the induced elements on M by supressing $\sim$) satisfying:

(i) D is *paraholomorphic* i.e. $UD_p^\perp \subset D_p$ for each $p \in M$, and

(ii) the complementary orthogonal distribution $D^\perp: p \to D_p^\perp \subset T_p(M)$ is *anti-invariant* i.e. $UD_p^\perp \subset T_p^\perp(M)$ for each $p \in M$ ($T_p^\perp(M)$ is the normal space to M at p).

The distribution D (respectively $D^\perp$) is called the *horizontal* (respectively *vertical*) distribution.

Further, according to Kobayashi [2], we say that M is a *contact ξ-horizontal* (respectively *ξ-vertical*) CR submanifold if $\xi \in D_p$ (respectively $\xi \in D_p^\perp$) for each $p \in M$.

If the immersion $x: M \to \tilde{M}$ is improper and d is the *defect* of $M(d = \dim M - \text{rank of the mapping } x)$, then according to Rosca [6] and Goldberg and Rosca [7], M is *mixed isotropic* if one has

$$T_p(M) \cap T_p^\perp(M) \neq 0,\ T_p(M) \not\subset T_p^\perp(M),\ T_p^\perp(M) \not\subset T_p(M) \\ \iff d \neq 0,\ d \neq \dim M,\ d \neq \operatorname{codim} M. \tag{2.15}$$

3. CICR SUBMANIFOLDS.

Let $x: M \to \tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ be the improper immersion of a co-isotropic submanifold M in $\tilde{M}$. Then by definition one has $T_p^\perp(M) \subset T_p(M)$ and without loss of generality one may suppose $T_p^\perp(M) \subset S_p$. We agree to call S_p the *normal self-orthogonal* (abr. n.s.o.) space associated with x and assume that $\dim T_p^\perp(M) = \ell$ ($\ell < m$).

Consider now the two complementary differentiable distributions

$$D: p \to D_p = T_p(M)\setminus T_p^\perp(M);\ D^\perp: p \to D_p^\perp = T_p^\perp(M) \subset T_p(M).$$

Referring to (2.4), one has

$$UD_p \subset D_p,\ UD_p^\perp = T_p^\perp(M). \tag{3.1}$$

Therefore, one may say that any co-isotropic submanifold M of a pseudo-Sasakian manifold $\tilde{M}$ is a contact ξ-horizontal CR submanifold.

A CR submanifold which is co-isotropic will be called in the following a *CICR submanifold*.

Suppose that the manifold M under consideration is defined by

$$\omega^{r^*} = 0;\ r^*, s^* = 2m+1-\ell, \ldots, 2m\ . \tag{3.2}$$

Then one has $D_p = \{h_i, h_{i^*}, \xi\}$ and $D_p^{\perp} = \{h_r; r = m+1-\ell, \ldots, m\}$. Further, according to Rosca [13], we agree to call $D_p^{*\perp} = C_{S^*}T_p(M) \cap S_p^*$ the *transversal* vector space associated with the co-isotropic immersion x. Hence one may write $T_p(\tilde{M})|_M = D_p \oplus D_p^{\perp} \oplus D_p^{*\perp}$. On the other hand, referring to (2.13), one has

$$d\eta = d\tilde{\eta}|_M = 2 \sum_i \omega^i \wedge \omega^{i^*}\ ,$$

and one may say that D_p is a contact vector subspace $(\dim D_p = 2(m-\ell)+1)$ of $T_p(M)$.

If we denote by ψ the simple unit form (see Rosca [14]) which corresponds to D_p, one may write

$$\psi = (\wedge d\eta)^{m-\ell} \wedge \eta / 2^{m-\ell}(m-\ell)! \tag{3.3}$$

Clearly one has $d\psi = 0$. Therefore the ideal $J(D^{\perp}) = \{\psi \in \Lambda(M);\ \psi$ annihilates $D^{\perp},\ dJ \subset J\}$ is a *differentiable ideal*, and we conclude that the distribution $D^{\perp}$ is always involutive (as in the case of proper CR submanifold of $\tilde{M}(U, \xi, \tilde{\eta}, \tilde{g})$ — see Yano and Kon' [1]; Kobayashi [2] — and in the case of CICR submanifolds of a para Kaehlerian manifold — see Rosca [14].)

