Internat. J. Math. & Math. Sci.
Vol. 7 No. 3 (1984) 519-522



# FINITE DIMENSIONALITY IN SOCLE OF BANACH ALGEBRAS

**SIN-EI TAKAHASI**

Department of Mathematics
Ibaraki University
Mito, Ibaraki 310, Japan

(Received September 23, 1982)

ABSTRACT. It is shown that if the socle soc(A) of a semisimple Banach algebra A is norm-closed, then soc(A) is already finite dimensional. The proof makes use of the Al-Moajil theorem. However it is remarked that our main theorem is an extension of the Al-Moajil's.

KEY WORDS AND PHRASES. Compactrum, socle, minimal idempotent, minimal left ideal, annihilator.
1980 MATHEMATICS SUBJECTS CLASSIFICATION CODES. Primary 46H10; Secondary 47B05.

## 1. INTRODUCTION AND MAIN THEOREM.

Throughout the note, we will refer to the notations and terminologies in the Bonsall-Duncan's book [3]. Let A be a (complex) Banach algebra and let comp(A) be the compactrum of A, that is the set of all x in A such that the mapping : $a \to xax$ is a compact operator of A into itself. A. H. Al-Moajil [2] gives some characterizations of a finite dimensionality of a semisimple Banach algebra in terms of its compactrum and socle, which generalizes a theorem of A. W. Tullo [4]. Indeed his characterization is essentially the following

Theorem A (A. H. Al-Moajil [2]). Let A be a semisimple Banach algebra with lan(comp(A)) = {0}. Then A is finite dimensional if and only if soc(A) is norm-closed. Here lan(comp(A)) denotes the left annihilator of comp(A).

However in case of a semisimple Banach algebra A which does not satisfy the Al-Moajil condition : lan(comp(A)) = {0}, the closeness of soc(A) is not necessarily equivalent to the finite dimensionality of A. There exists an easy counter example. In fact let $A = C(\Omega)$ be the algebra of all continuous complex-valued functions on $= [0, 1] \cup \{2, 3, \ldots, n\}$, with supremum norm. As it is well-known that $comp(C([0, 1])) = \{0\}$, we have $comp(A) \cong C(\{2,3,\ldots n\}) \cong C^{n-1}$, where C denotes the field of complex numbers. Then $\mathrm{lan}(\mathrm{comp}(A)) \cong C([0,1])$ and so $\mathrm{lan}(\mathrm{comp}(A)) \neq \{0\}$. Also notice that $\mathrm{soc}(A) \cong C(\{2,3,\ldots,n\}) \cong C^{n-1}$, so that $\mathrm{soc}(A)$ is finite dimensional and hence norm-closed. But $A$ is of course infinite dimensional.

On the other hand, this counter example suggests to us the following statement.

THEOREM. Let $A$ be a semisimple Banach algebra with $\mathrm{soc}(A)$. Then $\mathrm{soc}(A)$ is finite dimensional if and only if $\mathrm{soc}(A)$ is norm-closed.

The purpose of this note is to prove the above theorem and to remark that our theorem is an extension of the Al-Moajil theorem. To do this, we will prepare some lemmas in the next section.

## 2. KNOWN RESULTS AND LEMMAS.

The next lemma can be seen in [3, pp. 155-156].

LEMMA 1. Let $A$ be a semiprime algebra. Then $L$ is a minimal left ideal of $A$ if and only if $L = Ae$, where $e$ is a minimal idempotent in $A$. The similar result holds for minimal right ideals. In particular, if $A$ has minimal left ideals, then $\mathrm{soc}(A)$ exists.

The next lemma appears in A. H. Al-Moajil [2].

LEMMA 2. Let $A$ be a semisimple Banach algebra. Then $\mathrm{comp}(A)$ is nonzero if and only if $\mathrm{soc}(A)$ exists, in this case $\mathrm{soc}(A) \subset \mathrm{comp}(A)$.

LEMMA 3. Let $A$ be a semisimple Banach algebra and $B$ a nonzero closed one-sided ideal of $A$. Then $B$ is not a radical algebra.

PROOF. Suppose that $B$ is radical and hence $B = \mathrm{rad}(B)$. Choose a nonzero element $b$ of $B$, so that $r(b) = 0$ from [3, Proposition 25.1(i)]. If $AB \subset B$, then $r(ab) = 0$ for all $a \in A$ and hence $b \in \mathrm{rad}(A)$ from [3, Proposition 25.1(ii)]. The semisimplicity of $A$ implies that $b = 0$, a contradiction. We therefore conclude that if $B$ is a closed left ideal of $A$, then $B$ is not radical. Of course, the same conclusion holds for closed right ideals.

