Internat. J. Math. & Math. Sci.
Vol. 7 No. 2 (1984) 361-370

THE EXTRAPOLATED SUCCESSIVE OVERRELAXATION (ESOR) METHOD FOR CONSISTENTLY ORDERED MATRICES

N. M. MISSIRLIS

Department of Applied Mathematics
University of Athens
Panepistimiopolis 621
Athens, Greece.

D. J. EVANS

Department of Computer Studies
Loughborough University of Technology
Loughborough, Leicestershire, U.K.

(Received September 30, 1982 and in revised form April 2, 1983)

ABSTRACT. This paper develops the theory of the Extrapolated Successive Overrelaxation (ESOR) method as introduced by Sisler in [1],[2],[3] for the numerical solution of large sparse linear systems of the form $Au=b$, when A is a consistently ordered 2-cyclic matrix with non-vanishing diagonal elements and the Jacobi iteration matrix B possesses only real eigenvalues. The region of convergence for the ESOR method is described and the optimum values of the involved parameters are also determined. It is shown that if the minimum of the moduli of the eigenvalues of B, $\underline{\mu}$ does not vanish, then ESOR attains faster rate of convergence than SOR when $1-\underline{\mu}^2<(1-\bar{\mu}^2)^{\frac{1}{2}}$, where $\bar{\mu}$ denotes the spectral radius of B.

KEY WORDS AND PHRASES. *Successive Overrelaxation (SOR) method, linear systems, consistently ordered matrices, first order iterative methods.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. AMS(MOS): 65F10, CR:5.14.

1. INTRODUCTION.

In [4] it is shown how one can explain the origin of the well known first order iterative methods for the numerical solution of linear systems of the form,

$$Au = b \ , \tag{1.1}$$

where A is a real non-singular matrix with non-vanishing diagonal elements of order N, using the preconditioning approach [5]. A result of this was the formulation of two first order iterative schemes, the Extrapolated Gauss-Seidel (EGS) and the Extrapolated Successive Overrelaxation (ESOR). The analysis of the former method, when A is a consistently ordered (2-cyclic matrix [6]) [7] and the iteration matrix B of the Jacobi method possesses only real eigenvalues with $\bar{\mu}=S(B)>1$, $S(.)$ denotes the spectral radius, revealed that its rate of convergence is twice as fast as that of the GS method [4]. Since ESOR was formulated following a similar approach, a study of its rate of convergence as well as its comparison with SOR [6],[7] is of vital importance.

Let us commence our analysis by assuming that A can be expressed as,

$$A = D-C_L-C_U , \tag{1.2}$$

where D is a diagonal matrix possessing the same diagonal elements as A and $-C_L, -C_U$ are the strictly lower and upper triangular parts of A, respectively. The ESOR method is defined by [8],[4],[9],

$$u^{(n+1)} = (1-\tau)u^{(n)} + \omega Lu^{(n+1)} + (\tau-\omega)Lu^{(n)} + \tau Uu^{(n)} + \tau c , \tag{1.3}$$

or

$$u^{(n+1)} = \mathcal{L}_{\tau,\omega}u^{(n)} + \tau(I-\omega L)^{-1}c , \tag{1.4}$$

where,

$$\mathcal{L}_{\tau,\omega} = (I-\omega L)^{-1}[(1-\tau)I+(\tau-\omega)L+\tau U] = I-\tau(I-\omega L)^{-1}D^{-1}A , \tag{1.5}$$

$$L = D^{-1}C_L , \quad U = D^{-1}C_U \quad \text{and} \quad c = D^{-1}b \tag{1.6}$$

with $\omega, \tau(\neq 0)$ real parameters. At this point one may note that when $\tau=\omega$ (1.3) (or (1.4)) yields the SOR method. Further, from (1.3) it is observed that for one complete ESOR iteration we need more than one matrix-vector multiplication. However, if during the nth iteration the vector $Lu^{(n)}$ is stored, it can be used in the following (n+1)st step. Thus, with the exception of the first iteration, the amount of work involved for the computation of one complete ESOR iteration is equivalent to that of an SOR one. In the remainder of this paper we (i) establish the convergence conditions of ESOR and (ii) determine the values for τ and ω which are optimum in the sense of minimising $S(\mathcal{L}_{\tau,\omega})$ when A is consistently ordered and the matrix B=L+U possesses real eigenvalues.

2. CONVERGENCE ANALYSIS.

Theorem 2.1: Let A be a consistently ordered 2-cyclic matrix with non-vanishing diagonal elements. If μ is an eigenvalue of B and λ satisfies,

$$(1-\lambda)^2 = \mu^2(1-\lambda\omega) , \tag{2.1}$$

then λ is an eigenvalue of the matrix,

$$\Lambda_\omega = (I-\omega L)^{-1}D^{-1}A . \tag{2.2}$$

Conversely, if λ is an eigenvalue of Λ_ω and if μ satisfies (2.1), then μ is an eigenvalue of B.

