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COINCIDENCE THEOREMS FOR SOME MULTIVALUED MAPPINGS

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ABSTRACT. Two coincidence theorems in a metric space are proved for a multi-valued mapping that commutes with a single-valued mapping and satisfies a general multi-valued contraction type condition.

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1. INTRODUCTION.

Following the Banach contraction mapping, Nadler [1] introduced the concept of multi-valued contraction mappings and established that a multi-valued contraction mapping possesses a fixed point in a complete metric space. Subsequently a number of fixed point theorems in metric spaces have been proved for multi-valued mappings satisfying contractive type conditions; e.g. see [2]-[10], [11-17] and [18-20]. Jungck [21] generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting mappings in a metric space. He also pointed out the potential of commuting mappings for generalizing fixed point theorems in [22] and [23]. One of the most general fixed point theorems for a generalized multi-valued contraction mapping appears in Ciric [4]. In this paper we combine the ideas of Ciric and Jungck to obtain two coincidence theorems for a multi-valued mapping.

Let (X,d) be a metric space. We shall follow the following notations and definitions.

$CL(X) = \{A : A$ is a nonempty closed subset of $X\}$,

$CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$,

$N(\varepsilon,A) = \{x \in X : d(x,a) < \varepsilon$ for some $a \in A$, $\varepsilon > 0\}$, $A \in CL(X)$,

and

$$H(A,B) = \begin{cases} \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon,B) \text{ and } B \subseteq N(\varepsilon,A)\}, & \text{if the infimum exists} \\ \infty, & \text{otherwise} \end{cases}$$

for each A, $B \in CL(X)$.

H is called the generalized Hausdorff distance function for $CL(X)$ induced by d. If $H(A,B)$ is defined for A, $B \in CB(X)$ then the pair (X,H) is a metric space and H is called the Hausdorff metric induced by d. $D(x,A)$ will denote the ordinary distance between $x \in X$ and A, a nonempty subset of X. Let f be a single-valued mapping from X to X and T a multi-valued mapping from X to the nonempty subsets of X.

Definition 1. ([10]). T and f are said to commute if for each $x \in X$, $f(T(x)) = fTx \subseteq Tfx = T(f(x))$.

Definition 2. ([21], [4]). An orbit for T at a point x_0 is a sequence $\{x_n : x_n \in Tx_{n-1}\}$.

Definition 3. ([4]). A space X is said to be T-orbitally complete iff every Cauchy sequence of the form $\{(x_{n_i} : x_{n_i} \in Tx_{n_i-1})\}$ converges in X.

Definition 4. If for a point $x_0 \in X$ there exists a sequence $\{x_n\}$ such that $fx_{n+1} \in Tx_n$, $n = 0,1,2,\ldots$, then $0_f(x_0) = \{fx_n : n = 1,2,\ldots\}$ is the orbit for (T,f) at x_0. We shall use $0_f(x_0)$ as a set and as a sequence as the situation demands. Further $0_f(x_0)$ is called a regular orbit for (T,f) if for each n,

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

Definition 5. A space X is called (T,f)-orbitally complete iff every Cauchy sequence of the form $\{fx_{n_i} : fx_{n_i} \in Tx_{n_{i-1}}\}$ converges in X.

An immediate consequence of this definition is that if the space X is complete then it is (T,f)-orbitally complete for any T and f. However, simple examples can be constructed to show that, if for some T and f, X is (T,f)-orbitally complete then X need not be complete. It is also obvious from the fact that Definitions 2 and 3 are obtained from Definitions 4 and 5 when f is an identity mapping, and it is known that T-orbital completeness need not imply the completeness of X.

Definition 6. If for a point $x_0 \in X$ there exists a sequence $\{x_n\}$ such that the sequence $0_f(x_0)$ converges in X then X is called (T,f)-orbitally complete with respect to x_0 or simply (T,f,x_0)-orbitally complete.

Definition 7. A multivalued mapping $T : X \to CL(X)$ is said to be asymptotically regular at x_0 if, for each sequence $\{x_n\}$, $x_n \in Tx_{n-1}$, $\lim d(x_n, x_{n+1}) = 0$

Let $\psi = \{\phi : R_+ \to R_+ \mid \phi$ is upper semicontinuous and nondecreasing$\}$.

2. MAIN THEOREMS.

THEOREM 1. Let T be a multi-valued mapping from a metric space X to $CL(X)$. If there exist a mapping $f : X \to X$ such that $TX \subseteq fX$, for each $x , y \in X$,

$$H(Tx,Ty) \le \phi(\max\{D(fx,Tx),D(fy,Ty),D(fx,Ty),D(fy,Tx),d(fx,fy)\}), \quad (2.1)$$

$$\phi(t) < qt \text{ for each } t > 0 \text{ , for some fixed } \quad (2.2)$$

$$0 < q < 1 \text{ , } \phi \in \psi \text{ ,}$$

there exists an $x_0 \in X$ such that T is asymptotically regular at x_0 , (2.3)

and

$$X \text{ is } (T,f,x_0)\text{-orbitally complete,} \quad (2.4)$$

then T and F have a coincidence point.

PROOF. Pick $x_0 \in x$ satisfying (2.3). We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows. Since $TX \subset fX$, choose $y_1 = fx_1 \in Tx_0$. If $Tx_0 = Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $y_1 = y_2$. If $Tx_0 \ne Tx_1$, from the definition of H one can choose $y_2 = fx_2 \in Tx_1$ such that $d(y_1,y_2) \le q^{-1}H(Tx_0,Tx_1)$. In general, choose $y_{n+2} = fx_{n+2} \in Tx_{n+1}$ such that $y_{n+1} = y_{n+2}$ if $Tx_n = Tx_{n+1}$, and $d(y_{n+1},y_{n+2}) \le q^{-1}H(Tx_n,Tx_{n+1})$ otherwise.