Consider the vector bundle $L^* = S^* \backslash D^{*\perp} \subset D$ over M. Then, as is known, the elements of $A^1(M, L^*) = SL^*\mathrm{Hom}(\Lambda^1 TM)$ (SL^*: space of sections) are 1-forms of M with values in L^*. Set $\alpha \in \{i, i^*, r\}$. Then by virtue of (3.2) the 1-forms $\theta_\alpha^{r^*}$ represent the *mixed* connection forms (Rosca [13]) associated with α. Then taking the exterior derivative of (3.2), one finds by (2.8), (2.10), and (2.11) that

$$\theta_\alpha^{r^*} = \theta_{i^*}^{r^*} \in A^1(M, L^*). \tag{3.4}$$

Next denote by $\ell_r = -\langle dp, \nabla h_r\rangle$ the second fundamental quadratic forms associated with x (coefficients of ℓ_r are symmetric covariant 2-tensors and depend only on the normal connection $\nabla^{\perp}$). It follows from (2.5), (2.7), and Cartan's lemma that

$$\ell_r = \gamma_{i^*j^*}^{r^*}\, \omega^{i^*} \otimes \omega^{j^*}. \tag{3.5}$$

Then the second fundamental quadratic vectorial form on M, i.e. $II = \sum_r \ell_r h_r$ (II is an M-morphism of $T_0^2(M)$ in $\tilde{M}$ and does not depend on $\nabla^{\perp}$) is expressed by

$$II = \gamma_{i^*j^*}^{r^*}\, \omega^{i^*} \otimes \omega^{j^*} \otimes h_r\ . \tag{3.6}$$

If $\nabla^{\perp}$ defines the connection in the normal bundle $T^{\perp}(M)$, then for any $X \subset T_p(M)$ and any $N \subset T_p^{\perp}(M)$ a basic formula for submanifolds is the Weingarten formula

$$\nabla_X N = -A_N(X) + \nabla_X^{\perp} N \tag{3.7}$$

In (3.7) $A_N(X)$ and $\nabla_X^{\perp} N$ are the tangential and the normal part of $\nabla_X N$ respectively.

Setting $N = N^r h_r$, one finds by (3.4) that

$$A_N(x) = \sum_i N^r \theta^{r^*}_{i^*}(X) h_i \in S \setminus D^\perp . \tag{3.8}$$

Then, as is known, the vector space

$$T_p = \{X \in T_p(M) : A_N(X) = 0\}$$

is called the *space of relative nullity*. It follows from (2.8) that $T_p = S_p \oplus \{\xi\}_p$ and one may write

$$T_p(M) = T_p \oplus (S^* \setminus D^{*\perp})_p . \tag{3.9}$$

Since $\dim T_p = m+1$, this integer represents the *index of relative nullity* (Gardner [15]).

We may also consider the following basic invariants of II. Setting $N = T^\perp_p(M)$, $T = T_p(M)$, one has according to Gardner [15]:

1) The *target rank*, $\dim_N II$, is the integer $r(p)$ which in the case under discussion is defined by

$$r(p) = \dim_N II(p) = \dim\{ \sum_r \gamma^{r^*}_{i^* j^*} h_r \} = \ell(\ell+1)/2. \tag{3.10}$$

2) The *source rank*, $\dim_T II$, is the integer $s(p)$ defined by

$$\begin{aligned} s(p) &= \dim_T II(p) = \dim\{\theta^{r^*}_{i^*}\} = \ell(m-\ell) \\ &= \operatorname{codim} M \cdot \dim L^*_p. \end{aligned} \tag{3.11}$$