Proof: The proof of this theorem is analogous to that of Theorem 5-2.2 in [7] and is therefore omitted. □

A sufficient and necessary condition for ESOR to converge is $S(\mathcal{L}_{\tau,\omega})<1$. Thus, if we let $\lambda=a+ib$, where a,b are real numbers and $i=\sqrt{-1}$, be an eigenvalue of Λ_ω, then ESOR converges iff $|1-\tau(a+ib)|<1$ or

$$\tau^2(a^2+b^2)<2\tau a , \tag{2.3}$$

which implies that

$$\tau a>0 . \tag{2.4}$$

Therefore, (2.3) is equivalent to either,

$$a>0 , \ 0<\tau<2a/(a^2+b^2) \quad \text{or} \ a<0 \ \text{and} \ 2a/(a^2+b^2)<\tau<0 \tag{2.5}$$

When b=0, then (2.5) becomes $a>0$, $0<\tau<2/a$ or $a<0$, $2/a<\tau<0$.

Theorem 2.2: If A is a consistently ordered 2-cyclic matrix with non-vanishing diagonal elements such that the matrix B possesses real eigenvalues μ_i, $i=1,2,\ldots,N$ with $\underline{\mu}=\min_i|\mu_i|\neq 0$ and $\bar{\mu}=\max_i|\mu_i|$, then ESOR converges iff $\bar{\mu}=S(B)<1$ and the parameters ω,τ lie in any of the corresponding domains given by Table 2.1.

ω-Domain*	τ-Domain*	Condition
$-\infty<\omega\leq\omega'(\bar{\mu})$	$0<\tau<2/\lambda_+(\bar{\mu})$	-
$\omega'(\bar{\mu})\leq\omega\leq 2$	$0<\tau<h(\omega,\underline{\mu})$	-
$2\leq\omega<2/\bar{\mu}^2$	$0<\tau<h(\omega,\bar{\mu})$	-
$2/\underline{\mu}^2<\omega\leq\omega''(\bar{\mu})$	$h(\omega,\underline{\mu})<\tau<0$	$1-\underline{\mu}^2<(1-\bar{\mu}^2)^{\frac{1}{2}}$
$\omega''(\bar{\mu})\leq\omega<+\infty$	$2/\lambda_-(\bar{\mu})<\tau<0$	"

*$h(\omega,\mu)$ and $\lambda_\pm(\mu)$ are given by (2.12) and (2.7), respectively, whereas $\omega'(\mu)$ and $\omega''(\mu)$ are defined by (2.9).

TABLE 2.1

Proof: If μ,λ are eigenvalues of B and Λ_ω, respectively, then,

$$\lambda^2-(2-\omega\mu^2)\lambda+1-\mu^2 = 0 . \tag{2.6}$$

The roots of (2.6) are given by,

$$\lambda_\pm(\mu) = (2-\omega\mu^2\pm\Delta^{\frac{1}{2}})/2 , \tag{2.7}$$

where,

$$\Delta \equiv \Delta(\mu) = \mu^2[\omega^2\mu^2-4(\omega-1)] . \tag{2.8}$$

The kind of $\lambda_\pm(\mu)$ (real or complex) depends upon the sign of Δ which in turn is determined according to the position of ω with respect to the quantities,

$$\omega'(\mu) = 2/[1+(1-\mu^2)^{\frac{1}{2}}] \text{ and } \omega''(\mu) = 2/[1-(1-\mu^2)^{\frac{1}{2}}] , \tag{2.9}$$

which obviously are the roots of Δ. In the sequel we examine the sign of Δ by distinguishing the following three basic cases. Case I: All $\lambda_\pm(\mu)$ are complex, Case II: All $\lambda_\pm(\mu)$ are real and Case III: Some $\lambda_\pm(\mu)$ are complex and others are real.

Case I: In this case $\lambda_\pm(\mu)$ are complex conjugate pairs and $\Delta(\mu)<0$ for all μ^2 such that $0<\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2$, hence $\omega'(\bar{\mu})\leq\omega\leq\omega''(\bar{\mu})$. Evidently, $|\lambda_\pm(\mu)|^2=1-\mu^2$ and $\mathrm{Re}\lambda_\pm(\mu)=(2-\omega\mu^2)/2$. Next, we also note that $\mathrm{Re}\lambda_\pm(\mu)>0$ implies $\omega<2/\bar{\mu}^2$, whereas in order for $\mathrm{Re}\lambda_\pm(\mu)<0$ we must have $2/\underline{\mu}^2<\omega''(\bar{\mu})$ or