LEMMA 4. Let $A$ be a semisimple Banach algebra with $\mathrm{soc}(A)$ and let $\min(A)$ be the set of all minimal idempotents of $A$. Then

$$\mathrm{lan}(\mathrm{comp}(A)) = \mathrm{lan}(\mathrm{soc}(A)) = \mathrm{lan}(\min(A)) = \mathrm{ran}(\min(A)).$$

In particular, $\mathrm{lan}(\mathrm{comp}(A)) = \mathrm{ran}(\mathrm{comp}(A))$. Here $\mathrm{ran}(\min(A))$ denotes the right annihilator of $\min(A)$.

PROOF. It is clear that $\mathrm{lan}(\mathrm{comp}(A)) \subset \mathrm{lan}(\mathrm{soc}(A)) = \mathrm{lan}(\min(A))$ from Lemma 2, the definition of $\mathrm{soc}(A)$ and Lemma 1. Now in order to show that $\mathrm{lan}(\mathrm{soc}(A)) \subset \mathrm{ran}(\min(A))$, assume, on the contrary, that $\mathrm{lan}(\mathrm{soc}(A)) \cap (A \setminus \mathrm{ran}(\min(A))) \neq \emptyset$ and take an element from this set, say $x$. Then since $x \notin \mathrm{ran}(\min(A))$, there is an element $e \in \min(A)$ with $ex \neq 0$ and hence $exA \neq \{0\}$ from the semisimplicity of $A$. Since $\{0\} \subsetneq exA \subset eA$ and $eA$ is a minimal right ideal of $A$ from Lemma 1, we have $exA = eA$, so that there exists an element $y \in A$ with $exy = e$. Note that $ye \in \mathrm{soc}(A)$, so that $xye = 0$ because $x \in \mathrm{lan}(\mathrm{soc}(A))$. However $e = exy = exye = 0$, a contradiction. We therefore conclude that $\mathrm{lan}(\mathrm{soc}(A)) \subset \mathrm{ran}(\min(A))$. Moreover the symmetric argument implies that $\mathrm{ran}(\min(A)) \subset \mathrm{lan}(\min(A))$ (consider the reversed algebra $\mathrm{rev}(A)$). In order to show that $\mathrm{lan}(\mathrm{soc}(A)) \subset \mathrm{lan}(\mathrm{comp}(A))$, we will apply the method which appears in the proof of [2, Lemma 3]. In fact suppose, on the contrary, that $\mathrm{lan}(\mathrm{soc}(A)) \cap (A \setminus \mathrm{lan}(\mathrm{comp}(A))) \neq \emptyset$ and take an element from this set, say $x$. Then since $x \notin \mathrm{lan}(\mathrm{comp}(A))$, there is an element $y \in \mathrm{comp}(A)$ with $xy \neq 0$. Set $J = \overline{xyA}$, the norm-closure of $xyA$. Then $J$ is a nonzero closed right ideal of $A$. Also $J$ is a compact Banach algebra from [2, Proposition 1] and it is not radical from Lemma 3. Then by [1, Theorem 4.3], $J$ contains a nonzero idempotent $e$ such that $eJe$ is finite dimensional. Since $eAe = e(eA)e \subset eJe$, it follows that $eAe$ is also finite dimensional and so $e \in \mathrm{soc}(A)$ from [1, Theorem 7.2]. Now since $e \in J$, we can write $e = \lim_{n\to\infty} xya_n$, $a_n \in A$ $(n = 1, 2, \ldots)$. But since each $ya_n e$ belongs to $\mathrm{soc}(A)$ and $x \in \mathrm{lan}(\mathrm{soc}(A))$, we have that $e = e^2 = \lim_{n\to\infty} xya_n e = 0$, a contradiction. We therefore conclude that $\mathrm{lan}(\mathrm{soc}(A)) \subset \mathrm{lan}(\mathrm{comp}(A))$ and hence

$$\mathrm{lan}(\mathrm{comp}(A)) = \mathrm{lan}(\mathrm{soc}(A)) = \mathrm{lan}(\min(A)) = \mathrm{ran}(\min(A)).$$

In particular the symmetric argument implies again that $\mathrm{lan}(\mathrm{comp}(A)) = \mathrm{ran}(\mathrm{comp}(A))$.

NOTE. By the above lemma, if $\mathrm{lan}(\mathrm{comp}(A)) = \{0\}$, then $\mathrm{ran}(\mathrm{comp}(A)) = \{0\}$. We then see that the conclusion of [2, Lemma 3] holds certainly for closed left ideals.

The next result is known in the structure theorem of a finite dimensional complex algebra (cf. [3, Proposition 26.7]), but we here give an alternative proof.