Furthermore it follows from (3.6) that

$$II(D^\perp, D^\perp) = 0, \tag{3.12}$$

$$II(D, D^\perp) = 0, \tag{3.13}$$

and

$$II(S,S) = 0. \tag{3.14}$$

Hence, from the above equations we may say that any CICR submanifold is

(i) *vertical totally geodesic,*

(ii) *mixed totally geodesic,*

(iii) *n.s.o. geodesic.*

Set $L = S \setminus D^\perp \subset D$ and consider the distributions

$$\begin{aligned} \Sigma_p &= L_p \oplus \{\xi\}_p \subset D_p \ , \\ \Sigma^*_p &= L^*_p \oplus \{\xi\}_p \subset D_p \ , \end{aligned} \tag{3.15}$$

each of dimension $m-\ell+1$. It follows from this that

$$\begin{aligned} U\Sigma_p &= L_p = \operatorname{orth} L_p ; \quad d\eta|_{\Sigma_p} = 0 \ , \\ U\Sigma^*_p &= L^*_p = \operatorname{orth} L^*_p ; \quad d\eta|_{\Sigma^*_p} = 0 \end{aligned} \tag{3.16}$$

and referring to Weinstein [16] and Rosca [13], we agree to call Σ_p and Σ^*_p the *principal contact Lagrangian distributions* of D_p.

Denote now by $M^{\perp}$ the maximal integral manifold of $D^{\perp}$ and by $T_p(M^{\perp})$ and $T_p^{\perp}(M^{\perp})$ the tangent and normal spaces of $M^{\perp}$ at any point $p \in M^{\perp}$. Obviously one has

$$T_p^{\perp}(M^{\perp}) = D_p^{\perp} \oplus D_p \tag{3.17}$$

and this implies $T_p(M^{\perp}) = D_p^{\perp} \subset T_p^{\perp}(M^{\perp})$ that is the submanifold $M^{\perp}$ is *isotropic* (dim $M^{\perp}$ = codim M = defect of $x = \ell$). Since $M^{\perp}$ is orientable, we choose an orientation on $M^{\perp}$ with the volume element τ and the star operator $*$.

Since the line element of $M^{\perp}$ is

$$dp = \omega^r \otimes h_r;\ r = m-\ell+1,\dots,m \tag{3.18}$$

(we denote the elements induced on $M^{\perp}$ by the same letters), one finds using the star isomorphism that

$$*dp = \sum_r (-1)^{r-(m-\ell+1)} \omega^{r-(m-\ell+1)} \wedge \dots \wedge \hat{\omega}^r \wedge \dots \wedge \omega^m \otimes h_{r^*} \tag{3.19}$$

(the roof ^ means omission). Hence, we may say that $*dp$ is a vectorial $(\ell-1)$-form on the transversal bundle $D^{*\perp}$.

Let $\Delta = d \circ \delta + \delta \circ d$ be the harmonic operator on $\Lambda T^* M^{\perp}$. Since dp given by (3.18) is closed, one has $\Delta p = (\dim M^{\perp})\Gamma$ where Γ is an invariant vector field. Using (2.7), (2.8), (2.10), (2.1), (3.2), and (3.4), we infer from (3.19) that

$$d*dp = (\Gamma^T + \Gamma_t) \otimes \tau \tag{3.20}$$

where we have set

$$\Gamma^T = -(\ell\xi + \sum_{i,r} \gamma_{ir}^r h_{i^*}) \in D_p \tag{3.21}$$

and

$$\Gamma_t = -\sum_r \gamma_{rr}^r h_{r^*} \in D_p^{*\perp}. \tag{3.22}$$

Since by (3.17) the vector field Γ^T is normal to $M^{\perp}$ (this can be easily checked by a direct computation), we define Γ^T/ℓ as the *almost mean curvature vector* of $M^{\perp}$.