$$1-\underline{\mu}^2<(1-\bar{\mu}^2)^{\frac{1}{2}} , \tag{2.10}$$

in which case $\omega>2/\underline{\mu}^2$. We therefore split the above interval of ω into the following: (i) $\omega'(\bar{\mu})\leq\omega\leq 2/\bar{\mu}^2$, (ii) $2/\underline{\mu}^2<\omega\leq\omega''(\bar{\mu})$ and examine each subcase separately. When ω lies in the interval given by (i), $\mathrm{Re}\lambda_\pm(\mu)>0$ and from (2.5) it follows that τ must lie in the range,

$$0<\tau<\min_{\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2} h(\omega,\mu) , \tag{2.11}$$

where,

$$h(\omega,\mu) = (2-\omega\mu^2)/(1-\mu^2) . \tag{2.12}$$

However, sign $[\partial h(\omega,\mu)/\partial\mu^2]=\text{sign}(2-\omega)$, hence for this subcase the ranges for ω and τ such that ESOR converges are,

$$\omega'(\bar{\mu})\leq\omega\leq 2 \text{ and } 0<\tau<h(\omega,\underline{\mu}) \quad , \tag{2.13}$$

or

$$2\leq\omega<2/\bar{\mu}^2 \text{ and } 0<\tau<h(\omega,\bar{\mu}) \quad . \tag{2.14}$$

Alternatively, if (ii) is valid then (2.10) must hold and the range of τ is (see (2.5)),

$$h(\omega,\underline{\mu})<\tau<0 \quad . \tag{2.15}$$

Case II: In this case we assume that all $\lambda_\pm(\mu)$ are real. If they are all positive, then $-\infty<\omega\leq\omega'(\underline{\mu})$ and (2.5) yields (b=0) $0<\tau<2/\lambda_+(\mu)$. But, $\lambda_+(\mu)$ is an increasing function of μ^2, hence $0<\tau<2/\lambda_+(\bar{\mu})$. Similarly, if all $\lambda_\pm(\mu)$ are negative we find $\omega''(\underline{\mu})\leq\omega<+\infty$ and $2/\lambda_-(\bar{\mu})<\tau<0$.

Case III: Since some of $\lambda_\pm(\mu)$ are complex and the others are real, ω lies in either of the following ranges (i) $\omega'(\underline{\mu})\leq\omega\leq\omega'(\bar{\mu})$ or (ii) $\omega''(\bar{\mu})\leq\omega\leq\omega''(\underline{\mu})$. Let us first suppose that ω lies in (i). Then, the real $\lambda_\pm(\mu)$ are positive and $0<\tau<2/\lambda_+(\mu)$ (Case II), whereas for the complex $\lambda_\pm(\mu)$, we have that their real parts are positive and $0<\tau<h(\omega,\mu)$ (Case I). However, it is readily verified that $0<2/\lambda_+(\mu)<h(\omega,\mu)$ implying that when ω lies in (i) the interval for τ is $0<\tau<2/\lambda_+(\bar{\mu})$. Alternatively, if ω lies in (ii), the real $\lambda_\pm(\mu)$ are negative iff $2/\underline{\mu}^2<\omega$, hence the range (ii) becomes $\max\{2/\underline{\mu}^2,\omega''(\bar{\mu})\}\leq\omega\leq\omega''(\underline{\mu})$, where equality on the left holds only when $2/\underline{\mu}^2<\omega''(\bar{\mu})$. Evidently for this range we have $2/\lambda_-(\mu)<\tau<0$ (Case II). For the complex $\lambda_\pm(\mu)$, the range of τ is $h(\omega,\mu)<\tau<0$ (Case I). But $h(\omega,\mu)<2/\lambda_-(\mu)<0$, therefore the necessary and sufficient conditions in this case are

$$\max\{2/\underline{\mu}^2,\ \omega''(\bar{\mu})\}\leq\omega\leq\omega''(\underline{\mu}) \quad \text{and } 2/\lambda_-(\bar{\mu})<\tau<0.$$

Summarising the results of the above cases I,II and III we easily conclude that the region of convergence for ESOR is described by the ranges given in Table 2.1. □

Corollary 2.3: Under the hypothesis of Theorem 2.2 and if $\underline{\mu}=0$, ESOR converges iff $\bar{\mu}<1$ and either,

$$-\infty<\omega\leq 1 \quad \text{and } 0<\tau<2/\lambda_+(\bar{\mu}) \quad , \tag{2.16}$$

or

$$1\leq\omega\leq 2 \quad \text{and } 0<\tau<2 \quad , \tag{2.17}$$

or

$$2\leq\omega<2/\bar{\mu}^2 \text{ and } 0<\tau<h(\omega,\bar{\mu}) \quad . \tag{2.18}$$

Proof: Following a similar treatment as in the proof of Theorem 2.2 we distinguish the following two basic cases.

