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INTEGRABILITY OF DERIVATIONS OF CLASSICAL SOLUTIONS OF DIRICHLET'S PROBLEM FOR AN ELLIPTIC EQUATION

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ABSTRACT. The present work is concerned with integrability properties of derivatives of classical solutions of Dirichlet's problem for a linear second-order elliptic equation Lu = f. With the aid of special weighted Hilbert spaces of locally square integrable functions, we determine the nature of singularities that f can have near the boundary, in order that such classical solutions are in the Sobolev space W^1. By means of an example it is shown that the obtained result is exact.

KEY WORDS AND PHRASES. *Linear second-order elliptic equation. Dirichlet's problem. Classical solutions. Sobolev spaces. Weighted Hilbert spaces of locally square integrable functions.*

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1. INTRODUCTION.

The question of whether the classical solution u(x) of Dirichlet's problem for an elliptic equation Lu = f is in the Sobolev space W^1 was studied in [2], [3] and [4]. The main result concerning this question is that $u \in W^1$ provided the coefficients of L are essentially bounded and $f \in L^2$. Here, we prove a result showing that u may be in W^1 even when f is not in the class L^2. With the aid of special weighted Hilbert spaces of functions, we determine exactly the class of functions f, for which $u \in W^1$.

Let $G \subset R^n$ be a bounded region, whose boundary ∂G is a closed (n-1)-dimensional surface in the class C^2. For p>0, we let $G^p = \{x \in G: d(x) > p\}$, where d(x) = dist(x,$\partial$G). As was shown in [5], there exist positive numbers m,b, depending only on G, and a function $r(x) \in C^2(\overline{G})$ such that

$$r(x) = d(x), \quad x \in G \setminus G^m,$$

$$bd(x) \le r(x) \le b^{-1}d(x), \quad x \in G.$$

Moreover, if $p \in [0,m]$, then G^p is a region with boundary ∂G^p in C^2, and the relation $x_p = x_p(x) = x - p\underline{N}(x)$, $x \in \partial G$, determines a C^1-diffeomorphism of ∂G onto ∂G^p. Here $\underline{N}(x)$ denotes the unit outward normal to ∂G at x.

In G we consider a non-self-adjoint operator defined by

$$Lu = - D_i(a_{ij}D_j u) + a_i D_i u + au,$$

where we used summation convention, that is, we sum over an index that appears twice, and $D_i = \partial/\partial x_i$. It is assumed that $a_{ij}, a_i, a \in C(G)$, and L is strictly elliptic in G, that is

$$\nu|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2;\ \nu,\mu = \text{const} > 0,\ X \in G, \tag{1.1}$$

for all real vectors $\xi = (\xi_1,\dots,\xi_n)$; $|\xi|^2 = \sum_{l=1}^{n} \xi_i^2$.

Throughout this paper, $u(x) \in C^2(G) \cap C(\bar{G})$ denotes a classical solution of the problem

$$\left.\begin{aligned} Lu &= f(x), \quad x \in G, \\ u|_{\partial G} &= 0, \end{aligned}\right\} \tag{1.2}$$

where $f \in C(G)$.

We shall employ usual notation $W^{k,2}(G)$ and $\overset{\circ}{W}{}^{k,2}(G)$ for Sobolev spaces [1], but, conventionally, without the index "2". By $L_s^2(G)$ we shall denote the Hilbert space of all measurable functions v(x) in G for which

$$\|v\|^2_{L_s^2(G)} = \int_G r^s(x)v^2(x)dx < \infty.$$

Lemma. If $v \in \overset{\circ}{W}{}^1(G)$, then $v \in L^2_{-2}(G)$ and

$$\|v\|_{L^2_{-2}(G)} \le C\,\|Dv\|_{L^2(G)}, \tag{1.3}$$

where $D = (D_1,\dots,D_n)$ and $C > 0$ depends only on m, b and the diameter of G^m.

The proof is similar to that of the lemma in [5].

2. MAIN RESULTS

Our main result is the following:

THEOREM. Let the operator L be strictly elliptic and have coefficients a_i, $a \in L^\infty(G)$. If problem (1.2) has a classical solution u, then for arbitrary $f \in L_s^2(G)$ with $s \le 2$, this solution is in $W^1(G)$.

Moreover,

$$\|u\|_{W^1(G)} \le C(\|f\|_{L_s^2(G)} + \|u\|_{L^2(G)}) \tag{2.1}$$

where C is independent of f and u.

Proof. Let $f^{(p)} \in L^2(G)$ be the function defined by

$$f^{(p)}(x) = \begin{cases} f(x), & x \in G^p, \\ 0, & x \in G \setminus G^p. \end{cases}$$

Recalling the properties of r(x) and using the absolute continuity of Lebesgue integral, we have

$$\lim_{p \to 0} \int_G r^s(f - f^{(p)})^2 dx = \lim_{p \to 0} \int_{G \setminus G^p} r^s f^2 dx = 0,$$

that is, as $p \to 0$ the functions $f^{(p)}$ converge to f in the $L_s^2(G)$-norm.