From (3.21) and (2.2) one easily finds $<\Gamma^T,\Gamma^T> = \ell^2$. Hence, one may say that $M^{\perp}$ is of *constant almost mean curvature*.

Denote by $\ell_{\Gamma} = <dp, \nabla\Gamma^T>$ the *mean quadratic differential* of $M^{\perp}$. By (3.18) and (3.21) an easy calculation gives

$$\ell_{\Gamma^T} = \sum_{i,r} (\sum_s \gamma_{is}^s)\omega^r \otimes \theta_{i^*}^{r^*}. \tag{3.23}$$

Since $\theta_{i^*}^{r^*} \in A^1(M, L^*)$, it follows from this that on $M^{\perp}$ one has $\theta_{i^*}^{r^*} = 0$, and this implies $\ell_{\Gamma^T} = 0$.

This above fact together with $|\Gamma| = \text{const}$ proves that Γ^T is a *geodesic section* on $M^{\perp}$.

THEOREM 1. Let M be any co-isotropic submanifold of a pseudo-Sasakian manifold $\tilde{M}$. Then M is a CR submanifold of $\tilde{M}$ whose vertical distribution $D^\perp$ is involutive and the leaves $M^\perp$ of $D^\perp$ are isotropic.

Further M possesses the following properties:

(i) it is $D^\perp$-totally geodesic;

(ii) it is mixed totally geodesic;

(iii) it is n.s.o. geodesic.

If $\dim \tilde{M} = 2m+1$ and $\operatorname{codim} M = \ell$, then the source rank and the index of relative nullity at each point $p \in M$ are $\ell(m-\ell)$ and $m+1$ respectively.

Finally the maximal integral manifold $M^\perp$ of $D^\perp$ is of constant almost mean curvature, and the almost mean curvature vector field is a geodesic section on $M^\perp$.

4. FOLIATE CICR SUBMANIFOLDS.

We shall now consider the quadratic forms $\ell_{r^*} = -\langle dp, \nabla h_{r^*}\rangle$ and agree to call $II_t = \sum_r \ell_{r^*} h_{r^*}$ the *transversal quadratic vectorial form* on M.

Using (2.2), (2.7), (2.10), and (3.2), we obtain

$$II_t = (\theta^r_i \otimes \omega^i - \eta \otimes \omega^r) \otimes h_{r^*} \,. \tag{4.1}$$

Let now X and Y be any vector fields on the horizontal distribution D_p. Then the equation

$$II_t(X,UY) = II_t(UX,Y) \tag{4.2}$$

gives

$$\gamma^r_{ij} = \gamma^r_{ji},\ \gamma^r_{ij^*} = 0 \,. \tag{4.3}$$

It is easy to see by (2.6) that (4.3) is equivalent to $[X,Y] \in D_p$ that is the distribution D is *involutive*.

We shall say in this case according to Bejancu [4] and Kobayashi [2] that the CICR submanifold M is *foliate*.

If M^T are the leaves of D, then, as it has been proven by Rosca [3], M^T are *invariant* and *minimal* submanifolds of $\tilde{M}$.

Denote by ϕ the simple unit form corresponding to the vertical distribution $D^\perp$:

$$\phi = \omega^{m-\ell+1} \wedge \ldots \wedge \omega^m \,. \tag{4.4}$$

Let us express that ϕ is *exterior recurrent* with $u \in \Lambda(M)$ as a *recurrence 1-form*. Hence, according to Datta [5] we must write

$$d\phi = u \wedge \phi. \tag{4.5}$$

If u is given by

$$u = \ell\eta - \sum_r \theta^r_r \,, \tag{4.6}$$

we say that M is *Ricci $D^\perp$-exterior recurrent*.