Case I: Assume that all $\lambda_\pm(\mu)$ are complex, then their real parts are positive since when $\mu=0$, then $\text{Re}\lambda_\pm(0)=1>0$. Thus, the convergence ranges of ω and τ are either (Case I of Theorem 2.2) $\omega'(\bar{\mu})\leq\omega\leq 2$ and $0<\tau<2$ or $2\leq\omega<2/\bar{\mu}^2$ and $0<\tau<h(\omega,\bar{\mu})$.

Case II: Suppose now that some $\lambda_{\pm}(\mu)$ are complex and the others are real. Reasoning as in Case I, ω lies in the range $-\infty<\omega\leq\omega'(\bar{\mu})$ since $2/\bar{\mu}^2<\omega''(\bar{\mu})$. Evidently, the real $\lambda_{\pm}(\mu)$ are positive hence $0<\tau<2/\lambda_{+}(\bar{\mu})$, whereas the complex $\lambda_{\pm}(\mu)$ have positive real parts and $0<\tau<2$. Next, since $2\leq 2/\lambda_{+}(\bar{\mu})$ iff $\omega\geq 1$, it follows that the convergence intervals for this case are given by (2.16) and (2.17). □

Corollary 2.4: Under the hypothesis of Corollary 2.3, SOR converges iff $\bar{\mu}<1$ and $0<\omega<2$.

Proof: If easily follows from Corollary 2.3. □

Corollary 2.5: Under the hypothesis of Theorem 2.2 and if $0<\underline{\mu}=\bar{\mu}=\mu<1$, ESOR converges iff $\mu<1$ and the parameters ω,τ lie in any of the corresponding domains given by Table 2.2.

ω-Domain	τ-Domain
$-\infty<\omega\leq\omega'(\mu)$	$0<\tau<2/\lambda_{+}(\mu)$
$\omega'(\mu)\leq\omega<2/\mu^2$	$0<\tau<h(\omega,\mu)$
$2/\mu^2<\omega\leq\omega''(\mu)$	$h(\omega,\mu)<\tau<0$
$\omega''(\mu)\leq\omega<+\infty$	$2/\lambda_{-}(\mu)<\tau<0$

TABLE 2.2

Proof: It is easily derived by following a similar approach to the proof of Theorem 2.2. □

3.3 OPTIMUM VALUES FOR τ AND ω.

Our aim in this section is to determine the optimum values τ_0,ω_0 of τ and ω respectively such that $S(\mathcal{L}_{\tau,\omega})$ is minimised. Let ζ be an eigenvalue of $\mathcal{L}_{\tau,\omega}$, then because of (1.5) we have the following relationship,

$$\zeta = 1-\tau\lambda, \tag{3.1}$$

where λ is an eigenvalue of Λ_{ω}. Next, we first minimise the expression,

$$\max_{\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2} |\zeta| \tag{3.2}$$

with respect to τ for the different ranges of ω. Secondly, we find the value of ω for which the above expression attains its minimum value. From (3.1) we obtain,

$$|\zeta|^2 = \tau^2(a^2+b^2)-2\tau a+1 , \tag{3.3}$$

or in terms of μ and ω, (3.3) yields,

$$|\zeta|^2 \equiv g(\tau,\omega,\mu^2) = \tau^2(1-\mu^2)-\tau(2-\omega\mu^2)+1 . \tag{3.4}$$

Evidently, $|\zeta|^2$ is minimised if we let,

$$\tau_0 \equiv q(\omega,\mu) = (2-\omega\mu^2)/[2(1-\mu^2)] , \tag{3.5}$$

and its corresponding value is given by the expression,

$$\min_{\tau} |\zeta|^2 \equiv f(\omega,\mu^2) = -\Delta/[4(1-\mu^2)] . \tag{3.6}$$

Next, we examine the behaviour of $g(\tau,\omega,\mu^2)$ as a function of μ^2. Since,

$$\text{sign}(\partial g/\partial\mu^2) = \text{sign}(\tau(\omega-\tau)) , \tag{3.7}$$

we distinguish six cases in order to determine the sign $(\tau(\omega-\tau))$ which are presented in Table 3.1 together with the maximum value of $g(\tau,\omega,\mu^2)$ with respect to μ^2 for each case.