Since $u \in C(\bar{G})$ and $u\setminus_{\partial G} = 0$, we can choose a decreasing sequence of numbers $\{p_k\}$ such that

$$|u| < \frac{1}{k}, \quad x \in G \setminus G^{p_k}. \tag{2.2}$$

We write G_k for G^{p_k} and f_k for $f^{(p_k)}$, $k = 1,2,\ldots$

Let q be sufficiently large positive number such that

$$q + \inf_G a - \frac{1}{4\nu} \sup_G \sum_{i=1}^{n} a_i^2 - \tfrac{1}{2} > 0 \tag{2.3}$$

and consider the problem

$$\left.\begin{aligned} Lu_k + qu_k &= f + qu, \quad x \in G_k, \\ u_k|\partial G_k &= 0. \end{aligned}\right\} \tag{2.4}$$

Since $f + qu \in L^2(G_k)$, problem (2.4) has a weak solution in $\overset{\circ}{W}{}^1(G_k)$[1,p. 175]. Such solution is understood to be a function in $\overset{\circ}{W}{}^1(G_k)$ satisfying the integral identity

$$\int_{G^k} [a_{ij} D_j u_k D_i v + (a_i D_i u_k + au_k + au_k + qu_k)v]dx = \int_{G^k} (f + qu)v\, dx \tag{2.5}$$

for all $v \in \overset{\circ}{W}{}^1(G_k)$. Taking $v = u_k$ in (2.5), and using (1.1) and the well known Cauchy inequality $|ab| \le \varepsilon a^2 + (1/4\varepsilon)b^2$, we obtain

$$(\nu - \varepsilon) \int_{G_k} |Du_k|^2 dx + (q + \inf_G a - \frac{1}{4\varepsilon} \sup_G \sum_{i=1}^{n} a_i^2 - \tfrac{1}{2}) \int_{G_k} u_k^2 dx$$

$$\le \left| \int_{G_k} fu_k dx \right| + \tfrac{1}{2}q^2 \int_{G_k} u^2 dx.$$

Letting $u_k^* = \begin{cases} u_k & x \in G_k, \\ 0, & x \in G \setminus G_k \end{cases}$ and recalling the definition of f_k, the last inequality can be rewritten as

$$(\nu - \varepsilon)\int_G |Du_k^*|^2 dx + (q + \inf_G a - \frac{1}{4\varepsilon} \sup_G \sum_{i=1}^{n} a_i^2 - \tfrac{1}{2}) \int_G u_k^{*2} dx$$

$$\le \left| \int_G f_k u_k^* dx \right| + \tfrac{1}{2}q^2 \int_G u^2 dx. \tag{2.6}$$

The first term on the right of (2.6) we estimate by using Cauchy inequality and (1.3) as follows:

$$\left|\int_G f_k u_k^* dx\right| \le \frac{1}{4\varepsilon} \int_G r^2 f_k^2\, dx + \varepsilon_1 \int_G (u_k^{*2}/r^2) dx$$

$$\le \frac{1}{4\varepsilon_1} (\max_G r^{2-s}) \int_G r^s f^2 dx + \varepsilon_1 C \int_G |Du_k^*|^2 dx. \tag{2.7}$$

Since $2 - s \ge 0$, the function r^{2-s} is bounded, and it follows from (2.3), (2.6) and (2.7) with sufficiently small ε, ε_1 that

$$\int_G (|Du_k^*|^2 + u_k^{*2}) dx \le C \left(\int_G r^s f^2 d_x + \int_G u^2 dx\right) \le K, \tag{2.8}$$

where C depends only on m, b, s, G and the coefficients of L. Hence, K is independent of k. Consequently, there is a subsequence of $\{u_k^*\}$ weakly converging in the metric of $W^1(G)$ to some function $w \in \overset{o}{W}{}^1(G)$. Without loss of generality, we can assume that the sequence itself weakly converges to w. In view of (2.8), we have

$$\|w\|^2_{W^1(G)} \le C(\|f\|^2_{L^2_s(G)} + \|u\|^2_{L^2(G)}). \tag{2.9}$$

Since $u \in C^2(\bar{G}_k)$ and u_k is a weak solution of problem (2.4), the function $u - u_k$ is a weak solution in $W^1(G_k)$ of the problem

$$\left.\begin{array}{l} Lv_k + qv_k = 0, \quad x \in G_k, \\ v \,|\partial G_k = u|\partial G_k \end{array}\right\}$$

The conditions imposed in our theorem and the fact that $q + a > 0$; cf.(2.3), are sufficient to apply the weak maximum principle [1,p. 168] to the function $u - u_k$. Hence,

$$|u - u_k^*| < \max_{\partial G_k} |u|$$

almost everywhere (a.e.) in G_k. Taking (2.2) into consideration, we find that a.e. in G

$$|u - u_k^*| \quad \frac{1}{k},$$

that is, the sequence $\{u_k^*\}$ uniformly converges to u a.e. in G. But, as was shown above, the same sequence weakly converges to w in the metric of $W^1(G)$. Hence, $u = w$ a.e. on G, which completes the proof of the theorem.

Now we show that the condition $f \; L^2_s(G)$, $s \le 2$, is exact in the sense that, it cannot be weakened to allow functions f with singularities near the boundary of degree higher than the second. If f is in $L^2_{2+h}(G)$ for any $h > 0$, but it is not in $L^2_2(G)$, then the inclusion $u \in W^1(G)$ may be false. This may be seen from the following example.

Let B be the unit disk $\{|x| < 1\}$ in R^2. In B consider the function $u(x) = |x|^2(1 - |x|)^{\frac{1}{2}}$. It is easily verified that $u \in C^2(B) \cap C(\bar{B})$, $u|_{|x|=1} = 0$ and $u \notin W^1(B)$. At the same time $\Delta u \in L^2_{2+h}(B)$ for any $h > 0$; $\Delta = D_{11} + D_{22}$, while $\Delta u \notin L^2_2(B)$.

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