Then by (2.7), (2.9), (2.10) and (3.2) one derives from (4.5) and (4.6) in addition to condition (4.3) (which proves that M is foliate) the following relations

$$\gamma^r_{i0} = 0, \tag{4.7}$$

$$\sum_r \gamma^r_{ir} = 0 . \tag{4.8}$$

In the following we shall set

$$\gamma^{\perp} = \sum_r \theta^r_r \tag{4.9}$$

and agree to call $\gamma^{\perp}$ the *vertical component* of the Ricci 1-form $\gamma = \tilde{\gamma}|_M$ on M.

Let now X and Y be any vector fields of the contact Lagrangian distributions $\Sigma \subset D$. Taking into account (4.3) and (4.7), one finds $[X,Y] \subset \Sigma$, that is Σ is involutive. It is easily deduced that the same property holds for the contact Lagrangian distribution $\Sigma^* \subset D$.

Therefore one may say that if M is a Ricci $D^{\perp}$-exterior recurrent CICR submanifold, then it receives two contact Lagrangian foliations. Moreover, since ξ is geodesic on $\tilde{M}$ (see Rosca [3]), then if X (resp. X^*) is any constant vector field of Σ_p (resp. Σ^*_p), one finds by (4.7) that

$$\nabla_\xi X = 0 \quad (\text{resp. } \nabla_\xi X^* = 0).$$

Hence, any constant vector field of Σ_p or Σ^*_p is ξ-parallel.

Let now M be any CICR submanifold with the line element

$$dp = \omega^i \otimes h_i + \omega^{i^*} \otimes h_{i^*} + \eta \otimes \xi + \omega^r \otimes h_r \tag{4.10}$$

and the volume element

$$\tau = \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \phi \tag{4.11}$$

(M is defined by equations (3.2)). Taking the star isomorphism of (4.10), one has by (2.2) and (4.11)

$$\begin{aligned} {}^*dp = \textcircled{\Gamma} &= \sum_i (-1)^{i-1} \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \hat{\omega}^{i^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \phi \otimes h_{i^*} \\ &+ \sum_i (-1)^{i^*-1} \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \hat{\omega}^{i^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \phi \otimes h_i \\ &+ (-1)^{\ell} \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \phi \otimes \xi \\ &+ \omega^1 \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge (\sum_r (-1)^{r-(m-\ell+1)} \omega^{m-\ell+1} \wedge \ldots \wedge \hat{\omega}^r \wedge \ldots \wedge \omega^m \otimes h_{r^*}). \end{aligned} \tag{4.12}$$

We agree to define the vectorial $(2(m-\ell)+r)$-form $\textcircled{\Gamma}$ as the *improper* mean curvature form of M. Taking the exterior derivative of $\textcircled{\Gamma}$ and using (2.7) and (2.8), we obtain by a straight forward calculation that

$$d\textcircled{\Gamma} = (\Gamma^T + \Gamma^{\perp} + \Gamma_t) \otimes \tau . \tag{4.13}$$

In (4.13) we have set

$$\Gamma^T = -\sum_{i^*}\sum_r (\gamma^r_{ri}+\gamma^r_{ir})h_{i^*} - \sum_i (\sum_r \gamma^r_{ri^*})h_i - (\sum_r \gamma^r_{r0})\xi, \tag{4.14}$$

$$\Gamma^\perp = \sum_i \gamma^r_{ii^*} h_r, \tag{4.15}$$

$$\Gamma_t = -\sum_{s^*}(\sum_r \gamma^r_{rs})h_{s^*}, \quad r,s = m-\ell+1,\dots,m. \tag{4.16}$$

Putting

$$\Gamma = \Gamma^T + \Gamma^\perp + \Gamma_t, \tag{4.17}$$

we agree to say that the invariant vector field Γ is the *improper* mean curvature vector of M, and Γ^T, $\Gamma^\perp$, Γ_t are the horizontal, vertical and transversal components of Γ respectively.