ω-Domain	τ-Domain	$\tau(\omega-\tau)$	$\max\{g(\tau,\omega,\mu^2)\}$
$-\infty<\omega<0$	$-\infty<\tau<\omega$	<0	$g(\tau,\omega,\underline{\mu}^2)$
	$\omega\leq\tau<0$	≥ 0	$g(\tau,\omega,\bar{\mu}^2)$
	$0<\tau<+\infty$	<0	$g(\tau,\omega,\underline{\mu}^2)$
$0\leq\omega<+\infty$	$-\infty<\tau<0$	<0	$g(\tau,\omega,\underline{\mu}^2)$
	$0<\tau\leq\omega$	≥ 0	$g(\tau,\omega,\bar{\mu}^2)$
	$\omega<\tau<+\infty$	<0	$g(\tau,\omega,\underline{\mu}^2)$

TABLE 3.1

In addition, we note that,

$$\text{sign}(\partial g/\partial\omega) = \text{sign}(\tau) . \tag{3.8}$$

For the case where λ is real, it is known (see e.g.[10]) that $S(L_{\tau,\omega})$ is minimised at

$$\tau = \tau_0 = 2/(\bar{\lambda}+\underline{\lambda}) , \tag{3.9}$$

where,

$$\bar{\lambda} = \begin{cases} \max\limits_{\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2} \{\lambda\} , & \text{if } \lambda>0 \\ \min\limits_{\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2} \{\lambda\} , & \text{if } \lambda<0 \end{cases} , \quad \underline{\lambda} = \begin{cases} \min\limits_{\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2} \{\lambda\} , & \text{if } \lambda>0 \\ \max\limits_{\underline{\mu}^2\leq\mu^2\leq\bar{\mu}^2} \{\lambda\} , & \text{if } \lambda<0 \end{cases} \tag{3.10}$$

and its corresponding value is given by the expression

$$S(L_{\tau_0,\omega}) = |k(\Lambda_\omega)-1|/[k(\Lambda_\omega)+1], \quad k(\Lambda_\omega) = \bar{\lambda}/\underline{\lambda} . \tag{3.11}$$

A simple study of the behaviour of $\lambda_\pm(\mu)$ as a function of μ^2 reveals that,

$$\text{sign}(\partial\lambda_+(\mu)/\partial\mu^2) = +1 \text{ and } \text{sign}(\partial\lambda_-(\mu)/\partial\mu^2) = -1. \tag{3.12}$$

Theorem 3.1: Let A be a consistently ordered 2-cyclic matrix with non-vanishing diagonal elements such that the matrix B possesses real eigenvalues μ_i, $i=1,2,\ldots,N$ with $\underline{\mu}=\min\limits_i|\mu_i|\neq 0$, $\bar{\mu}=\max\limits_i|\mu_i|<1$. Then, the expressions for $S(L_{\tau_0,\omega})$ for the different ranges of ω are presented in Table 3.2. Moreover, if $1-\underline{\mu}^2<(1-\bar{\mu}^2)^{\frac{1}{2}}$, $S(L_{\tau,\omega})$ is minimised for,

$$\omega_0 = \omega'(\bar{\mu}) = 2/[1+(1-\bar{\mu}^2)^{\frac{1}{2}}] , \quad \tau_0 = (2-\omega_0\underline{\mu}^2)/[2(1-\underline{\mu}^2)], \tag{3.13}$$

and its corresponding value is given by the expression,

$$S(L_{\tau_0,\omega_0}) = [\underline{\mu}(\bar{\mu}^2-\underline{\mu}^2)^{\frac{1}{2}}]/[(1-\underline{\mu}^2)^{\frac{1}{2}}(1+(1-\bar{\mu}^2)^{\frac{1}{2}})] . \tag{3.14}$$

Proof: Following a similar treatment as in Theorem 2.2 we distinguish again three basic cases. Next, for each case we determine the expression of $S(L_{\tau_o,\omega})$ via (3.6) or (3.11) for the different ranges of ω.

Case I: Let us assume that all $\lambda_\pm(\mu)$ are complex, then ω lies in the range $\omega'(\bar{\mu})\leq\omega\leq\omega''(\bar{\mu})$. According to Table 3.1 we consider three subcases.

(i) $-\infty<\tau<0$. This implies $\mathrm{Re}\lambda_\pm(\mu)=(2-\omega\mu^2)/2<0$ or $2/\underline{\mu}^2<\omega$, hence this subcase exists if $2/\underline{\mu}^2<\omega''(\bar{\mu})$ or

$$1-\underline{\mu}^2<(1-\bar{\mu}^2)^{\frac{1}{2}} \quad . \tag{3.15}$$

Therefore, the range of ω becomes $2/\underline{\mu}^2<\omega\leq\omega''(\bar{\mu})$, whereas from (3.5) and (3.6) it follows that

$$\tau_o = q(\omega,\underline{\mu}) \text{ and } S(L_{\tau_o,\omega}) = \{-\underline{\Delta}/[4(1-\underline{\mu}^2)]\}^{\frac{1}{2}} \ , \tag{3.16}$$

where $\underline{\Delta}=\Delta(\underline{\mu})$.