From (2.3), $\lim d(y_n,y_{n+1}) = 0$. We wish to show that $\{y_n\}$ is Cauchy. It is sufficient to show that $\{y_{2n}\}$ is Cauchy. Suppose $\{y_{2n}\}$ is not Cauchy. Then there exists a positive ε such that, for each integer $2k$, there exist integers $2n(k)$, $2m(k)$ satisfying $2k \le 2n(k) < 2m(k)$, such that

$$d(y_{2n(k)} , y_{2m(k)}) > \varepsilon . \quad (2.5)$$

For each integer $2k$, let $2m(k)$ denote the smallest integer exceeding $2n(k)$ for which (2.5) is satisfied. Thus

$$d(y_{2n(k)} , y_{2m(k)-2}) \le \varepsilon . \quad (2.6)$$

For each integer $2k$, with $d_i = d(y_i,y_{i+1})$,

$$\varepsilon < d(y_{2n(k)} , y_{2m(k)}) \le d(y_{2n(k)} , y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1} .$$

Using (2.3) and (2.6) it follows that

$$\lim_k d(y_{2n(k)} , y_{2m(k)}) = \varepsilon . \quad (2.7)$$

Using the triangular inequality,

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2m(k)-1} ,$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2n(k)} + d_{2m(k)-1} .$$

From (2.3), (2.6) and (2.7) it follows that

$$\lim_k d(y_{2n(k)}, y_{2m(k)-1}) = \lim_k d(y_{2n(k)+1}, y_{2m(k)-1}) = \varepsilon .$$

For each integer $2k$ define $p(2k) = d(y_{2n(k)}, y_{2m(k)})$, $q(2k) = d(y_{2n(k)+1}, y_{2m(k)-1})$, and $r(2k) = d(y_{2n(k)}, y_{2m(k)-1})$. Then

$$\begin{aligned} p(2k) &\le d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)}) \\ &\le d_{2n(k)} + q^{-1}H(Tx_{2n(k)}, Tx_{2m(k)-1}) \\ &\le d_{2n(k)} + q^{-1}\phi(\max\{D(fx_{2n(k)}, Tx_{2n(k)}) , D(fx_{2m(k)-1}, Tx_{2m(k)-1}) , \\ &\qquad D(fx_{2n(k)}, Tx_{2m(k)-1}) , D(fx_{2m(k)-1}, Tx_{2n(k)}) , \\ &\qquad d(fx_{2n(k)}, fx_{2m(k)-1})\}) \\ &\le d_{2n(k)} + q^{-1}\phi(\max\{d_{2n(k)}, d_{2m(k)-1}, p(2k) , q(2k) , q(2k)\}) . \end{aligned}$$

Since ϕ is upper semicontinuous, taking the limit as $k \to \infty$ yields

$$\varepsilon \le q^{-1}\phi(\max\{0,0,\varepsilon,\varepsilon,\varepsilon\}) = q^{-1}\phi(\varepsilon) < \varepsilon ,$$

a contradiction.

Thus $\{y_n\}$ is Cauchy, and since fX is (T,f,x_0)-orbitally complete, $\{y_n\}$ converges to a point u in X . Hence there exists a point z in fX such that $u = fz$. Then

$$\begin{aligned} D(fz,Tz) &\le d(fz,fx_{n+1}) + D(fx_{n+1},Tz) \\ &\le d(fz,fx_{n+1}) + H(Tx_n,Tz) \\ &\le d(fz,fx_{n+1}) + \phi(\max\{(D(fx_n,Tx_n) , D(fz,Tz) , \\ &\qquad D(fx_n,Tz) , D(fz,Tx_n) , d(fx_n,fz)\}) \\ &\le d(fz,fx_{n+1}) + \phi(\max\{(d(fx_n,fx_{n+1}) , D(fz,Tz) , d(fx_n,fz) \\ &\qquad + D(fz,Tz) , d(fz,fx_{n+1}) , d(fx_n,fz)\}) . \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$D(fz,Tz) \le \phi(\max\{0,D(fz,Tz),D(fz,Tz),0,0\}) < qD(fz,Tz) ,$$

which implies $fz \in Tz$.

If, in (2.1) the terms $D(fx,Ty)$, $D(fy,Tx)$ are replaced by $[D(fx,Ty) + D(fy,Tx)]/2$, then $\{fx_n\}$ can be proved to be a Cauchy sequence without the assumption of the asymptotic regularity of T .

Replacing the condition $TX \subseteq fX$ by orbital regularity one obtains the following.

THEOREM 2. Let $T : X \to CL(X)$. If there exists a selfmap f of X such that (2.1),

(2') $\phi(t) < t$ for each $t > 0$, $\phi \in \psi$, and

(3') there exists a sequence $\{x_n\}$ such that the orbit $0_f(x_0)$ is regular and asymptotically regular, and X is (T,f,x_0)-orbitally complete,

then T and f have a coincidence point.

PROOF. Examining the proof of Theorem 1, the only change is to note that the regularity of the orbit $0_f(x_0)$ allows one to replace the inequality $d(y_n,y_{n+1}) \le g^{-1}H(Tx_n,Tx_{n+1})$ with the stronger inequality $d(y_n,y_{n+1}) \le H(Tx_n,Tx_{n+1})$.

If f is not the identity mapping, then a commuting T and f need not have a common fixed point. An example illustrating this fact appears in [19], where the commutativity of T and f is defined by $fTx = Tfx$, X not necessarily a metric space.

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The theorems of this paper generalize the corresponding results in [21], and the open question of [21] still remains; namely, what additional conditions will guarantee the existence of a common fixed point for T and f ?

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