LEMMA 5. If $A$ is a semisimple finite dimensional Banach algebra, then it has an identity element.

PROOF. Note that $A = \mathrm{comp}(A)$ from the finite dimensionality of $A$ and $\mathrm{lan}(A) = \{0\}$ from the semisimplicity of $A$, so that $\mathrm{lan}(\mathrm{comp}(A)) = \{0\}$. Therefore the second half of the proof of [2, Theorem] implies directly the desired conclusion.

3. PROOF OF MAIN THEOREM.

Let denote by $B$ the norm-closure of $\mathrm{soc}(A)$ in $A$. Then $B$ is a closed two-sided ideal of $A$. Since $A$ is semisimple, it follows that $B$ is also semisimple. We first claim that $\mathrm{soc}(A)$ exists and $\mathrm{soc}(A) = \mathrm{soc}(B)$. Actually, choose a minimal left ideal $L$ of $A$ arbitrarily. Then by Lemma 1, we can write $L = Ae$, where $e$ is a minimal idempotent in $A$. But $\{0\} \subsetneq eBe \subset eAe$ and $eAe$ is one-dimensional and so $eBe = eAe$. In other words, $e$ is also a minimal idempotent in $B$. Again by Lemma 1, $Be$ is a minimal left ideal of $B$ and so $\mathrm{soc}(B)$ exists. Also since $e \in \mathrm{soc}(A)$ and $Ae \subset B$, $L = Ae = (Ae)e \subset \mathrm{soc}(B)$. We thus obtain that $\mathrm{soc}(A) \subset \mathrm{soc}(B)$. Conversely choose a minimal left ideal $M$ of $B$ and write $M = Bf$ for some minimal idempotent $f$ in $B$. But since $B$ is a two-sided ideal of $A$, $fAf \subset B$ and so $fAf = fBf$. Thus $f$ is also a minimal idempotent in $A$. Then $f \in \mathrm{soc}(A)$ and hence $M = Bf \subset \mathrm{soc}(A)$. In other words, $\mathrm{soc}(B) \subset \mathrm{soc}(A)$ and so $\mathrm{soc}(A) = \mathrm{soc}(B)$.

Now it is clear that if $\mathrm{soc}(A)$ is finite dimensional, then it is norm-closed, and hence assume conversely that $\mathrm{soc}(A)$ is norm-closed. Since $\mathrm{soc}(B)$ exists from the above argument, it follows from Lemma 2 that $\mathrm{soc}(B) \subset \mathrm{comp}(B)$ and so

$$B = \overline{\mathrm{soc}(A)} = \mathrm{soc}(A) = \mathrm{soc}(B) \subset \mathrm{comp}(B) \subset B,$$

where the bar denotes the norm-closure in A. Then B = soc(B) = comp(B). But since B is semisimple, lan(B) = {0}, so that lan(comp(B)) = {0}. Therefore Theorem A implies that B is finite dimensional and so is soc(A). The proof is complete.

## 4. REMARK.

In this section we see that our theorem implies the Al-Moajil theorem. Indeed if A is finite dimensional, then by [1, Theorem 7.2], A = soc(A) and hence soc(A) is norm-closed. Then it is sufficient to show that if the socle of a semisimple Banach algebra A with lan(comp(A)) = {0} is norm-closed, then A is already finite dimensional. Then assume that soc(A) is norm-closed and lan(comp(A)) = {0}. By our theorem, soc(A) is a semisimple finite dimensional Banach algebra and so it has an identity element e from Lemma 5. Let x be any element of A. Then x(1 - e) belongs to lan(soc(A)) and hence lan(comp(A)) from Lemma 4. Hence we obtain that x(1 - e) = 0. Also since soc(A) is a two-sided ideal of A, it follows that e is a central element of A. Then e is an identity element of A and so A = soc(A). Therefore A is finite dimensional.

ACKNOWLEDGMENT. The author was partially supported by the Grant-in-Aid for Scientific Research C-57540005 and 58540005 from the Ministry of Education.

## REFERENCES

1. ALEXANDER, J. C. Compact Banach algebras, Proc. London Math. Soc. 18(1968), 1-18.

2. AL-MOAJIL, A. H. The compactrum and finite dimensionality in Banach algebras, Internat. J. Math. & Math. Sci. 5(1982), 275-280.

3. BONSALL, F. F. and DUNCAN, J. Completely normed algebras, Springer-Verlag, Berlin Heidelberg New York (1973).

4. TULLO, A. W. Conditions on Banach algebras which imply finite dimensionality, Proc. Edinburgh Math. Soc. 20(1976), 1-5.