(ii) $0<\tau\leq\omega$. Similarly, in this case $\omega''(\bar{\mu})\leq\omega<2/\bar{\mu}^2$. Moreover, since $\tau_o\leq\omega$ or $2/(2-\bar{\mu}^2)\leq\omega$, it follows that the range of ω becomes $2/(2-\bar{\mu}^2)\leq\omega<2/\bar{\mu}^2$, whereas

$$\tau_o = q(\omega,\bar{\mu}) \quad \text{and} \quad S(L_{\tau_o,\omega}) = \{-\bar{\Delta}/[4(1-\bar{\mu}^2)]\}^{\frac{1}{2}} \ . \tag{3.17}$$

where $\bar{\Delta}=\Delta(\bar{\mu})$.

(iii) $0<\omega<\tau<+\infty$. Again $\omega'(\bar{\mu})\leq\omega<2/\bar{\mu}^2$ and since $\omega<\tau_o$, it follows that $\omega<2/(2-\underline{\mu}^2)$. Therefore, for this case to exist $\omega'(\bar{\mu})<2/(2-\underline{\mu}^2)$ or (3.15) must hold. Evidently, the range of ω is $\omega'(\bar{\mu})\leq\omega<2/(2-\underline{\mu}^2)$ while (3.16) holds for this subcase also.

Case II: Here we consider the case where all $\lambda_\pm(\mu)$ are real. If they are positive, then $-\infty<\omega\leq\omega'(\underline{\mu})$ and $S(L_{\tau_o,\omega})$ is given by (3.11) with $\bar{\lambda}=\lambda_+(\bar{\mu})$ and $\underline{\lambda}=\lambda_-(\bar{\mu})$ thus,

$$S(L_{\tau_o,\omega}) = \bar{\Delta}^{\frac{1}{2}}/(2-\omega\bar{\mu}^2) \ . \tag{3.18}$$

where τ_o is determined by (3.9).

Alternatively, if all $\lambda_\pm(\mu)$ are negative $\omega''(\underline{\mu})\leq\omega<+\infty$ and,

$$S(L_{\tau_o,\omega}) = \bar{\Delta}^{\frac{1}{2}}/(\omega\bar{\mu}^2-2) \ . \tag{3.19}$$

Case III: Assume now that some of $\lambda_\pm(\mu)$ are complex and the others are real. This implies that ω lies in either of the following intervals: (i) $\omega'(\underline{\mu})\leq\omega\leq\omega'(\bar{\mu})$ or (ii) $\omega''(\bar{\mu})\leq\omega\leq\omega''(\underline{\mu})$. If ω lies in (i), then all real $\lambda_\pm(\mu)$ are positive and all complex $\lambda_\pm(\mu)$ have positive real parts. Therefore, in the real case (see Case II) $S(L_{\tau_o,\omega})=\bar{\Delta}^{\frac{1}{2}}/(2-\omega\mu^2)$, whereas in the complex case $S(L_{\tau_o,\omega})=\{-\Delta/[4(1-\mu^2)]\}^{\frac{1}{2}}$. However, since $(2-\omega\mu^2)^2\leq4(1-\mu^2)$ it follows that $S(L_{\tau_o,\omega})$ is given by (3.16). Similarly, when ω lies in (ii) we find that $S(L_{\tau_o,\omega})$ is given by (3.17).

Summarising our results so far we can construct Table 3.2 where the expressions of $S(L_{\tau_o,\omega})$ and the corresponding ranges of ω are presented.

Case	ω-Domain	Condition	$S(L_{\tau_0,\omega})$
1	$-\infty<\omega\leq\omega'(\bar{\mu})$	-	$\bar{\Delta}^{\frac{1}{2}}/(2-\omega\bar{\mu}^2)$
2	$\omega'(\bar{\mu})\leq\omega<2/(2-\underline{\mu}^2)$	*	$\{-\underline{\Delta}/[4(1-\underline{\mu}^2)]\}^{\frac{1}{2}}$
3	$2/(2-\bar{\mu}^2)\leq\omega<2/\bar{\mu}^2$	-	$\{-\bar{\Delta}/[4(1-\bar{\mu}^2)]\}^{\frac{1}{2}}$
4	$2/\underline{\mu}^2<\omega\leq\omega''(\bar{\mu})$	*	$\{-\underline{\Delta}/[4(1-\underline{\mu}^2)]\}^{\frac{1}{2}}$
5	$\omega''(\bar{\mu})\leq\omega<+\infty$	*	$\bar{\Delta}^{\frac{1}{2}}/(\omega\bar{\mu}^2-2)$

*implies that condition (3.15) holds.