On the other hand, if the vertical Ricci 1-form $\gamma^\perp$ vanishes, then the recurrence 1-form of equation (4.5) is $\ell\eta$. We shall say in this case that M is a *contact $D^\perp$-exterior recurrent CICR submanifold.*

We shall give now the following

DEFINITION. Let $x: M \to \tilde{M}$ be the improper immersion of a CICR submanifold M in a pseudo-Sasakian manifold $\tilde{M}$, and let Γ be the improper mean curvature vector associated with x. Then if the vertical component of Γ vanishes, we say that M is *almost minimal,* and if Γ vanishes, we say that M is *minimal.*

Referring now to (4.3) and (4.15), we see that if M is foliate, then it is almost minimal.

Furthermore, if M is Ricci $D^\perp$-exterior recurrent, then one readily derives that conditions (4.8) and $\sum_r \theta^r_r = 0$ imply $\Gamma = 0$, that is M is minimal.

It is easy to see that the converse is also valid.

THEOREM 2. Let M be a CICR submanifold and let II_t be the transversal quadratic vectorial form of M. Then the necessary and sufficient condition for M to be foliate is that for any vector fields X and Y of the horizontal distribution D one has $II_t(X,UY) = II_t(UX,Y)$, and in this case M is almost minimal. If M is Ricci $D^\perp$-exterior recurrent, then it receives two contact Lagrangian foliations and the necessary and sufficient condition for M to be minimal is that M be contact $D^\perp$-exterior recurrent.

5. CO-ISOTROPIC CONTACT ξ-VERTICAL CR SUBMANIFOLDS.

We shall consider now the improper immersion $x: M \to \tilde{M}$ where M is a contact *ξ-vertical* CICR submanifold of $\tilde{M}$, that is $\xi \in D^\perp_p$. As in Section 3, we suppose that M is defined by equations (3.2). Then the horizontal and vertical distributions at each point $p \in M$ are defined by $D_p = \{h_i, h_{i^*},\ i = 1,\dots,m-\ell;\ i^* = i+m\}$ and $D^\perp_p = \{h_r, \xi;\ r = m+1-\ell,\dots,m\}$ respectively.

In this case D_p is of even dimension (Kobayashi [2]); in the case under discussion $\dim D_p = 2(m-\ell))$ and its corresponding simple unit form ψ is equal to $(\Lambda d\eta)^{m-\ell}/2^{m-\ell}(m-\ell)!$.

It is easily deduced, that, as in Section 3, $J(D^{\perp}) = \{\eta\psi \in \Lambda(M);\ \psi$ annihilates $D^{\perp}\}$ is a differentiable ideal and this proves that $D^{\perp}$ is involutive.

Denote by $M^{\perp}$ the maximal integral manifold of $D^{\perp}$. The normal space $T_p^{\perp}(M^{\perp})$ at each point $p \in M^{\perp}$ in the case under discussion is defined by $(D_p^{\perp}\backslash\xi) \oplus D_p$.

On the other hand, since the tangent space $T_p(M^{\perp})$ at each point p is defined by $D_p^{\perp}$, it follows from this that we are in the situation of conditions (2.13). Therefore according to the definition given in Section 2, it follows that $M^{\perp}$ is a mixed isotropic submanifold of $\tilde{M}$.

Denote now by $\phi = \omega^{m-\ell+1}\Lambda\ldots\Lambda\omega^{m}\Lambda\eta$ the simple unit form which corresponds to $D_p^{\perp}$. Taking into account (2.12), one readily finds that the ideal $J(D) = \{\phi \in \Lambda M;\ \phi$ annihilates D, $dJ \subset J\}$, that is the ideal $J(D)$ is not a differentiable ideal.

Thus we conclude that the distribution D can not be involutive.

THEOREM 3. Let $x: M \to \tilde{M}$ be the improper immersion of a contact ξ-vertical CICR submanifold M in $\tilde{M}$. Then:

(i) the vertical distribution $D^{\perp}$ is always involutive, and leaves of $D^{\perp}$ are mixed isotropic;

(ii) there does not exist a foliate ξ-vertical CICR submanifold of $\tilde{M}$.

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