TABLE 3.2

By studying the behaviour of $S(L_{\tau_0,\omega})$ as a function of ω, for all the cases of Table 3.2 we can easily find that it is minimised either at $\omega_0=\omega'(\bar{\mu})$ or at $\omega_0=\omega''(\bar{\mu})$. A simple comparison of the two candidate minimum values of $S(L_{\tau_0,\omega})$ reveals that the one for $\omega_0=\omega'(\bar{\mu})$ is the smallest. □

Corollary 3.2: Under the hypothesis of Theorem 3.1 and if $\underline{\mu}=0$, then $S(L_{\tau,\omega})$ is minimised at $\tau_0=\omega_0=\omega'(\bar{\mu})$ and its corresponding value is,

$$S(L_{\omega_0,\omega_0}) = \omega_0-1 = [1-(1-\bar{\mu}^2)^{\frac{1}{2}}]/[1+(1-\bar{\mu}^2)^{\frac{1}{2}}] \quad . \qquad (3.20)$$

Proof: Since we always have $Re\lambda_{\pm}(\mu)>0$ and $\tau>0$ it follows (see (3.7)) that $\text{sign}(\partial g/\partial\mu^2)=\text{sign}(\omega-\tau)$ and $g(\tau,\omega,\mu^2)$ is an increasing function of ω (see (3.8)). In the sequel we distinguish two cases.

Case I: Suppose that all $\lambda_{\pm}(\mu)$ are complex, then ω lies in the range $\omega'(\bar{\mu})\leq\omega\leq2/\bar{\mu}^2$ and,

$$\max_{0\leq\mu^2\leq\bar{\mu}^2}|\zeta|^2 = \begin{cases} g(\tau,\omega,0) = (\tau-1)^2 , & \text{if } \omega\leq\tau \\ g(\tau,\omega,\bar{\mu}^2) , & \text{if } \omega\geq\tau . \end{cases} \qquad (3.21)$$

Case II: In this case we assume that some of $\lambda_{\pm}(\mu)$ are complex and the others are real. Then, $S(L_{\tau_0,\omega})$ is given by (3.16) (see Case III of Theorem 3.1) and ω lies in the range $-\infty<\omega\leq\omega'(\bar{\mu})$. The results of cases I and II are summarised in Table 3.3 (note that $\min_{\tau\leq\omega}(\tau-1)=\omega-1$)

Case	ω-Domain	$S(L_{\tau_0,\omega})$
1	$-\infty<\omega\leq\omega'(\bar{\mu})$	$\bar{\Delta}^{\frac{1}{2}}/(2-\omega\bar{\mu}^2)$
2	$\omega'(\bar{\mu})\leq\omega\leq\tau$	$\omega-1$
3	$\omega'(\bar{\mu})\leq\tau\leq\omega<2/\bar{\mu}^2$	$\{-\bar{\Delta}/[4(1-\bar{\mu}^2)]\}^{\frac{1}{2}}$

TABLE 3.3

Studying the behaviour of $S(L_{\tau_0,\omega})$ as a function of ω we easily arrive at the conclusion that its minimum is attained at $\tau_0=\omega_0=\omega'(\bar{\mu})$ (Case 2). □

Corollary 3.3: Under the hypothesis of Corollary 3.2, $S(L_{\omega,\omega})$ is minimised at $\omega_0=\omega'(\bar{\mu})$ and its corresponding value is given by (3.20).

Proof: It follows immediately from Corollary 3.2. □

Corollary 3.4: Under the hypothesis of Theorem 3.1 and if,

$$0<\underline{\mu} = \bar{\mu} = \mu<1, \tag{3.22}$$

then for either $\omega_0=\omega'(\mu)$ and $\tau_0=1/(1-\mu^2)^{\frac{1}{2}}$ or $\omega_0=\omega''(\mu)$ and $\tau_0=-1/(1-\mu^2)$, we have,

$$S(L_{\tau_0,\omega_0}) = 0 . \tag{3.23}$$

Proof: Following a similar approach we can construct Table 3.4. By studying the behaviour of $S(L_{\tau_0,\omega})$ as a function of ω, we easily conclude that its minimum is attained for $\omega_0=\omega'(\mu)$ and $\omega_0=\omega''(\mu)$. Since, these values of ω_0 are the roots of $\Delta(\mu)$ (3.23) is obtained. Finally, the optimum values of τ are obtained by using (3.5). □

4. FINAL REMARKS AND CONCLUSIONS.

In Table 2.1 we present sufficient and necessary conditions for ESOR to converge. Although the same problem was also tackled independently in [11] and [9], our approach differs from those in the afore-mentioned references. As a result we extended the convergence region found in [9] (see Theorem 2A) by adding the first and the last two conditions of Table 2.1. On the other hand, Corollary 2.3 shows that ESOR converges for a wider range of parameters τ and ω than SOR, whose convergence conditions are obtained as a by-product of the whole analysis. In addition, by modifying slightly the simple approach followed in Section 2, we were able to find the optimum values for the parameters τ and ω in terms of $\bar{\mu}$ and $\underline{\mu}$. At this point it should be mentioned that these results have also been obtained in [10]. However, the detailed analysis was not presented since "a tremendous number of cases" ([11]p.184) had to be examined. The conclusion from our analysis is that ESOR attains a faster rate of convergence than SOR when A is a consistently ordered matrix and B possesses real eigenvalues such that $\underline{\mu}\neq 0$, $\bar{\mu}<1$ and $1-\underline{\mu}^2<(1-\bar{\mu}^2)^{\frac{1}{2}}$. The ESOR's superiority under these conditions is due to the fact that in this method one can fully exploit the spectrum of the eigenvalues of B to achieve the best possible results whereas such a possibility is precluded in SOR. This is emphasised in Corollary 3.4, where under the special condition (3.22) an exceptionally fast rate of convergence is obtained by employing the ESOR method. Further, we note that the basic criterion of using ESOR is to check whether the value of $\underline{\mu}$ is away from zero. This requirement is derived from the fact that if $\underline{\mu}\to 0^+$ (Corollary 3.2) we have that $S(L_{\tau_0,\omega_0})\to S(L_{\omega_0,\omega_0})$, whereas if $\underline{\mu}\to\bar{\mu}$ (Corollary 3.4) we have that $S(L_{\tau_0,\omega_0})\to 0^+$. The quantity $\underline{\mu}$ is necessarily zero whenever the matrix A is consistently ordered of odd order, whereas when A is of even order we clearly need a formula for its estimation. Of course, the determination of $\underline{\mu}$ is the additional work required in the ESOR method (as compared with SOR) and may incur some extra computational effort. On the other hand, condition (3.15) yields a lower bound for $\underline{\mu}$ $(1-(1-\bar{\mu}^2)^{\frac{1}{2}}<\underline{\mu}^2)$, where if we let $\mu=1-\varepsilon, \varepsilon<<1$, then $\underline{\mu}\gtrsim 1-(\varepsilon/2)^{\frac{1}{2}}$. To realise the information that this result offers, let us take $\varepsilon=0.1$, then $\underline{\mu}\gtrsim 0.77639$. This corresponds to the situation where all the eigenvalues of B are scattered inside the intervals (-1.0,-0.77639) and (0.77639,1.0). Although, such cases do not often arise in practical problems, this should not obscure the overall performance evaluation of ESOR since the method is a generalised version of SOR and as such it is expected to exhibit its real power in more general problems. This was shown recently in [12] where the superiority of ESOR over SOR was established for linear systems with positive definite coefficient matrix. Finally, a simple numerical experiment carried out in [9] shows that ESOR converges twice as fast as SOR for a special non-cyclic matrix.

REFERENCES

[1] SISLER, M., *"Über ein zweiparametriges Iterationsverfahren"*, Apl.Math.18, (1973), 325-332.

[2] SISLER, M., *"Über die Optimierung eines zweiparametrigen Iterationsverfahrens"*, Ibid. 20, (1975), 126-142.

[3] SISLER, M., *"Bemerkungen zur Optimierung eines zweiparametrigen Iterationverfahren"*, Ibid. 21, (1976), 213-220.

[4] MISSIRLIS, N.M. and EVANS, D.J., *"On the Convergence of some Generalised Preconditioned Iterative Methods"*, SIAM J.Numer.Anal. 18, (1981), 591-596.

[5] EVANS, D.J. and MISSIRLIS, N.M., *"The Preconditioned Simultaneous Displacement Method (PSD method)"*, MACS 22, (1980), 256-263.

[6] VARGA, R.S., *"Matrix Iterative Analysis"*, Prentice Hall, Englewood Cliffs, New Jersey, (1962).

[7] YOUNG, D.M., *"Iterative Solution of Large Linear Systems"*, Academic Press, New York, (1971).

[8] HADJIDIMOS, A., *"Accelerated Overrelaxation Method"*, Math.Comp. 32, (1978), 149-157.

[9] NEITHAMMER, W., *"On Different Splittings and the Associated Iteration Method"*, SIAM J.Numer.Anal., 16, (1979), 186-200.

[10] FORSYTHE, G.E. and WASOW, W.R., *"Finite Difference Methods for Partial Differential Equations"*, John Wiley & Sons Inc., New York (1960).

[11] AVDELAS, G. and HADJIDIMOS, A., *"Optimum Accelerated Overrelaxation Method in a Special Case"*, Math.Comp. 36, (1981), 183-187.

[12] GAITANOS, N., HADJIDIMOS, A. and YEYIOS, A., *"Optimum Accelerated Over-relaxation (AOR) Method for Systems with Positive Definite Coefficient Matrix"*, SIAM J.Numer.Anal. 20, (1983), 774-783.